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Applications of the Fictitious Time Integration Method Using a New Time-Like Function

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Abstract: In this paper, a new time-like function with the incorporation of the fictitious time integration method (FTIM) is proposed. The new time-like function is modified from the original time-like function in the FTIM by adding a control parameter *m*, which dramatically improves the performance of the FTIM for solving highly nonlinear boundary value problems (BVPs) and plays as an important controller to assure the convergence of the solution during the time integration process. The requirements and the characteristics of the new time-like function are presented by examining the FTIM through the perspective of the new time-like function in which the limitation and the possible paths of improvement can be clarified. Several applications, including two- dimensional and three-dimensional BVPs using the FTIM with the new time-like function, are conducted. Results obtained demonstrate that the FTIM with the proposed time-like function can significantly improve accuracy as well as convergence. With the advantages and ease of numerical implementation, the new time-like function proposed in this study is seen to be a better alternative for the FTIM.

Keywords: Fictitious Time Integration Method (FTIM), time-like function, boundary value problem, three-dimensional, numerical method.

1 Introduction

A novel time integration method named the fictitious time integration method (FTIM) has been proposed by Liu and Atluri (2008c). Based on a novel continuation method, the FTIM embeds the linear or nonlinear algebraic equations into a system of nonautonomous first order ordinary differential equations (FOODEs) by introducing a time-like or fictitious variable. Since many numerical methods, such as

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the meshless, the finite difference and the finite element methods, lead to the solution of a system of linear or nonlinear algebraic equations [Zhu, Zhang and Atluri (1999), Atluri (2002), Atluri and Shen (2002), and Atluri, Liu and Han (2006)], it is essentially important to solve algebraic equations. The FTIM was first used to solve a nonlinear system of algebraic equations by introducing fictitious time [Liu (2008a, 2008f)], such that it is a mathematically equivalent system in the augmented n+1-dimensional space as the original algebraic equation system is in the original n-dimensional space. The fixed point of these evolution equations, which is the root for the original algebraic equation, is obtained by applying numerical integrations on the resultant ordinary differential equations, which do not require the information of derivative of nonlinear algebraic equations and their inverse. Based on a time marching algorithm, Liu (2008b) introduced the use of the FTIM to solve the nonlinear obstacle problems. The FTIM has also been adopted to tackle two-dimensional quasilinear elliptic boundary value problems by Liu (2008d). Six examples including Laplace, Poisson, reaction diffusion, Helmholtz, the minimal surface, as well as explosion equations were tested therein. It is interesting that the FTIM can easily deal with the nonlinear boundary value problems and has high efficiency as well as high accuracy. Liu and Atluri (2008e) proposed the idea of using the FTIM for solving a nonlinear optimization problem (NOP) under multiple equality and inequality constraints. The Kuhn-Tucker optimality conditions are used to transform the NOP into a mixed complementarity problem. With the aid of NCP-functions a set of nonlinear algebraic equations are obtained; then the FTIM is used to solve these nonlinear equations. Furthermore, Liu (2008f, 2009a) proposed the use of the FTIM for solving the discretized inverse Sturm-Liouville problems and m-point boundary value problems (BVPs).

The FTIM has recently been demonstrated as an important numerical tool for its ability to solve a certain class of problems more effectively than Newton-like methods [Dennis and More (1974, 1977), and Spedicato and Huang (1997)] in that the FTIM does not need to calculate the Jacobian matrix and its inverse and is thus very time saving, and that the FTIM is insensitive to the guessing of initial conditions, and it is thus easy to find the roots. It is believed that this new method may bring a major revolution to the computational fields. With the aim of improving the convergence and ease of implementation of the numerical algorithm, there exists a continued interest in the FTIM. In this study, a new time-like function with the incorporation of the FTIM is proposed. The new time-like function is modified from the original time-like function in the FTIM by adding a control parameter m, which dramatically improves the performance of FTIM for solving highly nonlinear BVPs and gives an important control to assure the convergence of the solution during the time integration process.

The requirements and the characteristics of the new time-like function are presented by examining the FTIM through the perspective of the new time-like function in which the limitation and the possible paths of improvement can be clarified. Several examples including two-dimensional and three-dimensional BVPs using the FTIM with the new time-like function have been conducted. With the advantages and the ease of numerical implementation, the new time-like function proposed in this study is seen to be a better alternative for the FTIM.

2 FTIM with the New Time-Like Function

Based on the idea of introducing a fictitious time, the FTIM can transform a system of linear or nonlinear algebraic equations into an ODEs system. The fictitious time function used in the transformation was named the time-like function. Liu and Atluri (2008c) have first pointed out that the time-like function, $q(t)$, has to be differentiable, $q(0) = 1$, and $q(\infty) = \infty$. In addition, they also mentioned that the time-like function may have other choices if one can provide better improvement than the original. Later, Liu and Atluri (2009b) have solved a system of ill-posed linear algebraic equations which may result from the discretization of a first-kind linear Fredholm integral equation by the FTIM. In their study, a more general form of the time-like function, $q(\tau) = (1+\tau)^{\gamma}$, $0 \le \gamma \le 1$, which uses a fictitious time variable τ, was addressed. In this study, the new time-like function based on the general form with the incorporation of the FTIM is proposed and the formulation of the new time-like function is described as follows.

2.1 A new time-like function

Let us consider the following algebraic equations:

$$F_i(x_1, \ldots, x_n) = 0, \; i = 1, \ldots, n \tag{1}$$

In this study, a modified FTIM is proposed based on the theory of the present FTIM by introducing a new time-like function, $q(t)$, instead of the original one:

$$y_i(t) = q(t)x_i, \; i = 1, \ldots, n \tag{2}$$

$$q(t) = (1+t)^m, \, 0 \le m \le 1 \tag{3}$$

where t is a variable which is independent of x_i. If we let $m = 1$, Eq. (3) becomes the original FTIM time function. Taking the derivative of Eq. (2) with respect to t, we have

$$\dot{y}_i = \frac{dy_i}{dt} = m(1+t)^{m-1}x_i \tag{4}$$

We multiply Eq. (1) by a non-zero coefficient, v, leading to

$$0 = -vF_i(x_1,\dots,x_n). \tag{5}$$

Using Eq. (2), we have

$$0 = -vF_i\left(\frac{y_1}{(1+t)^m},\dots,\frac{y_n}{(1+t)^m}\right). \tag{6}$$

Recalling that $\dot{y}_i = m(1+t)^{m-1}x_i$ by Eq. (4), and adding it on both the sides of the above equation we obtain

$$\dot{y}_i = m(1+t)^{m-1}x_i - vF_i\left(\frac{y_1}{(1+t)^m},\dots,\frac{y_n}{(1+t)^m}\right) \tag{7}$$

Then, using $x_i = y_i/(1+t)^m$, we can write Eq. (7) as an ODEs system for y_i:

$$\dot{y}_i = m(1+t)^{m-1}\frac{y_i}{(1+t)^m} - vF_i\left(\frac{y_1}{(1+t)^m},\dots,\frac{y_n}{(1+t)^m}\right) \tag{8}$$

Finally, multiplying each equation by the integrating factor $1/(1+t)^m$ and using Eq. (8) again we obtain

$$\frac{\dot{y}_i}{(1+t)^m} = \frac{m(1+t)^{m-1}}{(1+t)^m}\frac{y_i}{(1+t)^m} - \frac{v}{(1+t)^m}F_i\left(\frac{y_1}{(1+t)^m},\dots,\frac{y_n}{(1+t)^m}\right) \tag{9}$$

It is known that

$$\frac{d}{dt}\left(\frac{y_i(t)}{(1+t)^m}\right) = \frac{\dot{y}_i}{(1+t)^m} - \frac{m(1+t)^{m-1}}{(1+t)^m}\frac{y_i}{(1+t)^m} \tag{10}$$

Using Eq. (10), Eq. (9) can be rewritten as

$$\frac{d}{dt}\left(\frac{y_i(t)}{(1+t)^m}\right) = \frac{-v}{(1+t)^m}F_i(x_1,\dots,x_n),\ i = 1,\dots,n \text{ and } 0 < m \le 1 \tag{11}$$

Further using $y_i(t) = (1+t)^m x_i$, Eq. (11) becomes

$$\dot{x}_i = \frac{-v}{(1+t)^m}F_i(x_1,\dots,x_n),\ i = 1,\dots,n \text{ and } 0 < m \le 1 \tag{12}$$

We may employ a forward Euler scheme to integrate Eq. (12) by starting from a chosen initial condition. The following equation represents for the numerical integration scheme of Euler methods [Press, Teukolsky, Vetterling and Flannery (2007)]:

$$x_i^{k+1} = x_i^k - \frac{h\nu}{(1+t_k)^m} F_i\left(x_1^k, \ldots, x_n^k\right), \; i = 1, \ldots, n \text{ and } 0 < m \leq 1 \tag{13}$$

where h is the time step size and $x_i^k = x_i(t_k)$ is the value of x_i at the k-th discrete time $t_k = kh$.

2.2 Requirements of the new time-like function

To understand the requirements of the new time-like function proposed, we first consider a simple algebraic equation as

$$F(x) = x^2 - 1 = 0 \tag{14}$$

Using Eq. (13), we have

$$x^{k+1} = x^k - \frac{h\nu}{(1+t_k)^m} F(x) \tag{15}$$

The numerical procedure starts from an initial value of x_0 which can be arbitrarily chosen. Then, the FTIM integrates Eq. (15) from $t = 0$ to a selected final time. In the numerical integration process, the convergence criterion of x_i is written as

$$\sum_{i=1}^{n} \left(x_i^{k+1} - x_i^k\right)^2 \leq \varepsilon^2 \tag{16}$$

where ε is a given convergent criterion. If at a certain time the above criterion is satisfied, then the solution of x_i is obtained.

To reach the criterion of Eq. (16), two possible states may be found from Eq. (15). The first one is that the value of $F(x)$ tends to zero and the solution of $F(x)$ is found. On the contrary, if the convergence goes slow due to the fact that the equation involves high nonlinearity or a large system of equations exists, a long fictitious time may be required to approach the solution. A large t_k contributed by a long fictitious time may cause the inverse of time-like function, $q(t)$, to become a very small value. In the worst scenario, the criterion of convergence of Eq. (16) may be reached due to the fact that $h\nu/(1+t_k)^m$ of Eq. (15) is close to zero with very large t_k but the solution of x_i does not satisfy the equation of $F(x) = 0$ at all.

2.3 Characteristics of the new time-like function

Since the numerical integration process needs to integrate Eq. (15) from $t = 0$ to a final time to reach the solution, the inverse of $q(t)$ decreases continuously with time. It is, therefore, important to control the inverse of $q(t)$ to assure that Eq. (15) is satisfied only when the value of $F(x)$ closes to zero but not the inverse of $q(t)$. The methodology of the original FTIM does not have a direct control for manipulating the speed of convergence from $q(t)$. Accordingly, the new time-like function introduces a new parameter m which can be used to control the inverse of $q(t)$ along with the time integration process. In fact, the new parameter is to control and to increase the convergence as to prevent the latter situation addressed above.

A simple numerical test was conducted by applying the new time-like function to solve Eq. (14) using the same parameters of h =0.1, ν = 1.0, and the initial guess of $x = 5$. Fig. 1 shows the comparison between the coefficient of $1/q(t)$ in Eq. (15) with m values of 1 and 0.1 versus the number of fictitious time steps. It can be seen that the value of $1/q(t)$ for $m = 0.1$ decreases slower than that of $m = 1$. This demonstrates the possibility for making $F(x) = 0$ but not $1/q(t)$ in Eq. (15).

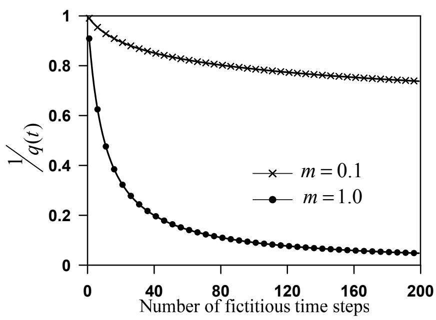


Figure 1: Comparison of $1/q(t)$ with m values of 1 and 0.1 versus the number of fictitious time steps.

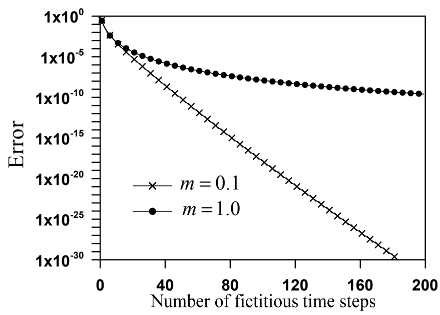


Figure 2: Comparison of error using $q(t)$ with m values of 1.0 and 0.1 versus the number of fictitious time steps.

Fig. 2 demonstrates the error, which is defined as the absolute value of the difference between calculated solution and the exact one, versus the number of fictitious time steps. From the results obtained, it is found that the parameter m in the new time-like function dramatically increases the speed of convergence.

Liu and Atluri (2008c) have applied the FTIM to solve the following simple algebraic equation.

$$F(x) = x^3 - 3x^2 + 2x = 0 \tag{17}$$

In the study, we adopted the new time-like function to solve Eq. (17). The roots of Eq. (17) are 0, 1, and 2. In this example, we compute the root of 2 for the above equation using the FTIM with the original and the new time-like functions by the same parameters of h =0.1, ν = 0.2, and the initial guess of $x = 5$.

Fig. 3 shows the speed of convergence to the exact solution of $x = 2$ for the FTIM with the original and the new time-like functions. The results show that the FTIM with the new time-like function can reach the exact solution of $x = 2$ within only 200 steps which is much faster than the original FTIM. Fig. 4 is the comparison of error versus the iteration number for the time-like function, $q(t)$ with m values of 0.01, 0.1, and 1. It also demonstrates that the error reduced faster using a smaller value of m at the same iteration number. However, it should be noted that the original FTIM performs well at the beginning of the evolution. It can be seen in Fig. 4 that the error of the original FTIM, i.e. the m value of 1, is smaller than those of the m values of 0.01 and 0.1 while the iteration number is less than 40.

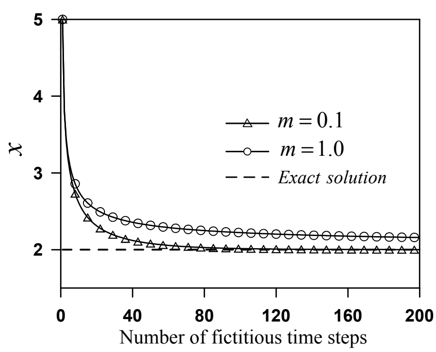


Figure 3: Speed of convergence to the exact solution of $x = 2$ for the FTIM using $q(t)$ with m values of 1.0 and 0.1.

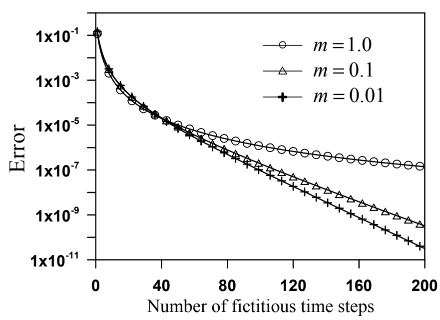


Figure 4: Comparison of error using $q(t)$ with m values of 0.01, 0.1, and 1.0 versus the number of fictitious time steps.

Fig. 5 shows that the evolution of the coefficients, $1/q(t)$, versus the number of time step for m values of 0.01, 0.1, and 1. Since the integration of the FTIM starts from $t = 0$ to a selected final time, $1/q(t)$ will decrease as time proceeds. A very small $1/q(t)$ may lead the adjustment of x_i from the k-th to *(k+1)*-th step to become infinitesimal which may strongly affect the convergence. In the worst scenario, the convergence to approach the solution may nearly stop when $1/q(t)$ is close to zero. This is why we introduce the m value in the new time-like function to control the value of $1/q(t)$ in the numerical integration process. Fig. 5 demonstrates that $1/q(t)$ can be controlled by using the m value. When we introduce a small value of

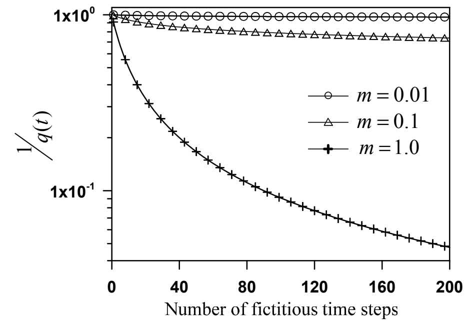


Figure 5: Decrease of $1/q(t)$ with m values of 0.01, 0.1, and 1.0 versus the number of fictitious time steps.

m, the value of $1/q(t)$ during the numerical integration process can be efficiently controlled. Another question may be raised about how small the m value should be. According to the numerical tests as shown in Fig.5, we found that it would be good enough for the efficient control of the value of $1/q(t)$ not to become a too small value to overshade $F(x)$ if the value of m is of 0.01. For the value of m less than 0.1, the improvement of the control does not affect too much in our test case. However, the m value can not be too small because a very small value of m may cause the solution to be divergent.

With the incorporation of the new time-like function into the FTIM, it is found that the convergence can be dramatically increased if we adopted a small value of m. The value of m also plays an important role in controlling the criterion of convergence so that $F(x) \approx 0$ can be satisfied.

2.4 Application to BVPs

Two-dimensional quasilinear elliptic BVPs were first solved by Liu (2008d) using the FTIM. Based on the new time-like function described in section 2.1, the formulation of three-dimensional BVPs for the quasilinear elliptical PDE can be derived as following:

$$\Delta u(x,y,z) = G(x,y,z,u,u_x,u_y,u_z,...),\ (x,y,z) \in \Omega \tag{18}$$

$$u(x,y,z) = H(x,y,z),\ (x,y,z) \in \Gamma \tag{19}$$

where Δ is a Laplacian operator, Γ is the boundary of the interested domain Ω, and G and H are given functions. To solve the three-dimensional BVPs using the FTIM, we adopted a semi-discretization technique by introducing a finite difference discretization [Duffy (2006)] at the three-dimensional discrete grid point (i,j,k)

and then we may employ a forward Euler scheme by starting from a chosen initial conditions:

$$u_{i,j,k}^{n+1} = u_{i,j,k}^{n} - \frac{hv}{(1+t_n)^m}\Bigg(\frac{1}{(\Delta x)^2}\left(u_{i+1,j,k} - 2u_{i,j,k} + u_{i-1,j,k}\right)$$
$$+ \frac{1}{(\Delta y)^2}\left(u_{i,j+1,k} - 2u_{i,j,k} + u_{i,j-1,k}\right) + \frac{1}{(\Delta z)^2}\left(u_{i,j,k+1} - 2u_{i,j,k} + u_{i,j,k-1}\right)$$
$$-G(x_i, y_j, z_k, u_{i,j,k}, \frac{u_{i+1,j,k} - u_{i-1,j,k}}{2\Delta x}, \frac{u_{i,j+1,k} - u_{i,j-1,k}}{2\Delta y}, \frac{u_{i,j,k+1} - u_{i,j,k-1}}{2\Delta z}, \ldots)\Bigg) \tag{20}$$

Comparing Eq. (20) with Eq. (13) described above, we can find

$$F_{i,j,k} = \Bigg(\frac{1}{(\Delta x)^2}\left(u_{i+1,j,k} - 2u_{i,j,k} + u_{i-1,j,k}\right) + \frac{1}{(\Delta y)^2}\left(u_{i,j+1,k} - 2u_{i,j,k} + u_{i,j-1,k}\right)$$
$$+ \frac{1}{(\Delta z)^2}\left(u_{i,j,k+1} - 2u_{i,j,k} + u_{i,j,k-1}\right)$$
$$-G(x_i, y_j, z_k, u_{i,j,k}, \frac{u_{i+1,j,k} - u_{i-1,j,k}}{2\Delta x}, \frac{u_{i,j+1,k} - u_{i,j-1,k}}{2\Delta y}, \frac{u_{i,j,k+1} - u_{i,j,k-1}}{2\Delta z}, \ldots)\Bigg) \tag{21}$$

where h is a time step size, $0 < m \leq 1$, and $u_{i,j,k}^{n}$ is the value of $u_{i,j,k}$ at the n-th discrete fictitious time t_n.

To validate the above formulations, a one-dimensional boundary value problem was first solved using the new time-like function. Liu and Atluri (2008c) have first applied the FTIM to solve the following boundary value problem:

$$\Delta u(x) = \frac{3}{2}u(x)^2 \tag{22}$$

$$u(0) = 4, \quad u(1) = 1 \tag{23}$$

The exact solution is

$$u(x) = \frac{4}{(1+x)^2} \tag{24}$$

By introducing a finite difference discretization of u at the grid points, from Eq. (20) we can obtain

$$u_i^{n+1} = u_i^{n} - \frac{hv}{(1+t_n)^m}\left(\frac{1}{(\Delta x)^2}\left(u_{i+1,} - 2u_i + u_{i-1}\right) - \frac{3}{2}u_i^2\right) \tag{25}$$

where $u_0 = 4$, $u_{n+1} = 1$, $\Delta x = l/(n+1)$ is the grid length, and l is the total length. The initial guesses of u_i are 0.

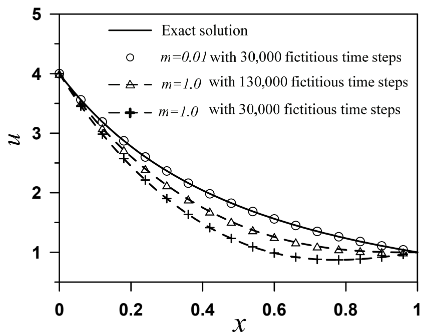


Figure 6: Results obtained using the FTIM with original and new time-like functions.

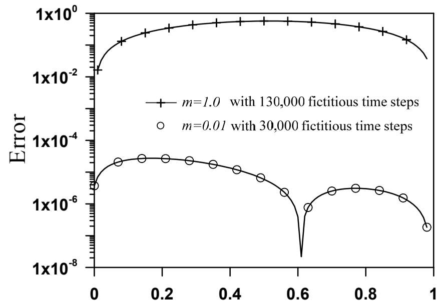


Figure 7: Error comparison of the FTIM using original and new time-like functions when the number of fictitious time steps is 30000.

Using the following parameters $n = 99$, $l = 10$, $h = 0.01$, $v = -0.1$, $m = 0.01$, the one-dimensional boundary value problem as shown in Eq. (22) can be solved by the FTIM with the new time-like function. Fig. 6 shows the results computed by the FTIM with the original time-like function and the new time-like function proposed. From the comparison of the results, it is apparent that the FTIM with the new time-like function reaches the final solution much faster than the original time-like function. The iteration number for approaching the solution is about 30,000. The error of the FTIM using the new time-like function can also reduce much smaller than the original FTIM as shown in Fig. 7. In this numerical example, we have found that the convergence of using the original time-like function goes extremely slow after the iteration number is greater than 30,000. The final value of the inverse of the original time-like function is close to 10^{-7} when the iteration number is 130,000. In this special case, we can find that the inverse of the original time-like function will eventually evolve into an infinitesimal value which may stop the process of the evolution of the fictitious time integration to approach the true solution. On the contrary, the new time-like function provides a better control to avoid the problem of a small value of the inverse of time-like function.

Though it should not be difficult for the FTIM using the original time-like function to reach the solution with the adjustment of the coefficient of v, it is sometimes hard to find an appropriate value of v.

3 Numerical illustrations

Now we are ready to use more examples to validate the performance of the FTIM by employing the new time-like function.

3.1 Example 1

The first example to be solved is a groundwater flow equation. This equation is often used if a three-dimensional unconfined flow field is reduced to a two-dimensional horizontal flow field by the invocation of the Dupuit-Forcheimer theory. This equation can be written as

$$\Delta u^2(x,y) = -N \tag{26}$$

where N is the infiltration rate which can be a function of position or a constant. The domain is given by $\Omega = \{(x,y)\,|0 \le x \le 9,\ 0 \le \mathrm{y} \le 9\}$. The boundary conditions are as following:

$$u(0,y) = 8,\ u(9,y) = 2 \tag{27}$$

$$\frac{\partial u}{\partial y}(x,\ 0) = 0,\ \frac{\partial u}{\partial y}(x,\ 9) = 0 \tag{28}$$

Based on Eq. (20) derived above, we can introduce a finite difference discretization of u at the grid points and obtain

$$u_{i,j}^{n+1} = u_{i,j}^{n} - \frac{hv}{(1+t_n)^m}\left(\frac{1}{(\Delta x)^2}\left(u_{i+1,j}^2 - 2u_{i,j}^2 + u_{i-1,j}^2\right)\right.$$
$$\left.+\frac{1}{(\Delta y)^2}\left(u_{i,j+1}^2 - 2u_{i,j}^2 + u_{i,j-1}^2\right) + N)\right) \tag{29}$$

where $\Delta x = \Delta y = 1$. The parameters are $n = 8\times 8$, $h = 0.01$, $v = -2$, $m = 0.01$, $N = 0$. Except for the boundary nodes, the initial values for the analysis were all set to zero. The analytical solution of Eqs. (26) to (28) can be written as follows, which is a one-dimensional solution due to the boundary condition given,

$$u^2 = u^2(0,y) - (u^2(0,y) - u^2(9,y))\frac{x}{9} \tag{30}$$

To compare with the analytical solution, the two-dimensional horizontal flow field can be reduced to an one-dimensional flow problem by assigning the boundary conditions of Eqs. (27) and (28). From the results obtained as shown in Fig. 8, it is evident that the numerical solution is convergent only within 750 steps with the maximum error of 10^{-13}. The error for each node of $u_{i,j}$ presented in Fig. 8 was computed using $\left|u_{computed} - u_{exact}\right|$. Comparing the computed results with analytical solutions as shown in Fig. 9, we can also find that the FTIM with the new time-like function is convergent within only few iterations and has very high accuracy.

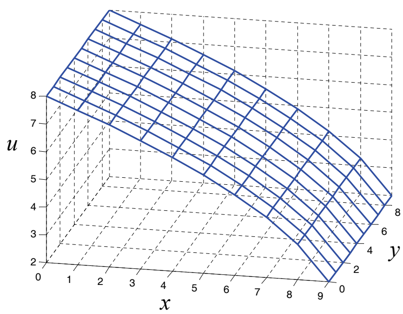


Figure 8: Results obtained using the FTIM with the new time-like function.

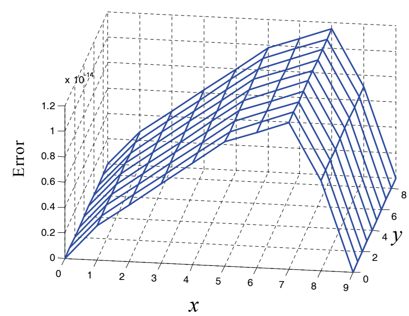


Figure 9: Error comparison of the computed results and the analytical solutions.

3.2 *Example 2*

The second example is a two-dimensional nonlinear Poisson equation:

$$\Delta u = 3u^2 \tag{31}$$

The analytical solution of Eq. (31) is

$$u(x,y) = \frac{4}{(1+x+y)^2} \tag{32}$$

Based on Eq. (20) derived above, we can introduce a finite difference discretization of u at the grid points and obtain

$$u_{i,j}^{n+1} = u_{i,j}^{n} - \frac{hv}{(1+t_n)^m}\left(\frac{1}{(\Delta x)^2}(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}) + \frac{1}{(\Delta y)^2}(u_{i,j+1} - 2u_{i,j} + u_{i,j-1}) - 3u_{i,j}^2\right) \tag{33}$$

where $\Delta x = \Delta y = l/(n+1)$. The domain is given by $\Omega = \{(x,y)\,|0 \le x \le 1,\ 0 \le y \le 1\}$. The boundary constraints can be obtained by using the exact solution of Eq. (32)

and directly assigned the Dirichlet values on the boundaries. The initial values for the analysis were all set to zero. The parameters used in this analysis are $n = 99 \times 99$, $l = 1$, $h = 0.01$, $v = -0.01$. In order to test the performance of our study method, the grid number was set to a large value with 100×100 rectangular mesh.

The one-dimensional numerical solution has been solved by Liu (2006). In his study, $n = 49$ was used and the numerical error was in the order of 10^{-4}. Figs. 10 and 11 show the computed results and the numerical error. Comparing the results with the exact solutions, good agreement can be found and the maximum and the average error are 1.6×10^{-5} and 8.33^{-7}, respectively. This numerical example also demonstrates the capability of the proposed method to solve large dimensional problems.

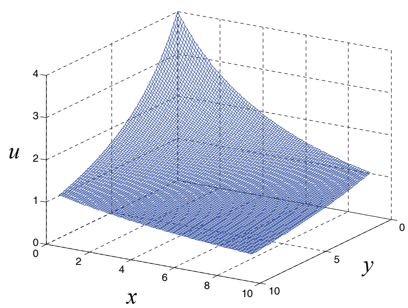


Figure 10: Results obtained using the FTIM with the new time-like function.

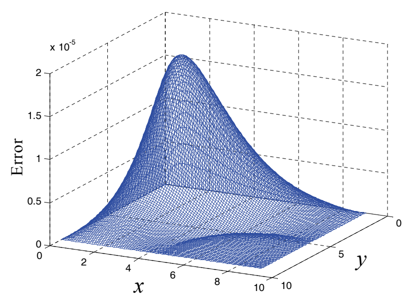


Figure 11: Error comparison of the computed results and the analytical solutions.

3.3 Example 3

The following example is a two-dimensional nonlinear Helmholtz equation:

$$\Delta u = k^2(u)u \tag{34}$$

where the $k^2(u)$ is assigned as $4u^2$. The analytical solution of Eq. (34) is

$$u(x,y) = \frac{1}{(x+y+1)} \tag{35}$$

Based on Eq. (20) derived above, we can introduce a finite difference discretization of u at the grid points and obtain

$$u_{i,j}^{n+1} = u_{i,j}^{n} - \frac{hv}{(1+t_n)^m}\left(\frac{1}{(\Delta x)^2}\left(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right) + \frac{1}{(\Delta y)^2}\left(u_{i,j+1} - 2u_{i,j} + u_{i,j-1}\right) - 4u_{i,j}^3\right) \quad (36)$$

where $\Delta x = \Delta y = 1$. The domain is given by $\Omega = \{(x,y)\,|0 \le x \le 13,\ 0 \le y \le 7\}$. The boundary constraints can be obtained by using the exact solution of Eq. (35) and directly assigned the Dirichlet values on the boundaries. Except for the boundary nodes, the initial values for the analysis were all set to zero. Using the following parameters $n = 12 \times 6$, $h = 0.01$, $v = -10$, this nonlinear Helmholtz equation was solved by the FTIM. Figs. 12 and 13 show the computed results and the numerical error. Comparing the results with the exact solutions, good agreement can be found and the maximum error is about 2.13×10^{-7}. This numerical example also demonstrates that along with the time evolution process, numerical solutions of very high accuracy to an order of 10^{-7} can be reached.

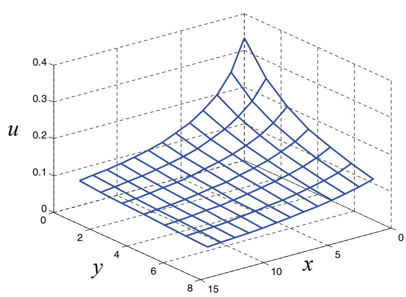


Figure 12: Results obtained using the FTIM with the new time-like function.

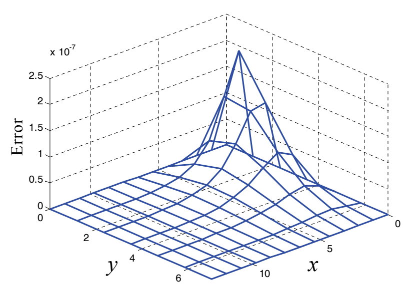


Figure 13: Error comparison of the computed results and the analytical solutions.

3.4 Example 4

The following example is a three-dimensional nonlinear Helmholtz equation:

$$\Delta u = k^2(u)u \quad (37)$$

where the $k^2(u)$ is assigned as $6u^2$. The analytical solution of Eq. (37) is

$$u(x,y,z) = \frac{1}{(x+y+z+1)} \quad (38)$$

It should be noted that the singularity is on $x+y+z=-1$. Based on Eq. (20) derived above, we can introduce a finite difference discretization of u at the grid points and obtain

$$u_{i,j,k}^{n+1}=u_{i,j,k}^{n}-\frac{hv}{(1+t_n)^m}\left(\frac{1}{(\Delta x)^2}\left(u_{i+1,j,k}-2u_{i,j,k}+u_{i-1,j,k}\right)\right.$$
$$\left.+\frac{1}{(\Delta y)^2}\left(u_{i,j+1,k}-2u_{i,j,k}+u_{i,j-1,k}\right)+\frac{1}{(\Delta z)^2}\left(u_{i,j,k+1}-2u_{i,j,k}+u_{i,j,k-1}\right)-6u_{i,j,k}^{3}\right) \tag{39}$$

Comparing Eq. (39) with Eq. (13) described above, we can find

$$F_{i,j,k}=\left(\left(\frac{1}{(\Delta x)^2}\left(u_{i+1,j,k}-2u_{i,j,k}+u_{i-1,j,k}\right)\right.\right.$$
$$\left.+\frac{1}{(\Delta y)^2}\left(u_{i,j+1,k}-2u_{i,j,k}+u_{i,j-1,k}\right)+\frac{1}{(\Delta z)^2}\left(u_{i,j,k+1}-2u_{i,j,k}+u_{i,j,k-1}\right)-6u_{i,j,k}^{3}\right) \tag{40}$$

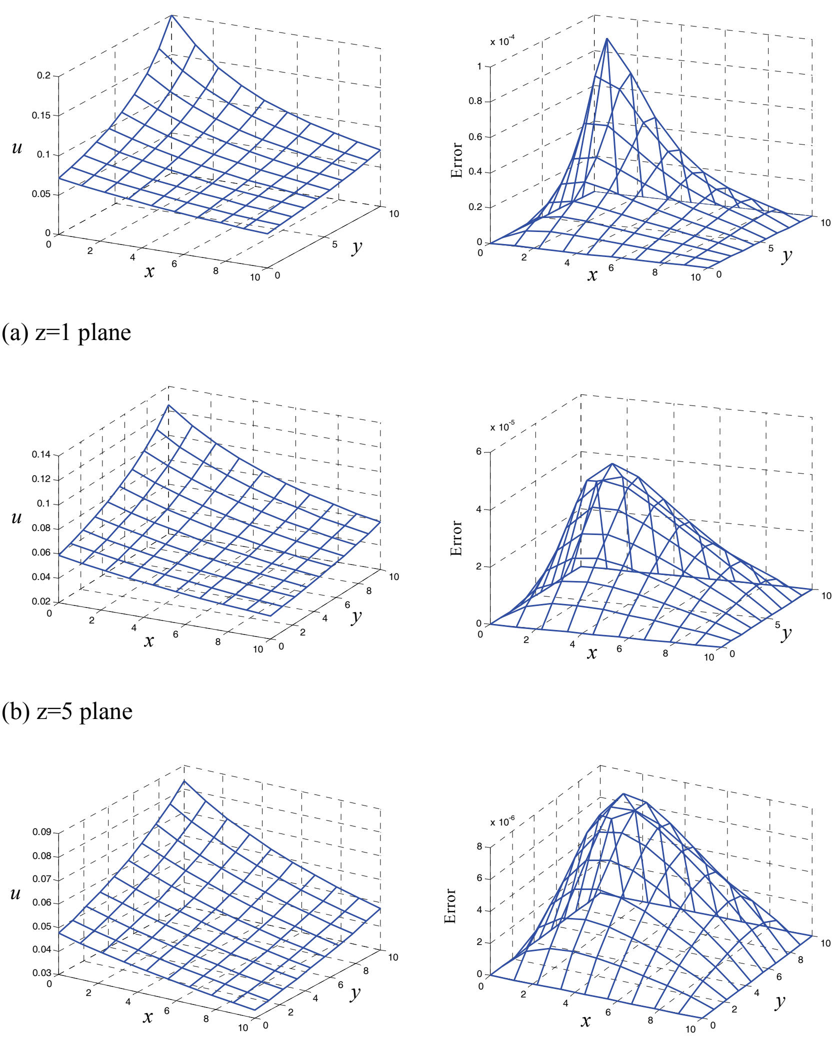
where $\Delta x=\Delta y=\Delta z=1$and the domain is given by

$\Omega=\{(x,y,z)\,|0\le x\le 9,\ 0\le \mathrm{y}\le 9,\ 0\le z\le 9\}$.

The boundary constraints can be obtained by using the exact solution of Eq. (28) and directly assigned the Dirichlet values on the boundaries. Using the following parameters $n=9\times 9\times 9$, $h=0.1$, $v=-10$, the three-dimensional nonlinear Helmholtz equation was solved by the FTIM as shown in Fig. 14. The comparison of the results and the exact solutions was also conducted as shown in Fig. 14. The maximum error is about 9.2×10^{-5}. From the results obtained, we can find that the FTIM with the new time-like function is very effective and gives very small error to an order of 10^{-6} for solving this three-dimensional highly nonlinear problem.

4 Conclusions

In this study, a new time-like function with the incorporation of the fictitious time integration method (FTIM) is proposed. The important fundamental concepts and the construct of the FTIM using the new time-like function were clearly addressed. Several examples including two-dimensional and three-dimensional BVPs using the FTIM with the new time-like function have been conducted. Findings from this study are drawn as follows.



(a) z=1 plane

(b) z=5 plane

(c) z=8 plane

Figure 14: Results obtained using the FTIM with the new time-like function (left) and error comparison (right) for z = 1, 5, and 8 planes.

In some specific situations, the original time-like function in FTIM may cause the problem of slow convergence when the fictitious time is large. Although the coefficient v proposed in the original FTIM can be used to increase the stability of numerical integration and the speed of convergence, however, it can be only effective in the first few steps and cannot be adjusted during the process of time integration. The new time-like function introduced in the FTIM provides a control of the inverse of $q(t)$ using the m value to assure that the $F(x)$ approaches to zero but not the inverse of $q(t)$.

Based on several numerical examples conducted in this study, it is apparent that the new time-like function can dramatically increase the convergence when a small value of m is used. Results from the numerical tests also show that a value of m less than 1 can efficiently control the inverse of $q(t)$. In addition, it is found that the value of m should be equal to or less than 0.1 for better performance. However, the m value can not be too small because a very small value of m may cause the solution to be divergent. Further study of the optimal value may be needed in the future.

Several numerical examples including a two-dimensional nonlinear flow equation, a two-dimensional Poisson equation and two-dimensional and three-dimensional nonlinear Helmholtz equations were analyzed in this study. Results obtained show that with the advantages and the ease of numerical implementation, the new time like function proposed in this study is seen to be a better alternative for the FTIM.

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