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Isoparametric FEM vs. BEM for Elastic Functionally Graded Materials

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Abstract: A Field Boundary Element Method (FBEM) for Functionally Graded Materials (FGM) is presented and compared with Isoparametric Finite Element Method. The presented formulation, using the Kelvin's fundamental solution, is able to analyse structures although no fundamental solution is actually known. Isoparametric FGM Finite Element Method is a well established tool for FGM structural analysis. The comparison shows that both FBEM and FEM give accurate results. In the paper, the solution of some examples for 2D plates are reported both using FEM and FBEM. Some comparisons with analytical results are discussed and accuracy of the solutions is highlighted. The comparison between FBEM and isoparametric FEM suggest that, in the case of FGM, methods have to be preferred on account of the analysis particular aspects likely for homogeneous problems.

Keywords: Functionally Graded Materials (FGM), Isoparametric Finite Element Method (FEM), FGM Field Boundary Element Method (FGM FBEM), Boundary Element Method (BEM).

1 Introduction

Functionally Graded Materials (FGM) structures are described by material properties that vary continuously within the body. The variability depends, in the majority of practical situations, on the microstructure. It consists of different component material phases whose lattice is designed accordingly to desired mechanical behaviour. Several works have been published concerning technological, mechanical and theoretical aspects of FGMs [Ichikawa (2001), Watanabe and Ziegler (2002), Pan, Gong, Chen (2003)] and more recently [Han at. all (2006), Dawson at all (2005)]. FGM has to be intended as a functional description of the behaviour of composites when the scale of the analysis is grater then characteristic length scale of the composite. In this way many multiscale analysis and composite formulations

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are strictly related to FGM [Raghavan, Ghosh (2004)]. Recently published papers deal with composites and many aspects are investigated. Usually twofold strategies are used to resolve FGM mechanics: the first possibility is to apply multiscale analysis. Namely the overall properties of FGM is derived by the computational investigation of the microstructure constituted by single phase elements [Sfantos and Aliabadi (2007)]. The second consists to assume overall properties from homogenization theory and to resolve the actual FGM as a material with variable parameters. Both strategies shall be used in order to solve structural problems although in some practical situations both methods give rise to cumbersome calculations.

From the numerical standpoint, generalized isoparametric formulation of Finite Element Method (FEM), proposed by Kim and Paulino [Kim and Paulino (2002)] extends the Finite Element Method to variable elasticity materials. The procedure can be applied to isotropic and anisotropic FGM and assumes the homogenization approach. Indeed the material properties are mapped onto the nodes of finite element description and are interpolated at the Gauss point level by isoparametric shape functions like displacements. Consequently, all FEM solution strategies can be applied, namely non linear problems and fracture mechanics of FGM can be addressed with well established procedures. The only difference from homogeneous approach is the way the stiffness matrices of elements are calculated. [Kim and Paulino (2001)]

Analogous procedure is applied to time dependent problems [Wang and Tian (2005)] assuming that FEM is coupled with Finite Differences Method (FDM), where FEM is used for the space discretization and FDM for time domain approximation.

The Boundary Element Method (BEM) is applied to FGM as well, provided that the Fundamental Solution (FS) of the material is known either analytically or numerically. The application of BEM to variable elasticity is well known since 1993 work of Sladek, Sladek and Makechova [Sladek et al. 1993] where boundary element method for nonhomogeneous elasticity is described. Some fundamental solutions of FGM, for potential problems, are obtained in the case of exponential variability of the constituent parameters [Sutradhar and Paulino (2004)]; fundamental solutions for nonhomogeneous elasticity in 2D and 3D can be found in several works, [Martin et al. (2001), Youn-Sha Chan et al. (2003)], those fundamental solutions are applied also to hypersingular BEM and Galerkin BEM, [Sutradhar, Paulino and Gray (2005)]. The analysis of BEM application to variable elasticity and heat transfer is also presented in the papers by Ang, Clements, Vadhati and Kusuma [Ang, Clements and Vahdati (2003)] and [Ang, Kusuma and Clements (1996)] where the FS's are calculated in analytical form for exponential variability of the material elasticity and thermal conductivity too. In previous cited works, by using variable material FS, the BEM based codes can be applied to FGM structures with slight modification with respect to the homogeneous case namely only the integral equation kernels differ from the homogeneous application ones, and pure boundary equations are formulated.

Starting from known FS, BEM is applied to fracture mechanics where the method presents its better performances. In the works by Zhang Sladek and Sladek [Zhang, Sladek and Sladek (2005)] elastostatic analysis of antiplane crack in FGM is proposed where exponential variability is assumed and Galerkin hyper-singular BEM is adopted; [Shiah and Tan (2000)] perform two dimensional anisotropic fracture analysis of thermoelastic problem and evaluate stress intensity factor. Moreover, a 2-D time domain boundary integral equation method for transient dynamic analysis of cracked orthotropic solids is presented in [Zhang (2002)].

Recently Aliabadi and coworkers [When et al. (2008)] apply Meshless Method to fracuture in FGMs. Moreover [Sladek at all (2008)] apply local Petrov Galerkin Method to Reissner-Mindlin shells under thermal loading. The shell is constituted by FGM along its thickness.

In engineering application, some times, variability of material may not be governed by simple analytical expressions and the calculation of FS may become impossible; in order to apply BEM to those problems, isotropic homogeneous elasticity fundamental solution can be used causing domain discretisation to be required. The strategy is outlined by Sladek and coauthors [Sladek, Sladek Markechova (1993)] that propose boundary element method for nonhomogeneous elasticity where domain elements are used. Generalised domain boundary element approach is developed in [Chen et al. (2000)] where one parameter variable isotropic elasticity is analysed, the resulting integral equation presents boundary integrals that retains the variable elastic modules, in this way more efficient formulation results. Analogously, Minutolo and coworkers [Fraldi et al. (2000)] derived the domain boundary equation for variable elasticity describing cancellous bone tissue. The cited results deals with materials that exhibit elasticity variation depending on a single function, precisely it is assumed that elastic modulus varies within the structure but Poisson ratio is constant.

More complicated two parameter or anisotropic material variation can be studied as well, provided that the proper domain part of the equation is accounted. Here, the elastic properties can vary independently giving rise to completely n-functional constitutive tensors where n is the number of independent elastic modules characterizing the class of material anisotropy.

To get the desired formulation, a reference material FS, that has no relationship with the actual FGM, is used [Minutolo et al. (2005)]. The simplest choice for FS is homogeneous isotropic Kelvin's one.

The resulting numerical method, that requires domain discretisation, can solve any variability of the structural material and can be applied to any problem where BEM is competitive compared to other numerical tools.

The possibility to reduce the domain integral to the boundary has been investigated by means of dual reciprocity method [Fraldi et al. (2002)] and by using radial basis [Gao et al. (2007)] where single function variation is analyzed.

In the paper, the behaviour of FEM and FBEM applied to FGM is analysed and the two methods are compared. The solved structural cases suggest the best application fields of both the methods.

In the following section the Field Boundary Element Method is recalled and the domain discretisation is described. The third section presents several applications of FEM and FBEM to FGM for 2D structures; comparison between the two methods is performed and some remarks upon the rate of convergence of solutions are also reported.

2 Functionally Graded Integral Equations

2.1 Displacement equation

The Boundary Integral Equation governing the mechanics of elastic solids is obtained by applying the reciprocal theorem between the solid Ω with body forces b_j, surface loads p_j on the loaded boundary and prescribed boundary displacement $u_j(x) = u_j^0(x)\ x \in \partial\Omega_u$ on the constrained boundary, and the unbounded space where point force acts at ξ. Provided that Ω and the unbounded space have the same elasticity law, the equation has the following form

$$\kappa_{lh}(\xi)u_h(\xi) = \int\limits_{\partial\Omega} G_{lj}(x,\xi)\,p_j(x)\,dS - \int\limits_{\partial\Omega} F_{lj}(x,\xi)\,u_j(x)\,dS + \int\limits_{\Omega} G_{lj}(x,\xi)\,b_j(x)\,dS \tag{1}$$

In Eq.(1), the matrix $\kappa_{lh}(\xi)$ contains the coefficients resulting from Cauchy Principal Value (CPV) integral of the singular kernel $F_{lj}(x,\xi)$ whose expression can be found, for instance, in [Aliabadi (2002)]. The equation kernels $G_{lj}(x,\xi)$ and $F_{lj}(x,\xi)$ are the displacement and the traction at any point $x \in \partial\Omega \cup \overset{\circ}{\Omega}$ due to the application of point force at $\xi \in \partial\Omega \cup \overset{\circ}{\Omega}$.

Eq.(1) could be obtained also when the constitutive parameters vary with the position, provided that the elastic properties of the actual body Ω and of the unbounded space are assumed to be the same; therefore $G_{lj}(x,\xi)$ and $F_{lj}(x,\xi)$ should be the FS corresponding to the actual material.

In order to deal with generic material variability, the proposed procedure assumes that the material of the unbounded space is homogeneous isotropic elastic, governed by the fourth order tensor, C°_{ijhk}, while the actual body material is characterized by the variable elasticity tensor $C_{ijhk}(x)$.

Let us define the elastic difference tensor

$$L_{ijhk}(x) = C^{\circ}_{jhk_i} - C_{ijhk}(x) \tag{2}$$

and apply the reciprocal theorem; it leads to the modified integral-differential equation containing an additional volume integral:

$$\kappa_{lh}(\xi)u_h(\xi) = \int_{\partial\Omega} G_{lj}(x,\xi)\, p_j(x)\, dS - \int_{\partial\Omega} F_{lj}(x,\xi)\, u_j(x) dS + \\ + \int_{\Omega} G_{lj}(x,\xi)\, b_j(x) dS + \int_{\Omega} L_{ijhk} B_{lij}(x,\xi)\, \varepsilon_{hk}(x) dV \tag{3}$$

The kernel $B_{lij}(x,\xi)$ is the strain of the fundamental solution, $\varepsilon_{hk}(x)$ is the actual infinitesimal strain in Ω and is the symmetrical part of the displacement gradient $\varepsilon_{hk} = \frac{u_{h,k}(x)+u_{k,h}(x)}{2}$.

To cancel the strain $\varepsilon_{hk}(x)$ from Eq.(3), the last term should be integrated by part. Notice that, since $B_{lij}(x,\xi)$ has strong singularity, the boundary integral deriving from integration by part should be calculated in the CPV sense.

Consequently the free term

$$A_{lh}(\xi) = L_{ijhk}(\xi) J_{lijk}(\xi) = L_{ijhk}(\xi) \lim_{\delta\to 0} \int_{S\cap\Omega} B_{lij}(x,\xi)\, n_k(x)\, dS \tag{4}$$

arises, where S is a sphere surface of radius δ centred atξ. The free term (4) depends on the fundamental solution of homogeneous isotropic elasticity and on the collocation point ξ. The value of $J_{lijk}(\xi)$ is listed in the Appendix for 2D case.

Finally the following Field Boundary Integral Equation results:

$$[\kappa_{lh}(\xi) - A_{lh}(\xi)]\, u_h(\xi) = \int_{\partial\Omega} G_{lj}(x,\xi)\, p_j(x)\, dS - \int_{\partial\Omega} F_{lj}(x,\xi)\, u_j(x) dS \\ + \int_{\Omega} G_{lj}(x,\xi)\, b_j(x) dV + \int_{\partial\Omega} \hat{F}_{lj}(x,\xi)\, u_j(x) dS + \int_{\Omega} \hat{b}_{lj}(x,\xi)\, u_j(x) dV \tag{5}$$

In Eq.(5) two new kernels are present, namely a traction like term:

$$\hat{F}_{lj}(x,\xi) = \hat{T}_{lij}(x,\xi)\, n_i(x) \tag{6}$$

and a body force like term:

$$\hat{b}_{lj}(x,\xi) = \frac{\partial}{\partial x_i}\left[\hat{T}_{lij}(x,\xi)\right] \tag{7}$$

where:

$$\hat{T}_{lhk}(x,\xi) = L_{ijhk}(x) B_{lij}(x,\xi)$$

is the stress difference between FGM and homogeneous material elastic space corresponding to FS strain. Consequently the new kernels Eq.(6) and Eq.(7) depend on the difference of the elasticity between the actual solid and the unbounded reference space, and vanish if $L_{ijhk}(x) = C^{\circ}_{jhk_i} - C_{ijhk}(x) = 0$, i.e. the actual body and the unbounded reference space are made of the same material.

It is noticeable that Eq.(5) holds even if the actual material is anisotropic, whether or not it is inhomogeneous; in the case of anisotropic homogeneous material, the elastic difference matrix,L_{ijhk} is constant with position.

The numerical solution of the Eq.(5) is accomplished by approximating the unknown fields, $u_j(x)$ and $p_j(x)$ on the boundary by means of boundary shape function N_{sj} and the displacement in the domain cells by domain shape functionsM_{sj}.

Following the above formulation a 2D FBEM code for FGM was implemented. Boundary and domain is modelled by quadratic isoparametric elements with three and eight node respectively.

The matrix form of the final equation is

$$[\mathbf{H}_{\kappa-A}]\begin{bmatrix}\mathbf{U}_b\\ \mathbf{U}_V\end{bmatrix} - \begin{bmatrix}\tilde{\mathbf{H}}_b & \tilde{\mathbf{H}}_{bV}\\ \tilde{\mathbf{H}}_{Vb} & \tilde{\mathbf{H}}_V\end{bmatrix}\begin{bmatrix}\mathbf{U}_b\\ \mathbf{U}_V\end{bmatrix} = \begin{bmatrix}\mathbf{G}_b\\ \mathbf{G}_V\end{bmatrix}\mathbf{P} + \begin{bmatrix}\mathbf{b}_b\\ \mathbf{b}_V\end{bmatrix} \tag{8}$$

where $\mathbf{U}_b$ and $\mathbf{U}_V$ are the nodal displacement vector for boundary and domain points respectively, $\mathbf{P}$ is the traction vector, $\mathbf{H}_{\kappa-A}$ contains the free terms. The remainder terms are the collection of the boundary and domain integrals,

$$
\begin{aligned}
\tilde{\mathbf{H}}_{bV} &= \left[\int_{\Omega} \hat{b}_{lj}\left(x,\xi^b\right) M_{sj} dV \right] \\
\tilde{\mathbf{H}}_{Vb} &= \int_{\partial\Omega} \hat{F}_{lj}\left(x,\xi^V\right) N_{sj}(x) dS - \int_{\partial\Omega} F_{lj}\left(x,\xi^V\right) N_{sj}(x) dS \\
\tilde{\mathbf{H}}_{V} &= \int_{\Omega} \hat{b}_{lj}\left(x,\xi^V\right) M_{sj} dV \\
\tilde{\mathbf{H}}_{b} &= \int_{\partial\Omega} \hat{F}_{lj}\left(x,\xi^b\right) N_{sj}(x) dS - \int_{\partial\Omega} F_{lj}\left(x,\xi^b\right) N_{sj}(x) dS \\
\mathbf{G}_{V} &= \int_{\partial\Omega} G_{lj}\left(x,\xi^V\right) N_{sj} dS \\
\mathbf{G}_{b} &= \int_{\partial\Omega} G_{lj}\left(x,\xi^b\right) N_{sj} dS \\
\mathbf{b}_{b} &= \int_{\Omega} G_{lj}\left(x,\xi^b\right) b_j dV \\
\mathbf{b}_{V} &= \int_{\Omega} G_{lj}\left(x,\xi^V\right) b_j dV
\end{aligned}
\tag{9}
$$

where $\xi^b \in \partial\Omega$ and $\xi^V \in \overset{\circ}{\Omega}$.

Consider that the diagonal terms of and are calculated from the strongly singular kernels $F_{lj}(x,\xi)$ and $\hat{F}_{lj}(x,\xi)$, and that contributes depending on the constitutive law variability do not increase the order of singularity of the equation with respect BEM for homogeneous structures. The numerical calculation of strongly singular integrals can be accomplished by considering rigid body equilibrium as for traditional BEM; since the equation

$$\mathbf{H}\mathbf{U}_R = 0 \tag{10}$$

holds for any rigid body motion $\mathbf{U}_R$ the body will undergo even in the presence of domain contributes.

2.2 Stress at internal points

The evaluation of the stress at internal points can be achieved collocating Eq.(3) on internal points; the coefficient on the left hand side becomes:

$$\kappa_{lh}(\xi) = \delta_{lh} \tag{11}$$

As in BEM plasticity for constant materials, Eq.(3), with the position Eq.(11), has to be derived with respect to ξ that yields to

$$\varepsilon_{lm}(\xi) = \int_{\partial\Omega} G^{\varepsilon}_{ljm}(x,\xi)\, p_j(x) dS - \int_{\partial\Omega} F^{\varepsilon}_{ljm}(x,\xi)\, u_j(x) dS + \int_{\Omega} G^{\varepsilon}_{ljm}(x,\xi)\, b_j(x)\, dS + \int_{\Omega} L_{ijhk} B^{\varepsilon}_{lmji}(x,\xi)\, \varepsilon_{hk}(x) dV + L_{ijhk}(\xi) J^{(}_{lmji}\xi)\varepsilon_{hk}(\xi) \tag{12}$$

Numerical solution of Eq.(12) leads to a set of simultaneous equations of the strain components at Gauss point of domain mesh.

The kernels G^{ε}_{lmj}, F^{ε}_{lmj}, B^{ε}_{lmij} in Eq.(12) are usually encountered in BEM plasticity analysis and can be found in [Aliabadi and Wrobel (2002), Banerjee (1993)].

Eq.(12) is valid for internal points, the term $J_{lmij}(\xi)$ is that in Eq.(4).

When ξ approaches the boundary, the first surface integral become strongly-singular and the second one hyper-singular at the same time the domain integral of B^{ε}_{lmij} becomes strongly-singular, consequently, Eq.(12) is no more applicable and the stress on the boundary has to be evaluated by ad hoc strategies

3 Results

In this section some applications of the proposed procedure are reported, and results are compared with Graded FEM analysis and analytical solutions for isotropic functionally graded plate (FGMs) of infinite length.

In the first example the isotropic functionally graded plate of finite length and width proposed by Kim & Paulino is considered. It is subjected to various loading conditions, as shown in Figure 1.

Generalized plane strain and exponential material variations has been considered.

In the plate, the Young's modulus has been assumed to vary from $E(0) = 1$ to $E(W) = 8$ exponentially along direction x, i.e:

$$E(x) = E(0)\exp(\beta x)$$

where $W = 9$ is the width of the plate and

$$\beta = \frac{1}{W}\log\left(\frac{E(W)}{E(0)}\right) = \frac{1}{9}\log 8$$

is the independent nonhomogeneity parameter. Consistent units have been employed here.

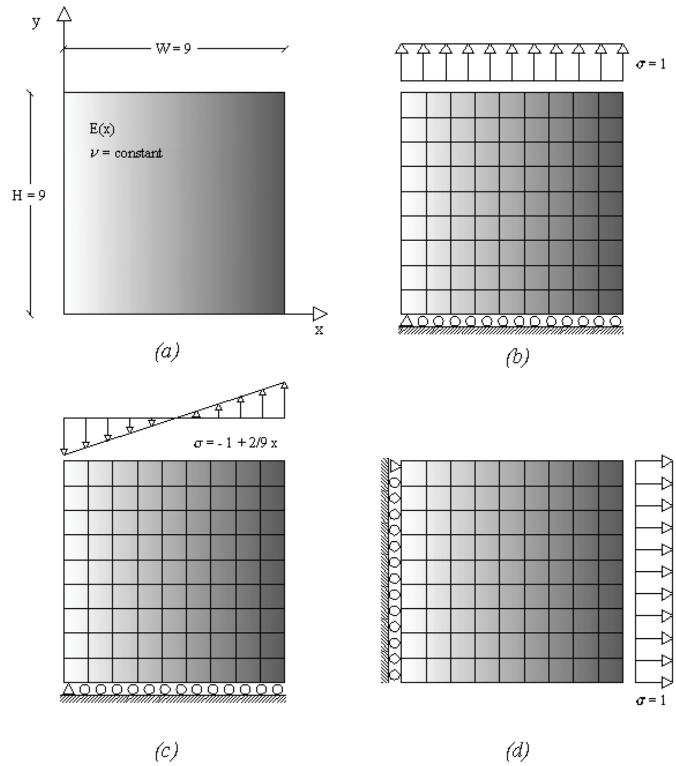


Figure 1: Isotropic FGM plate with material variation in the x-direction: (a) geometry, boundary conditions and material properties; (b) tension load perpendicular to material gradation; (c) bending load; (d) tension load parallel to material gradation.

For all loading conditions, the Poisson's ratio is constant and it has been chosen as follows: $\nu = 0.0$, $\nu = 0.3$ and $\nu = 0.49$in order to check its influence on the solution.

The relevant displacement values, obtained numerically by Graded BEM and the isoparametric Graded FEM procedure, have been compared with the analytical solutions for isotropic functionally graded plate (FGMs) of infinite length. In the tension loading perpendicular to material gradation and bending loading the displacement vertical component u_y has been considered as the relevant quantity. For tension loading parallel to material gradation the displacement horizontal component u_x has been considered as the relevant quantity.

The plate mesh, used for FEM analysis, consists of isoparametric eight nodes domain elements where 5x5 Gauss' points quadrature is employed; in FBEM analysis the same domain elements are used together whith three nodes quadratic boundary elements.

For tension loading perpendicular to material gradation and bending loading the applied stress resultants are definite by

$$N = \sigma_t W \quad \text{and} \quad M = \frac{\sigma_b W^2}{6}$$

respectively, where N is a membrane resultant along the $x = W/2$ line and M is the bending moment.

For these loading cases the analytical solutions for unbounded strip in terms of displacements becomes:

$$u_x(x,y) = \nu\left(\frac{A}{2}x^2 + Bx\right) - \frac{A}{2}y^2$$

$$u_y(x,y) = (Ax + B)\,y$$

Where the coefficients A and B for tension are:

$$A = \frac{\beta N}{2E(0)}\left[\frac{W\beta^2 e^{\beta W} - 2\beta e^{\beta W} + W\beta^2 + 2\beta}{e^{\beta W}\beta^2 W^2 - e^{2\beta W} + 2e^{\beta W} - 1}\right]$$

$$B = \frac{\beta N}{2E(0)}\left[\frac{e^{\beta W}\left[e^{\beta W}\left(-W^2\beta^2 + 3\beta W - 4\right) + W^2\beta^2 - 2\beta W + 8\right] - \beta W - 4}{\left(e^{\beta W} - 1\right)\left(e^{\beta W}\beta^2 W^2 - e^{2\beta W} + 2e^{\beta W} - 1\right)}\right]$$

respectively, while for bending are:

$$A = \frac{\beta^2 M}{E(0)}\left[\frac{\beta\left(1 - e^{\beta W}\right)}{e^{\beta W}\beta^2 W^2 - e^{2\beta W} + 2e^{\beta W} - 1}\right]$$

$$B = \frac{\beta^2 M}{E(0)}\left[\frac{\beta W e^{\beta W} - e^{\beta W} + 1}{e^{\beta W}\beta^2 W^2 - e^{2\beta W} + 2e^{\beta W} - 1}\right]$$

respectively.

Figure 2 shows the displacement vertical component u_y at cross sectional lines of the plate, as a function of horizontal abscissa. The solution obtained with Graded FBEM procedure matches the isoparametric Graded FEM solution. As expected, for $\nu = 0.0$ both numerical calculations are able to capture the analytical solution while for $\nu > 0.0$ both underestimate the displacement in comparison with analytical one.

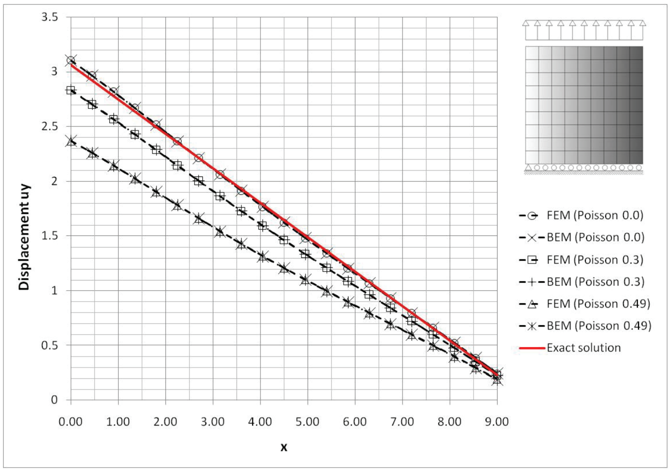


Figure 2: Displacement u_y for tension load perpendicular to material gradation

A similar result is obtained for the plate subjected to bending as well, as shown in Figure 3.

When the plate is subjected to tension parallel to material gradation, the applied stress resultant is definite by

$$N = \sigma_t H$$

where N is a membrane resultant along the $y = H/2$ line.

For this loading case the analytical solution in terms of displacements becomes:

$$u_x(x) = \frac{\sigma_t}{E(0)\,\beta}\left(1 - e^{-\beta x}\right)$$

Numerical solution obtained with Graded FBEM procedure and the Isoparametric Graded FEM formulation give rise to analogous results as shown in Figure 4.

In all loading condition the good agreement of the results, irrespective to the adopted numerical method, can be seen.

In order to highlight the influence of the direction of the load with respect to heterogeneity direction and the influence of the shape function representation of the Young modulus in isoparametric FEM, a second example has been presented.

Isoparametric FEM requires that elastic parameters should be evaluated at the nodes level and interpolated within the element by means of isoparametric shape function. As a consequence, the calculated stiffness of the element resulting from integral

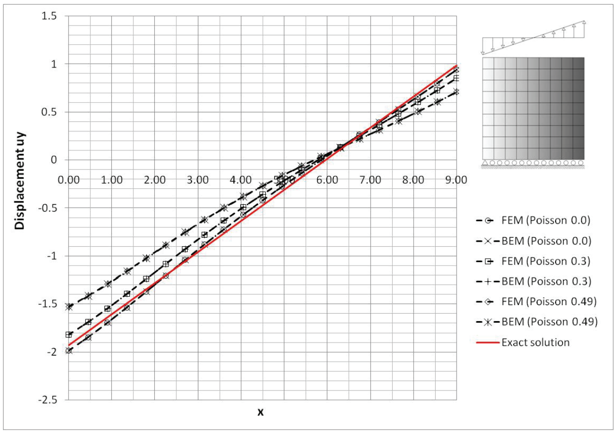


Figure 3: Displacement u_y for bending load

evaluation, is affected by an error that depends on the number of adopted Gauss' points.

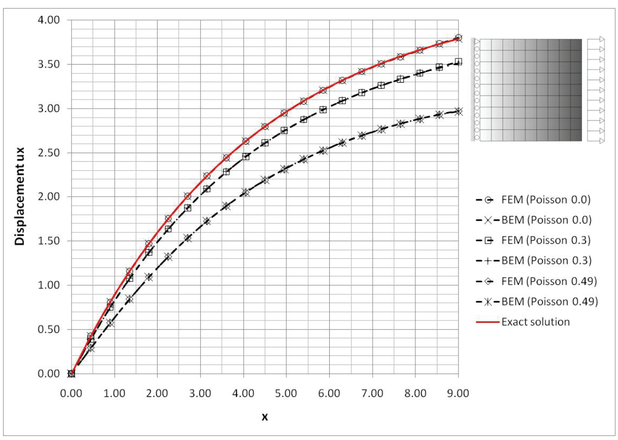


Figure 4: Displacement u_x for tension load parallel to material gradation

Moreover the introduction of shape function interpolation produces one more error due to Young's modulus mapping.

On the contrary, using overall function to describe the material heterogeneity overcome the problem although it requires that the analytical representation of the Young's modulus has to be known.

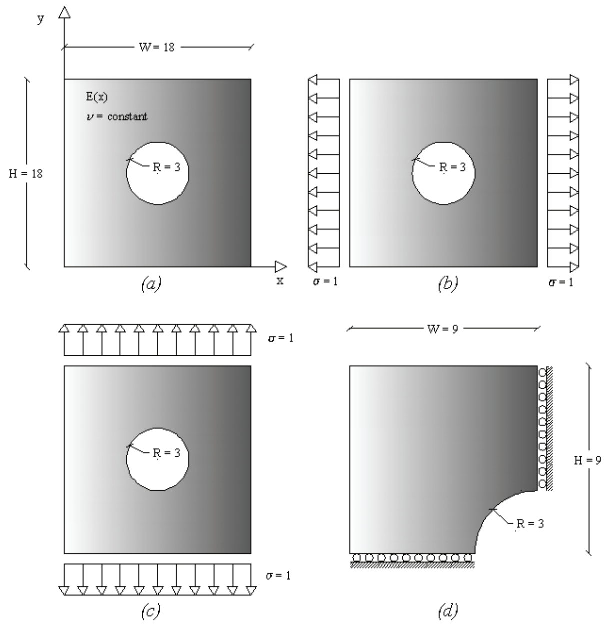


Figure 5: Isotropic FGM hole plate with material variation in the x-direction: (a) geometry, boundary conditions and material properties; (b) tension load parallel to material gradation; (c) tension load perpendicular to material gradation; (d) quarter of the plate performed

The second example concerns a rectangular plate with circular void at the centre (Figure 5(a)) subjected to unit axial loading along horizontal and vertical direction respectively (see Figure 5(b) and 5(c)) presenting horizontally variable Young's modulus.

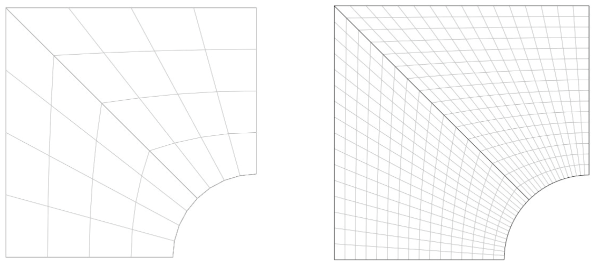
Due to symmetries, only a quarter of the plate has been considered, see Figure 5(d).

To check the influence of isoparametric mapping of the modulus, Comsol®Multiphisics has been used for comparisons. The program indeed performs analytical evaluation of the modulus at Gauss' points.

Several meshes have been analyzed assuming different numbers of equally spaced divisions on each side. Two typical meshes, the coarsest and the most refined respectively, are reported in Figure 6.

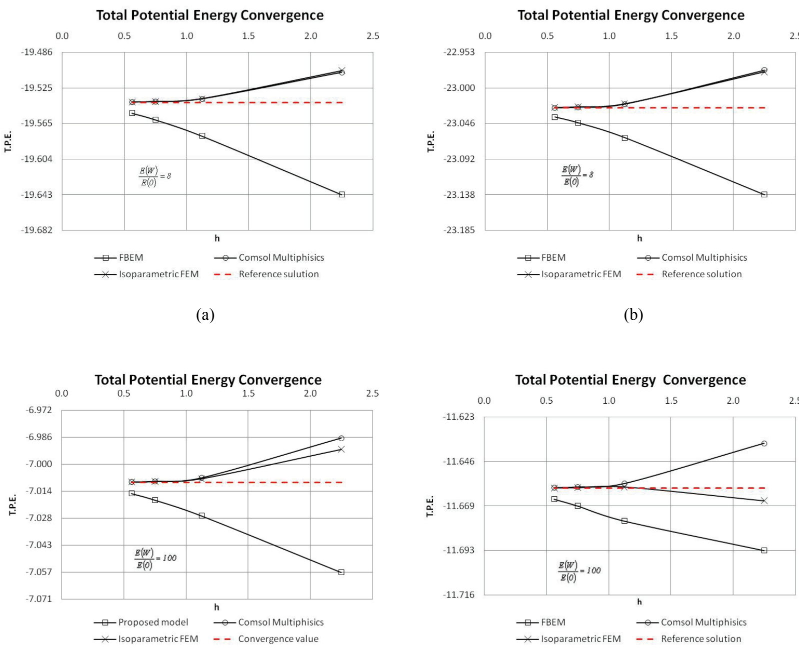
Moreover Comsol®Multiphisics has been used as "exact" reference solution when very fine mesh has been used, consisting of $h = 0.2$.

In Figure 7 the Total Potential Energy (TPE) at the solution is plotted versus the av-



(a) (b)

Figure 6: Two typical meshes analyzed: (a) coarse mesh with h = 2.25; (b) refined mesh with h = 0.56



(a) (b) (c) (d)

Figure 7: Total Potential Energy Convergence: (a) load applied perpendicular to the low gradient exponential material gradation; (b) load applied parallel to the low gradient exponential material gradation; (c) load applied perpendicular to the high gradient exponential material gradation; (d) applied parallel to the high gradient exponential material gradation

erage dimension of element mesh, h. Both FEM and Comsol®Multiphysics results show that TPE increases with h except when high gradient variation plate is loaded parallel to the gradation (Figure 7(d)).

In this case the TPE calculated from isoparametric Graded FEM procedures converges from below.

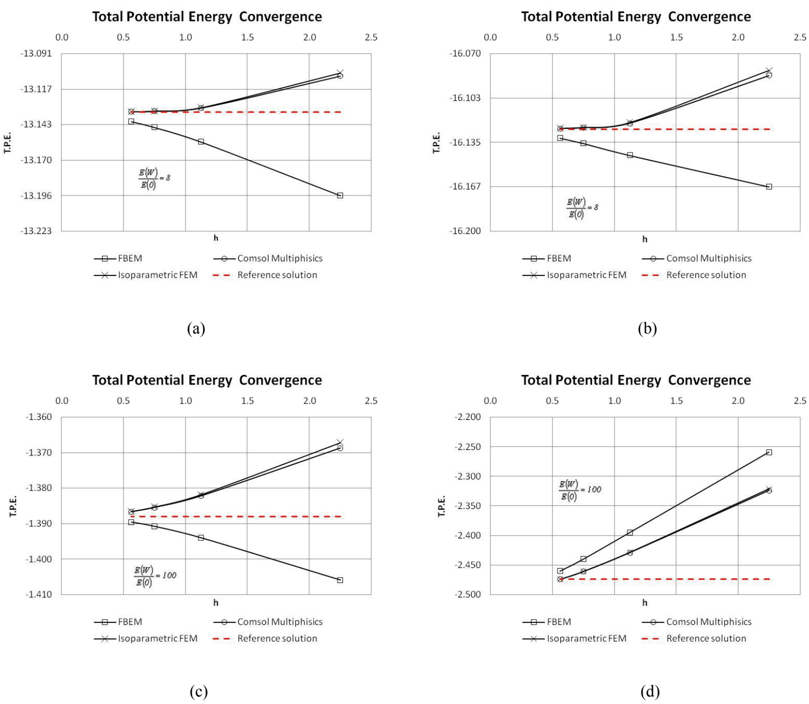


Figure 8: Total Potential Energy Convergence: (a) load applied perpendicular to the low gradient linear material gradation; (b) load applied parallel to the low gradient linear material gradation; (c) load applied perpendicular to the high gradient linear material gradation; (d) applied parallel to the high gradient linear material gradation

Hence it has to be stressed that monotonic convergence rate is violated when high gradient of gradation occurs along main load direction.

Rather different qualitative behaviour is encountered for convergence law of Graded BEM procedure. TPE does not present convergence from above since FBEM is not energy consistent method.

However monotone convergence from below is showed for any cases of exponentially variation of material.

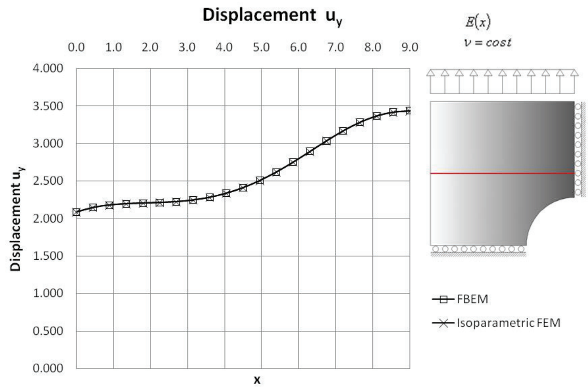


Figure 9: Displacement u_y for tension load perpendicular to material gradation

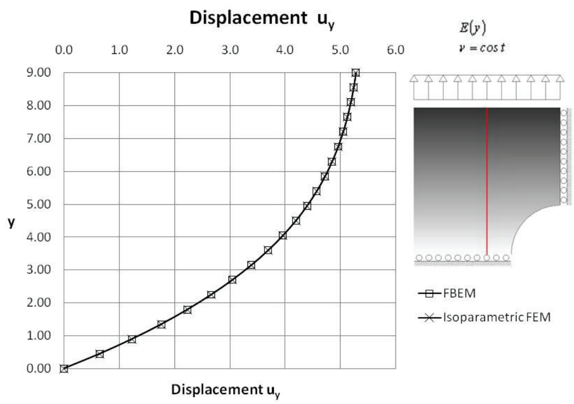


Figure 10: Displacement u_y for tension load parallel to material gradation

As third example the same plate has been analyzed assuming linear variation of Young's modulus both with low and high gradient of variation. The TPE calculated from isoparametric Graded FEM converges from above even when high gradient of variation occurs (see Figure 8(d)), accordingly with the expected behaviour.

This result depends on the fact that isoparametric mapping does not introduce errors since the linearity of Young's modulus variation law that is mapped exactly by quadratic shape functions.

Figure 9 shows the displacement vertical component u_y at cross sectional lines of

Table 1: Displacement u_y for tension load perpendicular to material gradation

Exponentional gradatio of material		
E = 1-8 direction gradation: x direction load: y cross section: x = 4.5		
x	**uy**	
	FBEM	**Isoparametric FEM**
0.00	2.0872	2.0864
0.45	2.1476	2.1467
0.90	2.1812	2.1803
1.35	2.1977	2.1968
1.80	2.2060	2.2050
2.25	2.2131	2.2120
2.70	2.2252	2.2240
3.15	2.2475	2.2460
3.60	2.2843	2.2825
4.05	2.3387	2.3366
4.50	2.4129	2.4107
4.95	2.5077	2.5055
5.40	2.6226	2.6200
5.85	2.7536	2.7508
6.30	2.8949	2.8919
6.75	3.0376	3.0345
7.20	3.1714	3.1680
7.65	3.2852	3.2818
8.10	3.3705	3.3670
8.55	3.4220	3.4183
9.00	3.4387	3.4350

the plate, as a function of horizontal abscissa when load is perpendicular to material gradation.

Figure 10 shows u_y as a function of vertical ordinate when load is parallel to material gradation. In both loadings, the obtained solution by means of Graded FBEM and by means isoparametric Graded FEM solution shows no meaningful differences.

By the last consideration, the good agreement of the results is obtained irrespective to the adopted numerical method also when irregular meshes are employed.

4 Conclusions

In the work an application of FBEM to FGM has been presented. FBEM has been compared with isoparametric FEM and Comsol®Multiphisics FEM program. The results show that FBEM and isoparametric FEM are in good agreement with analyt-

Table 2: Displacement u_y for tension load perpendicular to material gradation

Exponentional gradation of material		
E = 1-8 direction gradation: y direction load: y cross section: y = 4.5		
y	**uy**	
	FBEM	**Isoparametric FEM**
0.00	0.0000	0.0000
0.45	0.6484	0.6484
0.90	1.2355	1.2349
1.35	1.7647	1.7633
1.80	2.2410	2.2391
2.25	2.6691	2.6668
2.70	3.0530	3.0506
3.15	3.3953	3.3929
3.60	3.6989	3.6964
4.05	3.9660	3.9631
4.50	4.1987	4.1959
4.95	4.4007	4.3981
5.40	4.5753	4.5724
5.85	4.7250	4.7221
6.30	4.8528	4.8498
6.75	4.9611	4.9582
7.20	5.0523	5.0493
7.65	5.1282	5.1253
8.10	5.1908	5.1879
8.55	5.2417	5.2388
9.00	5.2828	5.2799

ical solution and with FEM using analytical mapping of material properties. However isoparametric FEM shows some drawbacks with respect to monotone convergence when great errors has to be expected from the material property mapping by shape functions. It has to be expected that when the mapping of the material properties is done by mean of low order shape functions, mesh refinement, although yielding to convergent solutions in term of displacement on a structure with constant material, does not preserve monotone convergence when the material vary within the body.

A well established result is that FBEM has not an energy representation so that no a priori monotonic convergence is enforced. Hence the results show energy convergence from below whit respect to element size.

Both FBEM and isoparametric FEM are seen to converge to the same results and no preference based on convergence and numerical behaviour have to accounted.

Finally, the proposed work suggests that the rate of variation of elastic properties has to be taken in to consideration in order to use the best nodal interpolation of material properties.

Acknowledgement: The authors acknowledge the contribute of MIUR, Italian Ministry of Education University and Research, PRIN2004, n.2004083253_004 and Tommaso Colella who developed some examples that formed part of its PhD dissertation work.

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Appendix A

For 2D plane strain elasticity, at internal points, is calculated over the entire circumference and results:

$$\begin{aligned}
&J_{1111} = J_{2222} = \frac{5-8\nu}{16\mu\left(1-\nu\right)},\\
&J_{1122} = J_{2211} = \frac{-1}{16\mu\left(1-\nu\right)},\\
&J_{1221} = J_{1212} = J_{2121} = J_{2112} = \frac{3-4\nu}{16\mu\left(1-\nu\right)},\\
&J_{1121} = J_{1112} = J_{1222} = J_{2111} = J_{2122} = J_{2212} = 0
\end{aligned} \tag{13}$$

if ξ is a smooth point of the boundary, S reduces to half-circle and becomes:

$$\begin{aligned}
&J_{1111} = J_{2222} = \frac{5-8\nu}{32\mu\left(1-\nu\right)},\\
&J_{1122} = J_{2211} = \frac{-1}{32\mu\left(1-\nu\right)},\\
&J_{1221} = J_{1212} = J_{2121} = J_{2112} = \frac{3-4\nu}{32\mu\left(1-\nu\right)},\\
&J_{1121} = J_{1112} = J_{1222} = J_{2111} = J_{2122} = J_{2212} = 0
\end{aligned} \tag{14}$$

On a right angle corner point, with the sides aligned with the positive axes of the reference frame, i.e. the circle portion subtend angles, α between 0 and $\frac{\pi}{2}$, J_{ijhk} is given by

$$
\begin{aligned}
&J_{1111} = J_{2222} = \frac{5-8\nu}{64\mu\left(1-\nu\right)},\\
&J_{1122} = J_{2211} = \frac{-1}{64\mu\left(1-\nu\right)}\\
&J_{1221} = J_{1212} = J_{2121} = J_{2112} = \frac{3-4\nu}{64\mu\left(1-\nu\right)}\\
&J_{1211} = J_{1222} = J_{2111} = J_{2122}\frac{1}{8\pi\mu}\\
&J_{1112} = J_{2221} = \frac{1-2\nu}{8\pi\mu\left(1-\nu\right)}\\
&J_{1121} = J_{2212} = 0
\end{aligned}
\tag{15}
$$

Equation (15) has to be modified when different reference angle are considered due to orientation of outward normal to the boundary.