Copyright © 2009 Tech Science Press CMES, vol.40, no.1, pp.1-28, 2009

Solving the Inverse Problems of Laplace Equation to Determine the Robin Coefficient/Cracks' Position Inside a Disk

Chein-Shan Liu[1]

Abstract: We consider an inverse problem of Laplace equation by recoverning boundary value on the inner circle of a two-dimensional annulus from the overdetermined data on the outer circle. The numerical results can be used to determine the Robin coefficient or crack's position inside a disk from the measurements of Cauchy data on the outer boundary. The Fourier series is used to formulate the first kind Fredholm integral equation for the unknown data $f(\theta)$ on the inner circle. Then we consider a Lavrentiev regularization, by adding an extra term $\alpha f(\theta)$ to obtain the second kind Fredholm integral equation. The termwise separable property of kernel function allows us to obtain a closed-form regularized solution, of which the uniform convergence and error estimation are proved. Then we apply this method to the inverse Cauchy problem, the unknown shape of zero-potential problem, the problem of detecting crack position, as well as the problem of unknown Robin coefficient. These numerical examples show the effectiveness of the new method in providing excellent estimates of the unknown data.

Keywords: Laplace equation, Inverse Cauchy problem, Fredholm integral equation, Lavrentiev regularization, Robin coefficient, Crack position

1 Inverse boundary value problem

The use of electrostatic image in the nondestructive testings of metalic disks leads to an inverse boundary value problem of Laplace equation in two-dimension. In order to detect the unknown shape of the inclusion inside a conducting metal we imposed overdetermined Cauchy data, for example the voltage and current, on the accessible exterior boundary. This amounts to solving an inverse Cauchy problem from available data on part of the boundary [Liu (2008a)]. This problem is well known to be highly ill-posed.

[1] Department of Civil Engineering, National Taiwan University, Taipei, Taiwan. E-mail: liuchei.nshan@msa.hinet.net

We consider a mathematical modeling of this problem. Given the Cauchy data $u(x,y)$ and $\partial u/\partial n(x,y)$ at the point $(x,y) \in \mathbb{R}^2$ with the unit outward normal $n(x,y)$ on the external circle Γ_1 with a radius r_1 of an annulus Ω, we consider the Cauchy problem of the Laplace equation $\Delta u(x,y) = 0$ in two dimensions to find the unknown function $u(x,y)$ on an internal circle Γ_2 with a radius $r_2 < r_1$. This problem setting can be used in the electrostatic image of the inverse problem in the human electro-cardiography [Johnston (2001)].

The problem we consider consists of the Laplace equation in a disk and the over-specified Cauchy data on boundary:

$$\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad r < r_1, \quad 0 \le \theta \le 2\pi, \tag{1}$$

$$u(r_1,\theta) = g(\theta), \quad 0 \le \theta \le 2\pi, \tag{2}$$

$$u_r(r_1,\theta) = h_1(\theta), \quad 0 \le \theta \le 2\pi, \tag{3}$$

where $g(\theta)$ and $h_1(\theta)$ are given functions obtained from measurements.

The inverse Cauchy problems may arise in the steady-state heat conduction inverse boundary value problems [Mera, Elliott, Ingham and Lesnic (2000)]. The situation is that there are many practical engineering applications where a part of the boundary is not accessible for temperature and heat flux measurements, and both of them are known on the other part. In order to get the whole temperature field of the body one may encounter the Cauchy problems.

On the other hand, for the data completion issues in elliptic inverse problems there are the tasks from the superfluous measurements made on the accessible boundary of a domain to recover either the Dirichlet or Neumann boundary data [Berntsson and Eldén (2001); Azaiez, Ben Abda and Ben Abdallah (2005); Leblond, Mahjoub and Partington (2006)], the Robin type exchange coefficient [Fasino and Inglese (1999); Chaabane and Jaoua (1999); Chaabane, Elhechmi and Jaoua (2004); Slodička and Van Keer (2004); Lin and Fang (2005)], the geometrical singularities [Brühl, Hanke and Pidcock (2001); Kress (2004); Chapko and Kress (2005)], or the geometrical shape of a constant temperature curve [Liu, Chang and Chiang (2008)].

The Cauchy problem is difficult to solve both numerically and analytically, since its solution, if exists, does not depend continuously on the given data. Therefore, we have to treat this type problem with a different numerical algorithm from that used in the direct problem, which compromises the accuracy and stability. When the influence matrix is highly ill-posed, Chang, Yeih and Shieh (2001) have shown that neither the traditional Tikhonov's regularization method nor the singular value decomposition method can yield acceptable numerical results for the inverse Cauchy problem of Laplace equation. A recent review of the Cauchy problems was given by Ben Belgacem and El Fekih (2005).

In this paper, we cast the Cauchy problem in an annular into the first kind Fredholm integral equation, and then we propose a Lavrentiev type regularization to transform it into the second kind Fredholm integral equation. By utilizing the separating characteristic of kernel function and eigenfunctions expansion techniques we can derive a closed-form regularized solution of the second kind Fredholm integral equation. This method was first used by Liu (2007a) to solve a direct problem of elastic torsion in an arbitrary plane domain, where it was called a meshless regularized integral equation method. Then, Liu (2007b, 2007c) extended it to solve the Laplace direct problem in arbitrary plane domains. The new method would provide us a semi-analytical solution, and renders a more compendious numerical implementation than other schemes to solve the inverse Cauchy problems.

Liu (2008b) has applied a modified Trefftz method to recover the unknown boundary data for the inverse Cauchy problem, but needs to consider a regularization technique by truncating the higher-mode components of the given data. Then, Liu (2008a) extended the modified Trefftz method by a simple collocation technique to treat the inverse Cauchy problem of Laplace equation in arbitrary plane domain. Furthermore, Liu (2008c) used the same technique to calculate the inverse Cauchy problem of biharmonic equation in arbitrary plane domain.

The method of fundamental solutions (MFS) utilizes the fundamental solutions as basis functions to expand the solution. While Jin and Zheng (2006) have applied the MFS to solve the inverse problem of Helmholtz equation, Marin and Lesnic (2005) have applied the MFS to solve the inverse Cauchy problem associated with a two-dimensional biharmonic equation. In order to tackle of the ill-posedness of MFS and the inherent ill-posed property of the inverse Cauchy problems, those authors proposed new numerical schemes with the regularization parameters determined by the L-curve method. Ling and Takeuchi (2008) have combined the MFS and boundary control technique to solve the inverse Cauchy problem of Laplace equation. Liu and Atluri (2008a) reformulated the inverse Cauchy problem of Laplace equation in a rectangle as an optimization problem, and applied a fictitious time integration method [Liu and Atluri (2008b)] to solve an algebraic equations system to obtain the data on an unspecified portion of boundary. When an extension to nonlinear inverse Cauchy problem is concerned with, they showed that good result can be obtained by using their method.

The remaining sections of this paper are organized as follows. In Section 2 we derive the second kind Fredholm intergral equation by a Lavrentiev regularization of the first kind Fredholm intergral equation. In Section 3 we derive a two-point boundary value problem, which can be used to derive a closed-form regularized solution of the second kind Fredholm intergral equation in Section 4. In Section 5 we prove the uniform convergence of the regularized solution, as well as give an error estimate. In Section 6 we use some numerical examples to test the new method. Then, we give conclusions in Section 7.

2 The Fredholm integral equation

We replace Eq. (3) by the following boundary condition:

$$u(r_2,\theta)=f(\theta),\ \ 0\leq\theta\leq 2\pi, \tag{4}$$

where $f(\theta)$ is an unknown function to be determined. The constant radius r_2 can be selected freely in addition $r_2<r_1$, such that the annular with radii r_1 and r_2 can cover the entire domain of the problem we consider.

We can write a series expansion of $u(r,\theta)$ satisfying Eqs. (1), (2) and (4):

$$u(r,\theta)=\frac{1}{2}(a_0+b_0\ln r)+\sum_{k=1}^{\infty}[(a_kr^k+b_kr^{-k})\cos k\theta+(c_kr^k+d_kr^{-k})\sin k\theta], \tag{5}$$

where

$$a_0=\frac{1}{\pi(\ln r_1-\ln r_2)}\left[\ln r_1\int_0^{2\pi}f(\xi)d\xi-\ln r_2\int_0^{2\pi}g(\xi)d\xi\right], \tag{6}$$

$$b_0=\frac{1}{\pi(\ln r_1-\ln r_2)}\left[\int_0^{2\pi}g(\xi)d\xi-\int_0^{2\pi}f(\xi)d\xi\right], \tag{7}$$

$$a_k=\frac{e_k}{r_2^k}\int_0^{2\pi}g(\xi)\cos k\xi d\xi-\frac{e_k}{r_1^k}\int_0^{2\pi}f(\xi)\cos k\xi d\xi, \tag{8}$$

$$b_k=e_kr_1^k\int_0^{2\pi}f(\xi)\cos k\xi d\xi-e_kr_2^k\int_0^{2\pi}g(\xi)\cos k\xi d\xi, \tag{9}$$

$$c_k=\frac{e_k}{r_2^k}\int_0^{2\pi}g(\xi)\sin k\xi d\xi-\frac{e_k}{r_1^k}\int_0^{2\pi}f(\xi)\sin k\xi d\xi, \tag{10}$$

$$d_k=e_kr_1^k\int_0^{2\pi}f(\xi)\sin k\xi d\xi-e_kr_2^k\int_0^{2\pi}g(\xi)\sin k\xi d\xi, \tag{11}$$

and

$$e_k:=\frac{1}{\pi\left[\left(\frac{r_1}{r_2}\right)^k-\left(\frac{r_2}{r_1}\right)^k\right]}. \tag{12}$$

Taking the differential of Eq. (5) with respect to r, leads to

$$u_r(r,\theta)=\frac{b_0}{2r}+\sum_{k=1}^{\infty}[(ka_kr^{k-1}-kb_kr^{-k-1})\cos k\theta+(kc_kr^{k-1}-kd_kr^{-k-1})\sin k\theta].$$

(13)

By imposing condition (3) on the above equation we obtain

$$\frac{b_0}{2r_1}+\sum_{k=1}^{\infty}[(ka_kr_1^{k-1}-kb_kr_1^{-k-1})\cos k\theta+(kc_kr_1^{k-1}-kd_kr_1^{-k-1})\sin k\theta]=h_1(\theta). \tag{14}$$

Substituting Eqs. (7)-(11) into Eq. (14) yields the first kind Fredholm integral equation:

$$\int_0^{2\pi}K(\theta,\xi)f(\xi)d\xi=h(\theta), \tag{15}$$

where

$$\begin{aligned}h(\theta):=&-h_1(\theta)+\frac{1}{2r_1\pi(\ln r_1-\ln r_2)}\int_0^{2\pi}g(\xi)d\xi\\&+\sum_{k=1}^{\infty}\left\{A_k\left(\int_0^{2\pi}g(\xi)\cos k\xi d\xi\cos k\theta+\int_0^{2\pi}g(\xi)\sin k\xi d\xi\sin k\theta\right)\right\}\end{aligned} \tag{16}$$

is the source function, and

$$K(\theta,\xi)=\frac{1}{2r_1\pi(\ln r_1-\ln r_2)}+\sum_{k=1}^{\infty}\left\{B_k\left[\cos k\theta\cos k\xi+\sin k\theta\sin k\xi\right]\right\} \tag{17}$$

is the kernel function. Here, we note that

$$A_k:=kr_1^{k-1}e_kr_2^{-k}+kr_1^{-k-1}e_kr_2^{k}, \tag{18}$$
$$B_k:=kr_1^{k-1}e_kr_1^{-k}+kr_1^{-k-1}e_kr_1^{k}=2r_1^{-1}ke_k. \tag{19}$$

In order to obtain $f(\theta)$ we have to solve the first kind Fredholm integral equation (15), which is however well known to be highly ill-posed [Tikhonov, Goncharsky, Stepanov and Yagola (1990)]. We assume that there exists a regularized parameter α, such that Eq. (15) can be regularized by

$$\alpha f(\theta)+\int_0^{2\pi}K(\theta,\xi)f(\xi)d\xi=h(\theta), \tag{20}$$

which is known as the second type Fredholm integral equation. The above regularization method to obtain a regularized solution by solving a singularly perturbed operator equation is usually called the Lavrentiev regularization method [Lavrentiev (1967)].

3 Two-point boundary value problem

We assume that the kernel function can be approximated by m terms with

$$K(\theta,\xi) = \frac{1}{2r_1\pi(\ln r_1 - \ln r_2)} + \sum_{k=1}^{m} \left\{ B_k \left[\cos k\theta \cos k\xi + \sin k\theta \sin k\xi\right] \right\}. \tag{21}$$

This assumption is for the convenience of our later derivation but is not necessary. When an analytical solution is obtained, we can let $m = \infty$ again.
By inspection we have

$$K(\theta,\xi) = \mathbf{P}(\theta)\cdot\mathbf{Q}(\xi), \tag{22}$$

where $\mathbf{P}$ and $\mathbf{Q}$ are $2m+1$-vectors given by

$$\mathbf{P} := \begin{bmatrix} \frac{1}{2r_1\pi(\ln r_1 - \ln r_2)} \\ B_1\cos\theta \\ B_1\sin\theta \\ B_2\cos 2\theta \\ B_2\sin 2\theta \\ \vdots \\ B_m\cos m\theta \\ B_m\sin m\theta \end{bmatrix}, \quad \mathbf{Q} := \begin{bmatrix} 1 \\ \cos\xi \\ \sin\xi \\ \cos 2\xi \\ \sin 2\xi \\ \vdots \\ \cos m\xi \\ \sin m\xi \end{bmatrix}. \tag{23}$$

The dot between $\mathbf{P}$ and $\mathbf{Q}$ in Eq. (22) denotes the inner product, which is sometimes written as $\mathbf{P}^{\mathrm{T}}\mathbf{Q}$, where the superscript T signifies the transpose. The sequence of functions in $\mathbf{Q}$ form an orthogonal Fourier bases system in $[0,2\pi]$.
With the aid of Eq. (22), Eq. (20) can be decomposed as

$$\alpha f(\theta) + \int_0^{\theta} \mathbf{P}^{\mathrm{T}}(\theta)\mathbf{Q}(\xi)f(\xi)d\xi + \int_{\theta}^{2\pi} \mathbf{P}^{\mathrm{T}}(\theta)\mathbf{Q}(\xi)f(\xi)d\xi = h(\theta). \tag{24}$$

Let us define

$$\mathbf{u}_1(\theta) := \int_0^{\theta} \mathbf{Q}(\xi)f(\xi)d\xi, \tag{25}$$

$$\mathbf{u}_2(\theta) := \int_{2\pi}^{\theta} \mathbf{Q}(\xi)f(\xi)d\xi, \tag{26}$$

and Eq. (24) can be expressed as

$$\alpha f(\theta) + \mathbf{P}^{\mathrm{T}}(\theta)[\mathbf{u}_1(\theta) - \mathbf{u}_2(\theta)] = h(\theta). \tag{27}$$

If $\mathbf{u}_1$ and $\mathbf{u}_2$ can be solved, we can calculate $f(\theta)$.
Taking the differentials of Eqs. (25) and (26) with respect to θ we obtain

$$\mathbf{u}_1'(\theta) = \mathbf{Q}(\theta) f(\theta), \tag{28}$$
$$\mathbf{u}_2'(\theta) = \mathbf{Q}(\theta) f(\theta). \tag{29}$$

Multiplying the above two equations by α, and inserting Eq. (27) for $\alpha f(\theta)$ we can derive

$$\alpha \mathbf{u}_1'(\theta) = \mathbf{Q}(\theta)\mathbf{P}^{\mathrm{T}}(\theta)[\mathbf{u}_2(\theta) - \mathbf{u}_1(\theta)] + h(\theta)\mathbf{Q}(\theta), \;\; \mathbf{u}_1(0) = \mathbf{0}, \tag{30}$$
$$\alpha \mathbf{u}_2'(\theta) = \mathbf{Q}(\theta)\mathbf{P}^{\mathrm{T}}(\theta)[\mathbf{u}_2(\theta) - \mathbf{u}_1(\theta)] + h(\theta)\mathbf{Q}(\theta), \quad \mathbf{u}_2(2\pi) = \mathbf{0}, \tag{31}$$

where the last two conditions follow from Eqs. (25) and (26) readily. The above two equations constitute a two-point boundary value problem of $4m+2$-dimensional ODEs.

4 A closed-form regularized solution

In this section we will find a closed-form solution of $f(\theta)$. From Eqs. (28) and (29) it can be seen that $\mathbf{u}_1' = \mathbf{u}_2'$, which means that

$$\mathbf{u}_1 = \mathbf{u}_2 + \mathbf{c}, \tag{32}$$

where $\mathbf{c}$ is a constant vector to be determined. By using the final condition in Eq. (31) we find that

$$\mathbf{u}_1(2\pi) = \mathbf{u}_2(2\pi) + \mathbf{c} = \mathbf{c}. \tag{33}$$

Substituting Eq. (32) into Eq. (30) we have

$$\alpha \mathbf{u}_1'(\theta) = -\mathbf{Q}(\theta)\mathbf{P}^{\mathrm{T}}(\theta)\mathbf{c} + h(\theta)\mathbf{Q}(\theta), \;\; \mathbf{u}_1(0) = \mathbf{0}. \tag{34}$$

Integrating the above equation and using the initial condition, one has

$$\mathbf{u}_1(\theta) = \frac{-1}{\alpha}\int_0^\theta \mathbf{Q}(\xi)\mathbf{P}^{\mathrm{T}}(\xi)d\xi\,\mathbf{c} + \frac{1}{\alpha}\int_0^\theta h(\xi)\mathbf{Q}(\xi)d\xi. \tag{35}$$

Taking $\theta = 2\pi$ in the above equation and imposing condition (33), one obtains a governing equation for $\mathbf{c}$:

$$\left(\alpha \mathbf{I}_{2m+1} + \int_0^{2\pi} \mathbf{Q}(\xi)\mathbf{P}^{\mathrm{T}}(\xi)d\xi\right)\mathbf{c} = \int_0^{2\pi} h(\xi)\mathbf{Q}(\xi)d\xi. \tag{36}$$

It is straightforward to write

$$\mathbf{c} = \left(\alpha \mathbf{I}_{2m+1} + \int_0^{2\pi} \mathbf{Q}(\xi)\mathbf{P}^{\mathrm{T}}(\xi)d\xi\right)^{-1} \int_0^{2\pi} h(\xi)\mathbf{Q}(\xi)d\xi. \tag{37}$$

On the other hand, from Eqs. (27) and (32) we have

$$\alpha f(\theta) = h(\theta) - \mathbf{P}(\theta)\cdot\mathbf{c}. \tag{38}$$

Inserting Eq. (37) for **c** into the above equation we obtain

$$\alpha f(\theta) = h(\theta) - \mathbf{P}^{\mathrm{T}}(\theta)\left(\alpha \mathbf{I}_{2m+1} + \int_0^{2\pi} \mathbf{Q}(\xi)\mathbf{P}^{\mathrm{T}}(\xi)d\xi\right)^{-1} \int_0^{2\pi} h(\xi)\mathbf{Q}(\xi)d\xi. \tag{39}$$

Due to the orthogonality of **P** and **Q**, the $(2m+1)\times(2m+1)$ matrix can be written as

$$\int_0^{2\pi} \mathbf{Q}(\xi)\mathbf{P}^{\mathrm{T}}(\xi)d\xi = \mathrm{diag}\left[\frac{1}{r_1(\ln r_1 - \ln r_2)}, \pi B_1, \pi B_1, \pi B_2, \pi B_2, \ldots, \pi B_m, \pi B_m\right], \tag{40}$$

where diag means a diagonal matrix.

Inserting Eq. (40) into Eq. (39) we thus obtain

$$\begin{aligned} f(\theta) = \frac{1}{\alpha}h(\theta) - \frac{1}{\alpha}\mathbf{P}^{\mathrm{T}}(\theta)\mathrm{diag}&\left[\frac{1}{\alpha + \frac{1}{r_1(\ln r_1 - \ln r_2)}}, \frac{1}{\alpha+\pi B_1}, \frac{1}{\alpha+\pi B_1}, \ldots,\right. \\ &\left.\frac{1}{\alpha+\pi B_m}, \frac{1}{\alpha+\pi B_m}\right] \int_0^{2\pi} h(\xi)\mathbf{Q}(\xi)d\xi. \end{aligned} \tag{41}$$

While Eq. (23) for **P** and **Q** is used, we can get

$$\begin{aligned} f(\theta) = \frac{1}{\alpha}h(\theta) &- \frac{1}{2\pi[\alpha^2 r_1(\ln r_1 - \ln r_2) + \alpha]} \int_0^{2\pi} h(\xi)d\xi \\ &- \frac{1}{\alpha}\sum_{k=1}^{m} \frac{B_k}{\alpha+\pi B_k} \int_0^{2\pi} (\cos k\theta \cos k\xi + \sin k\theta \sin k\xi)h(\xi)d\xi. \end{aligned} \tag{42}$$

For a given $h(\theta)$ obtained from Eq. (16), through some integrals one may employ the above equation to calculate $f(\theta)$ very efficiently.

Moreover, due to

$$\int_0^{2\pi} f(\xi)d\xi = \frac{r_1(\ln r_1 - \ln r_2)}{\alpha r_1(\ln r_1 - \ln r_2)+1}\int_0^{2\pi} h(\xi)d\xi, \tag{43}$$

$$\int_0^{2\pi} f(\xi)\cos k\xi d\xi = \frac{1}{\alpha+\pi B_k}\int_0^{2\pi} h(\xi)\cos k\xi d\xi, \tag{44}$$

$$\int_0^{2\pi} f(\xi)\sin k\xi d\xi = \frac{1}{\alpha+\pi B_k}\int_0^{2\pi} h(\xi)\sin k\xi d\xi, \tag{45}$$

we can insert the above three equations into Eqs. (6)-(11) to calculate all coefficients, which are given as follows:

$$a_0^\alpha = \frac{r_1\ln r_1}{\pi[1+\alpha r_1(\ln r_1-\ln r_2)]}\int_0^{2\pi} h(\xi)d\xi - \frac{\ln r_2}{\pi(\ln r_1-\ln r_2)}\int_0^{2\pi} g(\xi)d\xi, \tag{46}$$

$$b_0^\alpha = \frac{1}{\pi(\ln r_1-\ln r_2)}\int_0^{2\pi} g(\xi)d\xi - \frac{r_1}{\pi[1+\alpha r_1(\ln r_1-\ln r_2)]}\int_0^{2\pi} h(\xi)d\xi, \tag{47}$$

$$a_k^\alpha = \frac{e_k}{r_2^k}\int_0^{2\pi} g(\xi)\cos k\xi d\xi - \frac{e_k}{r_1^k(\alpha+\pi B_k)}\int_0^{2\pi} h(\xi)\cos k\xi d\xi, \tag{48}$$

$$b_k^\alpha = \frac{e_k r_1^k}{\alpha+\pi B_k}\int_0^{2\pi} h(\xi)\cos k\xi d\xi - e_k r_2^k\int_0^{2\pi} g(\xi)\cos k\xi d\xi, \tag{49}$$

$$c_k^\alpha = \frac{e_k}{r_2^k}\int_0^{2\pi} g(\xi)\sin k\xi d\xi - \frac{e_k}{r_1^k(\alpha+\pi B_k)}\int_0^{2\pi} h(\xi)\sin k\xi d\xi, \tag{50}$$

$$d_k^\alpha = \frac{e_k r_1^k}{\alpha+\pi B_k}\int_0^{2\pi} h(\xi)\sin k\xi d\xi - e_k r_2^k\int_0^{2\pi} g(\xi)\sin k\xi d\xi. \tag{51}$$

Then, from Eq. (5) by inserting the above regularized coefficients we can calculate $u^\alpha(r,\theta)$ by

$$u^\alpha(r,\theta) = \frac{1}{2}(a_0^\alpha + b_0^\alpha \ln r) + \sum_{k=1}^{\infty}[(a_k^\alpha r^k + b_k^\alpha r^{-k})\cos k\theta + (c_k^\alpha r^k + d_k^\alpha r^{-k})\sin k\theta], \tag{52}$$

where we use $u^\alpha(r,\theta)$ to stress that it is a regularized solution of u.

5 Error estimation

In the previous section we have derived a regularized solution $u^\alpha(r,\theta)$ of Eqs. (1)-(3) under the regularization in Eq. (20) with a regularized parameter $\alpha > 0$. We can prove the following main results. Because an analytic solution is already derived exactly, we let $m = \infty$ again.

Theorem 1: Assume that the data $g(\theta), h(\theta) \in L^2(0,2\pi)$. Then the sufficient and necessary condition that the inverse problem (1)-(3) has a solution is that

$$\sum_{k=1}^{\infty}\left[\left(\frac{1}{\pi}\int_0^{2\pi} g(\xi)\cos k\xi d\xi\right)^2+\left(\frac{1}{\pi}\int_0^{2\pi} g(\xi)\sin k\xi d\xi\right)^2\right]<\infty, \tag{53}$$

$$\sum_{k=1}^{\infty}\left[\left(\frac{1}{\pi^2 B_k}\int_0^{2\pi} h(\xi)\cos k\xi d\xi\right)^2+\left(\frac{1}{\pi^2 B_k}\int_0^{2\pi} h(\xi)\sin k\xi d\xi\right)^2\right]<\infty. \tag{54}$$

Proof: Taking $\alpha = 0$ in Eqs. (46)-(51) and inserting them into Eq. (52) we have a unique solution of Eqs. (1)-(3):

$$u(r,\theta)=\frac{1}{2}(a_0^\star+b_0^\star\ln r)+\sum_{k=1}^{\infty}[(a_k^\star r^k+b_k^\star r^{-k})\cos k\theta+(c_k^\star r^k+d_k^\star r^{-k})\sin k\theta], \tag{55}$$

where

$$a_0^\star=\frac{r_1\ln r_1}{\pi}\int_0^{2\pi}h(\xi)d\xi-\frac{\ln r_2}{\pi(\ln r_1-\ln r_2)}\int_0^{2\pi}g(\xi)d\xi, \tag{56}$$

$$b_0^\star=\frac{1}{\pi(\ln r_1-\ln r_2)}\int_0^{2\pi}g(\xi)d\xi-\frac{r_1}{\pi}\int_0^{2\pi}h(\xi)d\xi, \tag{57}$$

$$a_k^\star=\frac{e_k}{r_2^k}\int_0^{2\pi}g(\xi)\cos k\xi d\xi-\frac{e_k}{\pi r_1^k B_k}\int_0^{2\pi}h(\xi)\cos k\xi d\xi, \tag{58}$$

$$b_k^\star=\frac{e_k r_1^k}{\pi B_k}\int_0^{2\pi}h(\xi)\cos k\xi d\xi-e_k r_2^k\int_0^{2\pi}g(\xi)\cos k\xi d\xi, \tag{59}$$

$$c_k^\star=\frac{e_k}{r_2^k}\int_0^{2\pi}g(\xi)\sin k\xi d\xi-\frac{e_k}{\pi r_1^k B_k}\int_0^{2\pi}h(\xi)\sin k\xi d\xi, \tag{60}$$

$$d_k^\star=\frac{e_k r_1^k}{\pi B_k}\int_0^{2\pi}h(\xi)\sin k\xi d\xi-e_k r_2^k\int_0^{2\pi}g(\xi)\sin k\xi d\xi. \tag{61}$$

Inserting $r = r_1$ into Eq. (55) and noting Eq. (2), we have

$$\begin{aligned} g(\theta)&=u(r_1,\theta)\\ &=\frac{1}{2}(a_0^\star+b_0^\star\ln r_1)+\sum_{k=1}^{\infty}[(a_k^\star r_1^k+b_k^\star r_1^{-k})\cos k\theta+(c_k^\star r_1^k+d_k^\star r_1^{-k})\sin k\theta], \end{aligned} \tag{62}$$

where $g(\theta) \in L^2(0,2\pi)$. The above is a Fourier expansion of $g(\theta)$, and then by the Parseval equality we have

$$\frac{1}{4}(a_0^\star+b_0^\star\ln r_1)^2+\sum_{k=1}^{\infty}\left[(a_k^\star r_1^k+b_k^\star r_1^{-k})^2+(c_k^\star r_1^k+d_k^\star r_1^{-k})^2\right]=\|g(\theta)\|^2_{L^2(0,2\pi)}<\infty,$$

(63)

where $\|g(\theta)\|_{L^2(0,2\pi)}$ is the L^2-norm of $g(\theta)$ in the interval of $(0,2\pi)$. Then, by utilizing Eqs. (56)-(61) and (12) we can derive the condition (53).

Similarly, inserting $r = r_2$ into Eq. (55) and noting Eq. (4), we have

$$\begin{aligned} f(\theta) &= u(r_2,\theta) \\ &= \frac{1}{2}(a_0^\star + b_0^\star \ln r_2) + \sum_{k=1}^{\infty} [(a_k^\star r_2^k + b_k^\star r_2^{-k}) \cos k\theta + (c_k^\star r_2^k + d_k^\star r_2^{-k}) \sin k\theta], \end{aligned} \tag{64}$$

where $f(\theta) \in L^2(0,2\pi)$. The above is a Fourier expansion of $f(\theta)$, and then by the Parseval equality we have

$$\frac{1}{4}(a_0^\star + b_0^\star \ln r_2)^2 + \sum_{k=1}^{\infty} \left[(a_k^\star r_2^k + b_k^\star r_2^{-k})^2 + (c_k^\star r_2^k + d_k^\star r_2^{-k})^2 \right] = \|f(\theta)\|^2_{L^2(0,2\pi)} < \infty, \tag{65}$$

Then, by utilizing Eqs. (56)-(61) we can derive condition (54). □

Remark: The condition (53) is insufficient to guarantee that the series in Eq. (16) exists. From Eqs. (18) and (12) it follows that

$$A_k = \frac{ke_k}{r_1} \left[\left(\frac{r_1}{r_2}\right)^k + \left(\frac{r_2}{r_1}\right)^k \right] = \frac{k\left[\left(\frac{r_1}{r_2}\right)^k + \left(\frac{r_2}{r_1}\right)^k \right]}{r_1 \pi \left[\left(\frac{r_1}{r_2}\right)^k - \left(\frac{r_2}{r_1}\right)^k \right]}. \tag{66}$$

Obviously, A_k is divergent. Therefore, we need to impose

$$\sum_{k=1}^{\infty} A_k^2 \left\{ \left(\int_0^{2\pi} g(\xi) \cos k\xi d\xi \right)^2 + \left(\int_0^{2\pi} g(\xi) \sin k\xi d\xi \right)^2 \right\} < \infty. \tag{67}$$

This condition about the boundary data $g(\theta)$ is stronger than condition (53); however, it guarantees that the series in Eq. (16) is convergent.

Theorem 2: If the Cauchy data $g(\theta)$ and $h(\theta)$ satisfy Eqs. (53) and (54), and are also bounded in the interval $\theta \in [0,2\pi]$, i.e.,

$$|g(\theta)| \le C_1, \ \ |h(\theta)| \le C_2, \ \ \theta \in [0,2\pi], \tag{68}$$

where C_1 and C_2 are positive constants, then for any $\alpha > 0$ the regularized solution $u^\alpha(r,\theta)$ converges uniformly for all $r \in (r_2, r_1)$ and $\theta \in [0,2\pi]$.

Proof: From Eqs. (19) we have

$$\frac{e_k}{B_k} = \frac{r_1}{2k}. \tag{69}$$

Depending on the ratio $r_2/r_1 = \rho < 1$ there exists a positive number $C_0 \geq 1/(1-\rho^2) > 1$, such that the following inequality holds for all $k \geq 1$:

$$e_k = \frac{1}{\pi\left[\left(\frac{r_1}{r_2}\right)^k - \left(\frac{r_2}{r_1}\right)^k\right]} \leq \frac{C_0}{\pi}\rho^k. \tag{70}$$

We only require to check that

$$C_0\rho^k\left[\left(\frac{r_1}{r_2}\right)^k - \left(\frac{r_2}{r_1}\right)^k\right] = C_0(1-\rho^{2k}) \geq \frac{1-\rho^{2k}}{1-\rho^2} \geq 1, \;\; \forall k \geq 1.$$

In order to prove the uniform convergence of $u^\alpha(r,\theta)$, we only estimate the coefficients $a_k^\alpha r^k$ and $b_k^\alpha r^{-k}$ appeared in the series functions $a_k^\alpha r^k \cos k\theta$ and $b_k^\alpha r^{-k}\cos k\theta$, and the other two series $c_k^\alpha r^k \sin k\theta$ and $d_k^\alpha r^{-k}\sin k\theta$ can be estimated similarly.

From Eqs. (69) and (70), the following results are obtained by using $\alpha > 0$ and $r_2 < r < r_1$:

$$r^k r_1^{-k} e_k \leq \frac{C_0}{\pi}\rho^k\left(\frac{r}{r_2}\right)^k = \frac{C_0}{\pi}\left(\frac{r}{r_1}\right)^k, \tag{71}$$

$$r^{-k} r_2^k e_k \leq \frac{C_0}{\pi}\rho^k r^{-k} r_2^k = \frac{C_0}{\pi}\rho^k\left(\frac{r_2}{r}\right)^k, \tag{72}$$

$$\frac{e_k r^k}{r_1^k(\alpha+\pi B_k)} \leq \frac{e_k r^k}{\pi r_1^k B_k} = \frac{r_1 r^k}{2\pi k r_2^k} = \frac{r_1}{2\pi k}\left(\frac{r}{r_1}\right)^k, \tag{73}$$

$$\frac{e_k r_1^k}{r^k(\alpha+\pi B_k)} \leq \frac{C_0\rho^k r_1^k}{\pi r^k \alpha} = \frac{C_0}{\pi\alpha}\left(\frac{r_2}{r}\right)^k. \tag{74}$$

Therefore, from Eqs. (71)-(74) and Eqs. (48)-(51) it follows that

$$|a_k^\alpha r^k| \leq \frac{2\pi r^k e_k}{r_1^k(\alpha+\pi B_k)}C_2 + \frac{2\pi r^k e_k}{r_2^k}C_1 \leq \left[\frac{r_1 C_0 C_2}{k} + 2C_0C_1\right]\left(\frac{r}{r_1}\right)^k, \tag{75}$$

$$|b_k^\alpha r^{-k}| \leq 2\pi e_k r^{-k} r_2^k C_1 + \frac{2\pi e_k r^{-k} r_1^k}{\alpha+\pi B_k}C_2 \leq \left[2C_0C_1 + \frac{2C_0C_2}{\alpha}\right]\rho^k\left(\frac{r_2}{r}\right)^k, \tag{76}$$

$$|c_k^\alpha r^k| \leq \left[\frac{r_1 C_0 C_2}{k} + 2C_0C_1\right]\left(\frac{r}{r_1}\right)^k, \tag{77}$$

$$|d_k^\alpha r^{-k}| \leq \left[2C_0C_1 + \frac{2C_0C_2}{\alpha}\right]\rho^k\left(\frac{r_2}{r}\right)^k, \tag{78}$$

where C_1 and C_2 are the bounds of $g(\theta)$ and $h(\theta)$.
Hence, the series of $u^\alpha(r,\theta)$ defined by Eq. (52) is dominated by the following series

$$2C_0 \sum_{k=1}^{\infty} \left(\left[\frac{r_1 C_2}{k} + 2C_1 \right] \left(\frac{r}{r_1} \right)^k + \left[2C_1 + \frac{2C_2}{\alpha} \right] \rho^k \left(\frac{r_2}{r} \right)^k \right). \tag{79}$$

This series is convergent due to $r < r_1$, $\rho < 1$ and $r_2 < r$. Then, by the Weierstrass M-test [Apostol (1974)], the series in Eq. (52) converges uniformly with respect to r and θ whenever $r \in (r_2, r_1)$ and $\theta \in [0, 2\pi]$. This ends the proof. $\square$

Theorem 3: If the datum $h(\theta)$ satisfies condition (54) and there exists an $\varepsilon \in (0,1)$, such that, moreover,

$$\sum_{k=1}^{\infty} \frac{1}{(\pi^2 B_k)^{2(1+\varepsilon)}} \left[\left(\int_0^{2\pi} h(\xi) \cos k\xi \, d\xi \right)^2 + \left(\int_0^{2\pi} h(\xi) \sin k\xi \, d\xi \right)^2 \right] =: M^2(\varepsilon) < \infty, \tag{80}$$

then for any $\alpha > 0$ the regularized solution $u^\alpha(r,\theta)$ satisfies the following error estimation:

$$\| u^\alpha(r,\theta) - u(r,\theta) \|_{L^2(0,2\pi)} \le \alpha^\varepsilon M(\varepsilon). \tag{81}$$

Proof: From Eqs. (56)-(61) and (46)-(51) it follows that

$$\begin{aligned} u(r,\theta) - u^\alpha(r,\theta) = {} & \frac{\alpha}{2}(a_0^\diamond + b_0^\diamond \ln r) \\ & + \alpha \sum_{k=1}^{\infty} [(a_k^\diamond r^k + b_k^\diamond r^{-k}) \cos k\theta + (c_k^\diamond r^k + d_k^\diamond r^{-k}) \sin k\theta], \end{aligned} \tag{82}$$

where

$$a_0^\diamond := a_0^\star - a_0^\alpha = \frac{r_1^2 \ln r_1 (\ln r_1 - \ln r_2)}{\pi[1+\alpha r_1(\ln r_1 - \ln r_2)]} \int_0^{2\pi} h(\xi) d\xi, \tag{83}$$

$$b_0^\diamond := b_0^\star - b_0^\alpha = -\frac{r_1^2 (\ln r_1 - \ln r_2)}{\pi[1+\alpha r_1(\ln r_1 - \ln r_2)]} \int_0^{2\pi} h(\xi) d\xi, \tag{84}$$

$$a_k^\diamond := a_k^\star - a_k^\alpha = -\frac{e_k}{\pi r_1^k B_k(\alpha+\pi B_k)} \int_0^{2\pi} h(\xi) \cos k\xi d\xi, \tag{85}$$

$$b_k^\diamond := b_k^\star - b_k^\alpha = \frac{e_k r_1^k}{\pi B_k(\alpha+\pi B_k)} \int_0^{2\pi} h(\xi) \cos k\xi d\xi, \tag{86}$$

$$c_k^\diamond := c_k^\star - c_k^\alpha = -\frac{e_k}{\pi r_1^k B_k(\alpha+\pi B_k)} \int_0^{2\pi} h(\xi) \sin k\xi d\xi, \tag{87}$$

$$d_k^\diamond := d_k^\star - d_k^\alpha = \frac{e_k r_1^k}{\pi B_k(\alpha+\pi B_k)} \int_0^{2\pi} h(\xi) \sin k\xi d\xi. \tag{88}$$

In order to estimate the norm of $\|u(r,\theta) - u^\alpha(r,\theta)\|_{L^2(0,2\pi)}$, we only need to estimate the coefficients $a_k^\diamond r^k + b_k^\diamond r^{-k}$ appeared in the series functions $(a_k^\diamond r^k + b_k^\diamond r^{-k}) \cos k\theta$ in Eq. (82), and the other series $(c_k^\diamond r^k + d_k^\diamond r^{-k}) \sin k\theta$ can be estimated similarly. When $r_2 \le r \le r_1$, the following inequality is obvious,

$$0 \le \left[\left(\frac{r_1}{r}\right)^k - \left(\frac{r}{r_1}\right)^k\right] \le \left[\left(\frac{r_1}{r_2}\right)^k - \left(\frac{r_2}{r_1}\right)^k\right]. \tag{89}$$

From Eqs. (85) and (86) it follows that

$$a_k^\diamond r^k + b_k^\diamond r^{-k} = \frac{e_k}{\pi B_k(\alpha+\pi B_k)} \left[\left(\frac{r_1}{r}\right)^k - \left(\frac{r}{r_1}\right)^k\right] \int_0^{2\pi} h(\xi) \cos k\xi d\xi. \tag{90}$$

By using Eqs. (89) and (12) we have

$$|a_k^\diamond r^k + b_k^\diamond r^{-k}|^2 \le \frac{1}{(\pi^2 B_k)^2 (\alpha+\pi B_k)^2} \left(\int_0^{2\pi} h(\xi) \cos k\xi d\xi\right)^2, \tag{91}$$

and similarly we have

$$|c_k^\diamond r^k + d_k^\diamond r^{-k}|^2 \le \frac{1}{(\pi^2 B_k)^2 (\alpha+\pi B_k)^2} \left(\int_0^{2\pi} h(\xi) \sin k\xi d\xi\right)^2. \tag{92}$$

It follows that

$$
\begin{aligned}
&\|u(r,\theta)-u^{\alpha}(r,\theta)\|^2_{L^2(0,2\pi)} \leq \\
&\alpha^2 \sum_{k=1}^{\infty} \frac{1}{(\pi^2 B_k)^2(\alpha+\pi B_k)^2} \left[\left(\int_0^{2\pi} h(\xi)\cos k\xi d\xi \right)^2 + \left(\int_0^{2\pi} h(\xi)\sin k\xi d\xi \right)^2 \right] \\
&= \alpha^2 \sum_{k=1}^{\infty} (\pi^2 B_k)^{-2} [(\alpha+\pi B_k)^{\varepsilon}(\alpha+\pi B_k)^{1-\varepsilon}]^{-2} \\
&\qquad \left[\left(\int_0^{2\pi} h(\xi)\cos k\xi d\xi \right)^2 + \left(\int_0^{2\pi} h(\xi)\sin k\xi d\xi \right)^2 \right] \\
&\leq \alpha^2 \sum_{k=1}^{\infty} (\pi^2 B_k)^{-2} (\pi B_k)^{-2\varepsilon} \alpha^{-2+2\varepsilon} \\
&\qquad \left[\left(\int_0^{2\pi} h(\xi)\cos k\xi d\xi \right)^2 + \left(\int_0^{2\pi} h(\xi)\sin k\xi d\xi \right)^2 \right] \\
&= \alpha^{2\varepsilon} \sum_{k=1}^{\infty} (\pi^2 B_k)^{-2(1+\varepsilon)} \left[\left(\int_0^{2\pi} h(\xi)\cos k\xi d\xi \right)^2 + \left(\int_0^{2\pi} h(\xi)\sin k\xi d\xi \right)^2 \right] \\
&=: \alpha^{2\varepsilon} M^2(\varepsilon).
\end{aligned} \tag{93}
$$

Therefore, we complete the proof. □

The above **Theorems** are important that the present regularized solution is well behaved, and can approach the true solution of the ill-posed problem in Eqs. (1)-(3).

6 Numerical examples

6.1 Inverse Cauchy problem

In order to validate the performance of our numerical method we consider a typical benchmark example

$$\Delta u = 0, \ \ r_2 < r < r_1, \ \ 0 \leq \theta \leq 2\pi, \tag{94}$$

$$u(r_1,\theta) = g(\theta) = r_1^2 \cos 2\theta, \ \ 0 \leq \theta \leq 2\pi, \tag{95}$$

$$u_r(r_1,\theta) = h_1(\theta) = 2r_1 \cos 2\theta, \ \ 0 \leq \theta \leq 2\pi. \tag{96}$$

The datum to be retrieved is given by

$$u(r,\theta) = r^2 \cos 2\theta = x^2 - y^2. \tag{97}$$

Inserting Eqs. (95) and (96) into Eq. (16) we obtain

$$h(\theta) = (\pi A_2 r_1^2 - 2r_1)\cos 2\theta. \tag{98}$$

Substituting the above $g(\theta)$ and $h(\theta)$ into Eqs. (46)-(51) only the following coefficients are nonzero:

$$a_2^\alpha = \frac{\pi r_1^2 e_2}{r_2^2} - \frac{\pi e_2(\pi A_2 r_1^2 - 2r_1)}{r_1^2(\alpha + \pi B_2)}, \tag{99}$$

$$b_2^\alpha = \frac{\pi e_2 r_1^2(\pi A_2 r_1^2 - 2r_1)}{\alpha + \pi B_2} - \pi e_2 r_1^2 r_2^2, \tag{100}$$

and the other coefficients are all zero. Hence, from Eq. (52) we obtain a regularized solution:

$$u^\alpha(r,\theta) = (a_2^\alpha r^2 + b_2^\alpha r^{-2})\cos 2\theta. \tag{101}$$

For the comparison purpose we have fixed $r_1 = 1$ and $r_2 = 0.5$ in the following calculations. In Fig. 1 we compare the exact solution with the regularized solution under $\alpha = 10^{-8}$, with a fixed $r = 0.8$. It can be seen that the numerical error is in the order of 10^{-10}. In Fig. 2 we have compared the contours of isopotential with $u = 0.1, 0.3, 0.5, -0.2, -0.4, -0.6$. These curves are very well coincident with the curves obtained from the exact solutions. In these comparisons the present results are very excellent.

For the same problem, Saito, Nakada, Iijima and Onish (2005) have developed a high order finite difference scheme together with a multi-precesion arithmic system up to 50 decimal digits to calculate it. They have reported that the maximum error occurred at the internal circle is 9×10^{-6}. In our calculation if we let $\alpha = 10^{-8}$, a maximum error at the internal circle is 2.34×10^{-9}.

6.2 Zero-potential curve

As mentioned by Kress (2004), when the inner inclusion is a perfectly conducting body, the determination of its unknown shape leads to an inverse Dirichlet boundary value problem. The mathematical problem is that for the given data $g(\theta)$ and $h_1(\theta)$ on the exterior boundary, we attempt to determine the zero-potential curve, that is,

$$u = 0, \quad \text{on} \quad \Gamma, \tag{102}$$

where Γ is an unknown curve inside the domain Ω.

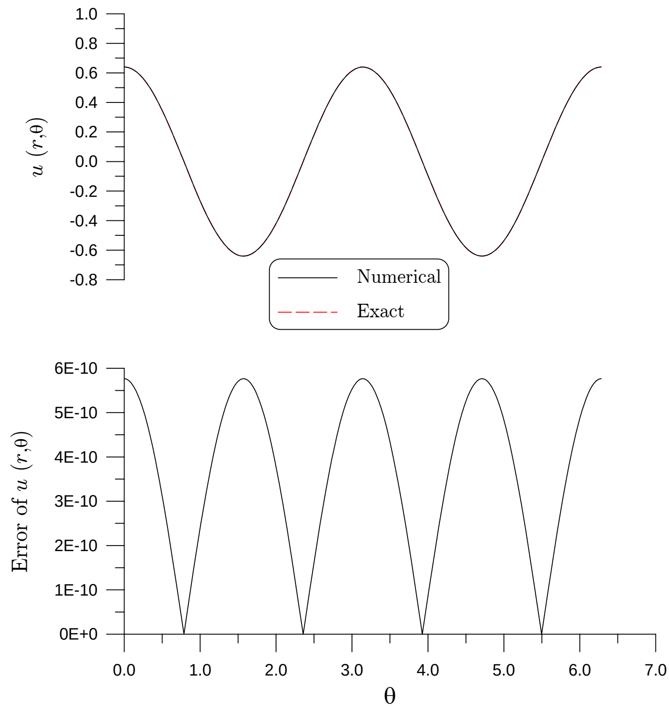

Figure 1: For the Cauchy type inverse problem we have compared regularized and exact solutions in (a), and the numerical error in (b) for a specific example.

In order to illustrate the unknown curve problem and compare our numerical result with exact solution, we first consider a direct problem with the following data:

$$u(r_1,\theta) = g(\theta) = \cos 2\theta + \beta\cos\theta, \;\; 0 \le \theta \le 2\pi, \tag{103}$$
$$u(r_2,\theta) = f(\theta) = -\cos 2\theta - \beta\sin\theta, \;\; 0 \le \theta \le 2\pi, \tag{104}$$

where β is a given constant. The solution of the direct problem would provide us $h_1(\theta)$, which is required in the inverse problem.
Then, from Eqs. (6)-(11) we obtain

$$a_1 = \frac{\beta\pi e_1}{r_2}, \;\; a_2 = \frac{\pi e_2}{r_1^2} + \frac{\pi e_2}{r_2^2}, \tag{105}$$
$$b_1 = -\beta\pi e_1 r_2, \;\; b_2 = -\pi e_2 r_1^2 - \pi e_2 r_2^2, \tag{106}$$
$$c_1 = \frac{\beta\pi e_1}{r_1}, \;\; d_1 = -\beta\pi e_1 r_1. \tag{107}$$

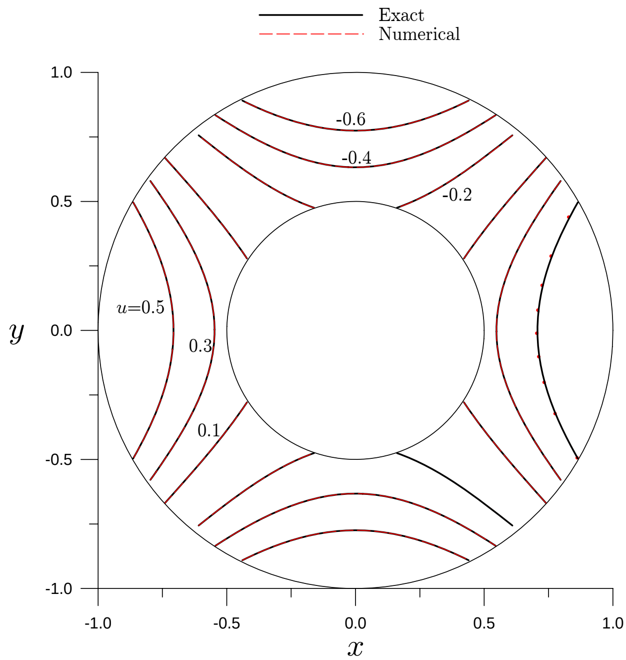

Figure 2: Comparing the isopotential curves of the same example in Fig. 1.

Therefore, we have a closed-form solution of the direct problem:

$$u(r,\theta) = \beta\pi e_1\left[\frac{r\sin\theta}{r_1} + \frac{r\cos\theta}{r_2}\right] - \beta\pi e_1\left[\frac{r_1\sin\theta}{r} + \frac{r_2\cos\theta}{r}\right] \\ + \pi e_2\left[\frac{1}{r_1^2} + \frac{1}{r_2^2}\right] r^2\cos 2\theta - \pi e_2(r_1^2 + r_2^2)r^{-2}\cos 2\theta. \tag{108}$$

The derivative of $u(r,\theta)$ with respect to r is

$$u_r(r,\theta) = \beta\pi e_1\left[\frac{\sin\theta}{r_1} + \frac{\cos\theta}{r_2}\right] + \beta\pi e_1\left[\frac{r_1\sin\theta}{r^2} + \frac{r_2\cos\theta}{r^2}\right] \\ + 2\pi e_2\left[\frac{1}{r_1^2} + \frac{1}{r_2^2}\right] r\cos 2\theta + 2\pi e_2(r_1^2 + r_2^2)r^{-3}\cos 2\theta. \tag{109}$$

Letting $r = r_1$ in Eq. (109), we obtain another Cauchy data of the inverse problem:

$$h_1(\theta) = D_1\cos\theta + D_2\cos 2\theta + D_3\sin\theta, \tag{110}$$

where

$$D_1 := \frac{\beta\pi e_1}{r_2} + \beta\pi e_1 \frac{r_2}{r_1^2}, \tag{111}$$

$$D_2 := 2\pi e_2 \left[\frac{1}{r_1^2} + \frac{1}{r_2^2}\right] r_1 + 2\pi e_2 (r_1^2 + r_2^2) r_1^{-3}, \tag{112}$$

$$D_3 := \frac{2\beta\pi e_1}{r_1}. \tag{113}$$

Inserting Eqs. (103) and (110) into Eq. (16) we obtain

$$h(\theta) = (\beta\pi A_1 - D_1)\cos\theta + (\pi A_2 - D_2)\cos 2\theta - D_3 \sin\theta. \tag{114}$$

Substituting the above $g(\theta)$ and $h(\theta)$ into Eqs. (46)-(51) we obtain

$$a_1^\alpha = \frac{\beta\pi e_1}{r_2} - \frac{\pi e_1(\beta\pi A_1 - D_1)}{r_1(\alpha + \pi B_1)}, \tag{115}$$

$$a_2^\alpha = \frac{\pi e_2}{r_2^2} - \frac{\pi e_2(\pi A_2 - D_2)}{r_1^2(\alpha + \pi B_2)}, \tag{116}$$

$$b_1^\alpha = \frac{\pi e_1 r_1(\beta\pi A_1 - D_1)}{\alpha + \pi B_1} - \beta\pi e_1 r_2, \tag{117}$$

$$b_2^\alpha = \frac{\pi e_2 r_1^2(\pi A_2 - D_2)}{\alpha + \pi B_2} - \pi e_2 r_2^2, \tag{118}$$

$$c_1^\alpha = \frac{\pi e_1 D_3}{r_1(\alpha + \pi B_1)}, \tag{119}$$

$$d_1^\alpha = -\frac{\pi e_1 r_1 D_3}{\alpha + \pi B_1}. \tag{120}$$

Hence, from Eq. (52) we obtain a regularized solution:

$$u^\alpha(r,\theta) = (a_1^\alpha r + b_1^\alpha r^{-1})\cos\theta + (a_2^\alpha r^2 + b_2^\alpha r^{-2})\cos 2\theta + c_1^\alpha r \sin\theta + d_1^\alpha r^{-1}\sin\theta. \tag{121}$$

From Eq. (108), by solving $u = 0$ we have a closed-form solution:

$$\beta\pi e_1 \left[\frac{r\sin\theta}{r_1} + \frac{r\cos\theta}{r_2}\right] - \beta\pi e_1 \left[\frac{r_1\sin\theta}{r} + \frac{r_2\cos\theta}{r}\right]$$
$$+ \pi e_2 \left[\frac{1}{r_1^2} + \frac{1}{r_2^2}\right] r^2 \cos 2\theta - \pi e_2 (r_1^2 + r_2^2) r^{-2} \cos 2\theta = 0. \tag{122}$$

On the other hand, from Eq. (121) we have a numerical solution by solving $u^\alpha = 0$:

$$(a_1^\alpha r + b_1^\alpha r^{-1})\cos\theta + (a_2^\alpha r^2 + b_2^\alpha r^{-2})\cos 2\theta + (c_1^\alpha r + d_1^\alpha r^{-1})\sin\theta = 0. \tag{123}$$

We consider two cases with $\beta = 0$ and $\beta = 0.05$ in Fig. 3. For the case of $\beta = 0$ the zero-potential curves are regular and the numerical solution with $\alpha = 10^{-5}$ coincides very well with the exact solution as shown in Fig. 3(a). For the case of $\beta = 0.05$ the zero-potential curve is discontinuous; however, the numerical solution with $\alpha = 10^{-5}$ also coincides very well with the exact solution as shown in Fig. 3(b).

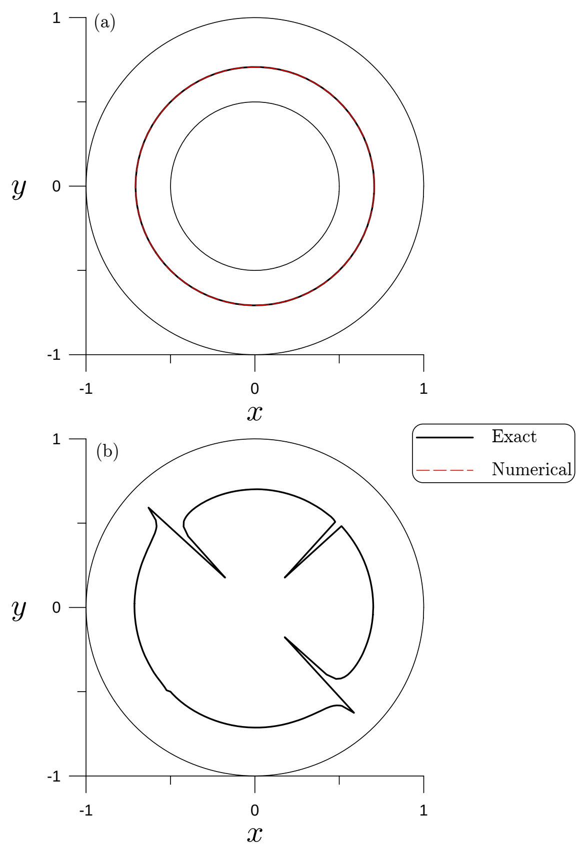

Figure 3: Plotting the zero-potential curves in (a) with $\beta = 0$ and in (b) with $\beta = 0.05$; the numerical and exact results coincide very well.

6.3 Detection of the position of cracks

The purpose of this problem is to detect the position of cracks inside the domain Ω through the measurements of the Cauchy data on the outer boundary.

In order to examine the performance of our numerical method we consider the following example:

$$u(r_1,\theta) = g(\theta) = \frac{a^2-b^2}{2(a^2+b^2)} r_1^2 \cos 2\theta + \frac{a^2b^2}{a^2+b^2}, \tag{124}$$

$$u_r(r_1,\theta) = h_1(\theta) = \frac{a^2-b^2}{a^2+b^2} r_1 \cos 2\theta. \tag{125}$$

The datum to be retrieved is given by

$$u(r,\theta) = \frac{a^2-b^2}{2(a^2+b^2)} r^2 \cos 2\theta + \frac{a^2b^2}{a^2+b^2}. \tag{126}$$

The two parameters $a > b$ determine the position and size of cracks, which are obtained by solving $u(r,\theta) = 0$.

Inserting Eqs. (124) and (125) into Eq. (16) we obtain

$$h(\theta) = \frac{a^2b^2}{r_1(\ln r_1 - \ln r_2)(a^2+b^2)} + \left[\frac{\pi A_2(a^2-b^2)}{2(a^2+b^2)} r_1^2 - \frac{a^2-b^2}{a^2+b^2} r_1\right] \cos 2\theta, \tag{127}$$

Substituting the above $g(\theta)$ and $h(\theta)$ into Eqs. (46)-(51) we obtain

$$a_0^\alpha = \frac{2\ln r_1 a^2b^2}{[1+\alpha r_1(\ln r_1 - \ln r_2)](\ln r_1 - \ln r_2)(a^2+b^2)} - \frac{2a^2b^2 \ln r_2}{(\ln r_1 - \ln r_2)(a^2+b^2)}, \tag{128}$$

$$b_0^\alpha = \frac{2a^2b^2}{(\ln r_1 - \ln r_2)(a^2+b^2)} - \frac{2a^2b^2}{[1+\alpha r_1(\ln r_1 - \ln r_2)](\ln r_1 - \ln r_2)(a^2+b^2)}, \tag{129}$$

$$a_2^\alpha = \frac{\pi r_1^2 e_2(a^2-b^2)}{2r_2^2(a^2+b^2)} - \frac{\pi e_2}{r_1^2(\alpha+\pi B_2)} \left[\frac{\pi A_2(a^2-b^2)}{2(a^2+b^2)} r_1^2 - \frac{a^2-b^2}{a^2+b^2} r_1\right], \tag{130}$$

$$b_2^\alpha = \frac{\pi e_2 r_1^2}{\alpha+\pi B_2} \left[\frac{\pi A_2(a^2-b^2)}{2(a^2+b^2)} r_1^2 - \frac{a^2-b^2}{a^2+b^2} r_1\right] - \pi e_2 r_1^2 r_2^2 \frac{a^2-b^2}{2(a^2+b^2)}. \tag{131}$$

Hence, from Eq. (52) we obtain a regularized solution:

$$u^\alpha(r,\theta) = \frac{1}{2}(a_0^\alpha + b_0^\alpha \ln r) + (a_2^\alpha r^2 + b_2^\alpha r^{-2}) \cos 2\theta. \tag{132}$$

Solving the above equation with $u^\alpha(r,\theta) = 0$ we can determine the position of cracks.

It can be seen that the regularized solution in Eq. (132) is much complicated than the exact solution in Eq. (126); however, it is easy to prove that the two solutions are identical when $\alpha = 0$. In Fig. 4 we compare the cracks' position calculated by the numerical method based on the regularized solution with that calculated from the exact solution, where the parameters used in this comparison are $a = 3$, $b = 1$, $r_1 = 8$ and $r_2 = 1$. We have applied the half-interval numerical method to solve $u^\alpha(r,\theta) = 0$ under an error tolerance with 10^{-5}. Even with $\alpha = 0.05$ it can be seen that the numerical results match very well with the exact solutions.

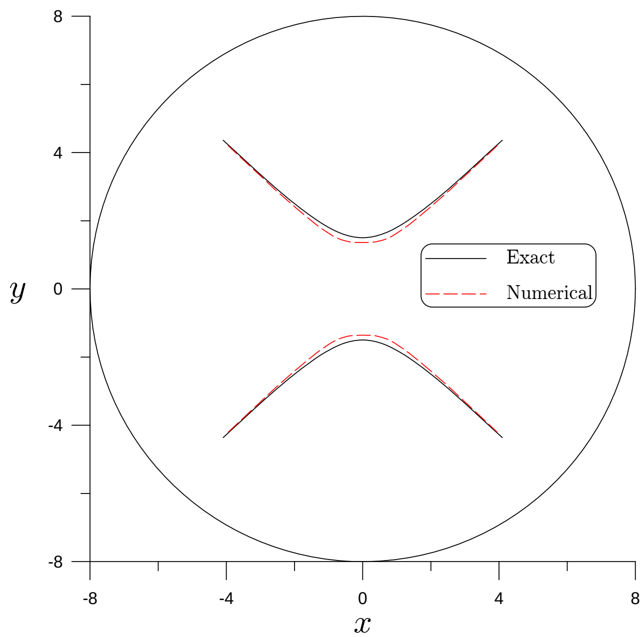

Figure 4: Displaying the position of two cracks, where the numerical and exact results almost coincide even $\alpha = 0.05$.

6.4 Robin type exchange coefficient problem

For the Robin type exchange coefficient problem the reader may refer the paper by Chaabane, Elhechmi and Jaoua (2004). Let $\mathbb{D}$ be a disc with a radius r_1 and boundary $\mathbb{T}$ and G be the annulus $G = \mathbb{D} \setminus \overline{s\mathbb{D}}$ for some fixed $0 < s < 1$. Given two functions g and h_1, find a function ϕ such that a solution u to

$$\Delta u = 0, \ \text{in} \ G, \tag{133}$$
$$u = g(\theta), \ \text{on} \ \mathbb{T}, \tag{134}$$
$$u_n = h_1(\theta), \ \text{on} \ \mathbb{T} \tag{135}$$

also satisfies

$$u_n + \phi u = 0, \ \text{on} \ s\mathbb{T}, \tag{136}$$

where $s\mathbb{T}$ is a circle on the annulus with a radius $r_3 = sr_1$ and n is the outward normal vector. The purpose of this problem is to find the unknown Robin coefficient function ϕ.

In order to illustrate the Robin type exchange coefficient problem and compare our numerical result with exact solution, we first consider a direct problem with the following data:

$$u(r_1,\theta) = g(\theta) = a + \cos\theta, \ 0 \le \theta \le 2\pi, \tag{137}$$
$$u(r_2,\theta) = f(\theta) = b + \sin\theta, \ 0 \le \theta \le 2\pi, \tag{138}$$

where $r_2 < r_3$ is a given radius. a and b are selected such that $g > f \ge 0, \ 0 \le \theta \le 2\pi$.

Then, from Eqs. (6)-(11) we obtain

$$a_0 = \frac{2b\ln r_1 - 2a\ln r_2}{\ln r_1 - \ln r_2}, \ b_0 = \frac{2(a-b)}{\ln r_1 - \ln r_2}, \tag{139}$$
$$a_1 = \frac{\pi e_1}{r_2}, \ b_1 = -\pi e_1 r_2, \tag{140}$$
$$c_1 = \frac{\pi e_1}{r_1}, \ d_1 = -\pi e_1 r_1, \tag{141}$$

and a closed-form solution of the direct problem follows:

$$\begin{aligned} u(r,\theta) = {} & \frac{b\ln r_1 - a\ln r_2}{\ln r_1 - \ln r_2} + \frac{(a-b)\ln r}{\ln r_1 - \ln r_2} \\ & + \pi e_1\left[\frac{r\sin\theta}{r_1} + \frac{r\cos\theta}{r_2}\right] - \pi e_1\left[\frac{r_1\sin\theta}{r} + \frac{r_2\cos\theta}{r}\right]. \end{aligned} \tag{142}$$

The derivative of $u(r,\theta)$ with respect to r is

$$u_r(r,\theta) = \frac{a-b}{r(\ln r_1 - \ln r_2)} + \pi e_1\left[\frac{\sin\theta}{r_1} + \frac{\cos\theta}{r_2}\right] + \pi e_1\left[\frac{r_1\sin\theta}{r^2} + \frac{r_2\cos\theta}{r^2}\right]. \tag{143}$$

Letting $r = r_1$ in Eq. (109), we obtain another Cauchy data of the inverse problem:

$$h_1(\theta) = \frac{a-b}{r_1(\ln r_1 - \ln r_2)} + D_1 \cos\theta + D_2 \sin\theta, \tag{144}$$

where

$$D_1 := \frac{\pi e_1}{r_2} + \frac{\pi e_1 r_2}{r_1^2}, \tag{145}$$

$$D_2 := \frac{2\pi e_1}{r_1}. \tag{146}$$

Inserting Eqs. (137) and (144) into Eq. (16) we obtain

$$h(\theta) = \frac{b}{r_1(\ln r_1 - \ln r_2)} + (\pi A_1 - D_1)\cos\theta - D_2 \sin\theta. \tag{147}$$

Substituting the above $g(\theta)$ and $h(\theta)$ into Eqs. (46)-(51) we obtain

$$a_0^\alpha = \frac{2b\ln r_1}{(\ln r_1 - \ln r_2)[1+\alpha r_1(\ln r_1 - \ln r_2)]} - \frac{2a\ln r_2}{\ln r_1 - \ln r_2}, \tag{148}$$

$$b_0^\alpha = \frac{2a}{\ln r_1 - \ln r_2} - \frac{2b}{(\ln r_1 - \ln r_2)[1+\alpha r_1(\ln r_1 - \ln r_2)]}, \tag{149}$$

$$a_1^\alpha = \frac{\pi e_1}{r_2} - \frac{\pi e_1(\pi A_1 - D_1)}{r_1(\alpha + \pi B_1)}, \tag{150}$$

$$b_1^\alpha = \frac{\pi e_1 r_1(\pi A_1 - D_1)}{\alpha + \pi B_1} - \pi e_1 r_2, \tag{151}$$

$$c_1^\alpha = \frac{\pi e_1 D_2}{r_1(\alpha + \pi B_1)}, \tag{152}$$

$$d_1^\alpha = -\frac{\pi e_1 r_1 D_2}{\alpha + \pi B_1}. \tag{153}$$

Hence, from Eq. (52) we obtain a regularized solution:

$$u^\alpha(r,\theta) = \frac{1}{2}(a_0^\alpha + b_0^\alpha \ln r) + (a_1^\alpha r + b_1^\alpha r^{-1})\cos\theta + (c_1^\alpha r + d_1^\alpha r^{-1})\sin\theta. \tag{154}$$

Inserting Eq. (142) for u and Eq. (143) for u_r into the following equation:

$$\phi = \frac{u_r}{u} \tag{155}$$

we can obtain the exact ϕ. Similarly, inserting Eq. (154) and its differential into the above equation we can obtain a numerical solution of ϕ by the regularized method. We compare the exact solutions with the regularized solutions under the parameters $a = 5$, $b = 2$, $r_3 = 0.8$ and $\alpha = 0.01$ in Fig. 5(a), and $a = 3$, $b = 1$, $r_3 = 0.75$ and $\alpha = 0.01$ in Fig. 5(b). The results are very good.

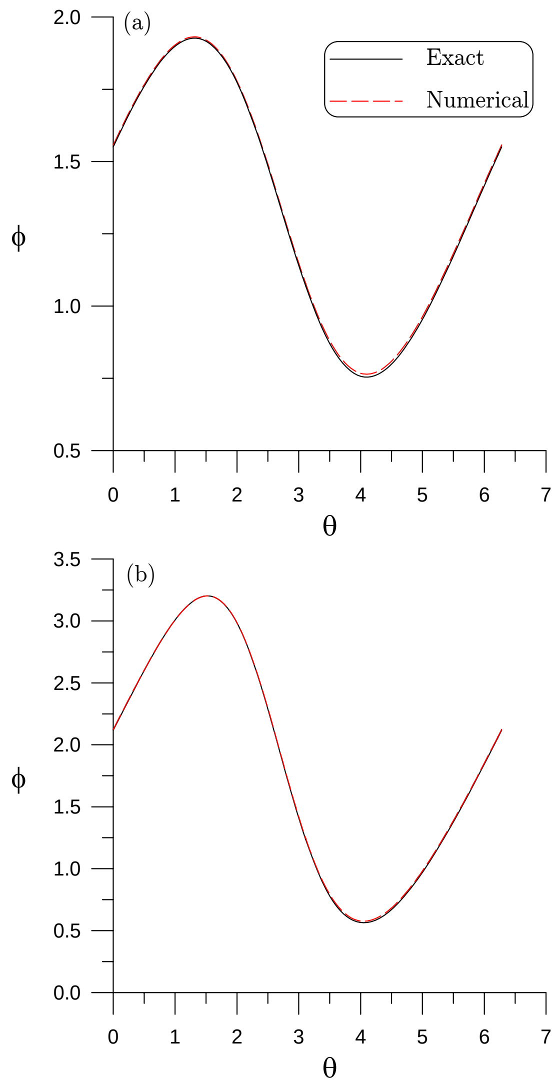

Figure 5: Comparing the Robin coefficients for two cases, where the numerical results match very well the exact results.

7 Conclusions

The idea of detecting unknown Robin coefficient or cracks' position in a disk is modeled by an inverse Cauchy problem of Laplace equation. We have shown that the reconstruction of an unknown inner boundary data on the inaccessible part can be reduced to a well-posed regularized second kind Fredholm integral equation. Then, by using the Fourier series expansion technique and a termwise separable property of kernel function, an analytical solution for approximating the exact solution is presented. The influence of regularized parameter on the perturbed solution is clear. The regularized solution was shown to be uniformly convergent to the exact solution, and the error estimation was provided. The numerical examples have shown that the new method could retrieve very well the missing boundary data, and very excellent numerical results were obtained.

Acknowledgement: Taiwan's National Science Council projects NSC-96-2221-E-019-027-MY3 and NSC-97-2221-E-019-009-MY3 granted to the author are highly appreciated.

References

Apostol, T. M. (1974): Mathematical Analysis. Second Edition, Addison-Wesley, Reading, Mass.

Azaiez, M.; Ben Abda, A.; Ben Abdallah,J. (2005): Revisiting the Dirichlet-to-Neumann solver for data completion and application to some inverse problems. *Int. J. Appl. Math. Mech.*, vol. 1, pp. 106-121.

Ben Belgacem, F.; El Fekih, H. (2005): On Cauchy's problem. I. a variational Steklov-Poincaré theory. *Inv. Prob.*, vol. 21, pp. 1915-1936.

Berntsson, F.; Eldén, L. (2001): Numerical solution of a Cauchy problem for the Laplace equation. *Inverse Problems*, vol. 17, pp. 839-854.

Brühl, M.; Hanke, M.; Pidcock, M. (2001): Crack detection using electrostatic measurements. *Math. Model. Numer. Anal.*, vol. 35, pp. 595-605.

Chaabane, S.; Jaoua, M. (1999): Identification of Robin coefficients by the means of boundary measurements. *Inverse Problems*, vol. 15, pp. 1425-1438.

Chaabane, S.; Elhechmi, C.; Jaoua, M. (2004): A stable method for the Robin inverse problem. *Math. Comp. Simul.*, vol. 66, pp. 367-383.

Chang, J.-R.; Yeih, W,; Shieh, M.-H. (2001): On the modified Tikhonov's regularization method for the Cauchy problem of the Laplace equation. *J. Mar. Sci. Tech.*, vol. 9, pp. 113-121.

Chapko, R.; Kress, R. (2005): A hybrid method for inverse boundary value problems in potential theory. *J. Inv. Ill-Posed Problems*, vol. 13, pp. 27-40.

Fasino, D.; Inglese, G. (1999): An inverse Robin problem for Laplace's equation: theoretical results and numerical methods. *Inverse Problems*, vol. 15, pp. 41-48.

Jin, B.; Zheng, Y. (2006): A meshless method for some inverse problems associated with the Helmholtz equation. *Comp. Meth. Appl. Mech. Engng.*, vol. 195, pp. 2270-2288.

Johnston, P. (2001): Computational Inverse Problems in Electrocardiography, Advances in Computational Bioengineering, Vol. 5, Johnston, P. Ed., WIT Press.

Kress, R. (2004): Inverse Dirichlet problem and conformal mapping. *Math. Comp. Simul.*, vol. 66, pp. 255-265.

Leblond, J.; Mahjoub, M.; Partington, J. R. (2006): Analytic extensions and Cauchy-type inverse problems on annular domains: stability results. *J. Inv. Ill-Posed Problems*, vol. 14, to appear.

Lin, F.; Fang, W. (2005): A linear integral equation approach to the Robin inverse problem. *Inverse Problems*, vol. 21, pp. 1757-1772.

Ling, L.; Takeuchi, T. (2008): Boundary control for inverse Cauchy problems of the Laplace equations. *CMES: Computer Modeling in Engineering & Sciences*, vol. 29, pp. 45-54.

Liu, C.-S. (2007a): Elastic torsion bar with arbitrary cross-section using the Fredholm integral equations. *CMC: Computers, Materials & Continua*, vol. 5, pp. 31-42.

Liu, C.-S. (2007b): A meshless regularized integral equation method for Laplace equation in arbitrary interior or exterior plane domains. *CMES: Computer Modeling in Engineering & Sciences*, vol. 19, pp. 99-109.

Liu, C.-S. (2007c): A MRIEM for solving the Laplace equation in the doubly-connected domain. *CMES: Computer Modeling in Engineering & Sciences*, vol. 19, pp. 145-161.

Liu, C.-S. (2008a): A highly accurate MCTM for inverse Cauchy problems of Laplace equation in arbitrary plane domains. *CMES: Computer Modeling in Engineering & Sciences*, vol. 35, pp. 91-111.

Liu, C.-S. (2008b): A modified collocation Trefftz method for the inverse Cauchy problem of Laplace equation. *Eng. Anal. Bound. Elem.*, vol. 32, pp. 778-785.

Liu, C.-S. (2008c): A highly accurate MCTM for direct and inverse problems of biharmonic equation in arbitrary plane domains. *CMES: Computer Modeling in Engineering & Sciences*, vol. 30, pp. 65-75.

Liu, C.-S.; Atluri, S. N. (2008a): A fictitious time integration method (FTIM) for solving mixed complementarity problems with applications to non-linear optimization. *CMES: Computer Modeling in Engineering & Sciences*, vol. 34, pp. 155-178.

Liu, C.-S.; Atluri, S. N. (2008b): A novel time integration method for solving a large system of non-linear algebraic equations. *CMES: Computer Modeling in*

Engineering & Sciences, vol. 31, pp. 71-83.

Liu, C.-S.; Chang, C. W.; Chiang, C. Y. (2008): A regularized integral equation method for the inverse geometry heat conduction problem. *Int. J. Heat Mass Transfer*, vol. 51, pp. 5380-5388.

Marin, L.; Lesnic, D. (2005): The method of fundamental solutions for inverse boundary value problems associated with the two-dimensional biharmonic equation. *Math. Comp. Model.*, vol. 42, pp. 261-278.

Mera, N. S.; Elliott, L.; Ingham, D. B.; Lesnic, D. (2000): The boundary element solution of the Cauchy steady heat conduction problem in an anisotropic medium. *Int. J. Num. Meth. Eng.*, vol. 49, pp. 481-499.

Lavrentiev, M. M. (1967): Some Improperly Posed Problems of Mathematical Physics. Springer, New York.

Saito, K.; Nakada, M.; Iijima, K.; Onish, K. (2005): A quasi-spectral method for Cauchy problem of 2/D Laplace equation on an annulus. *J. Phys. Conf. Ser.*, vol. 12, pp. 128-139.

Slodička, M.; Van Keer, R. (2004): A numerical approach for the determination of a missing boundary data in elliptic problems. *Appl. Math. Comp.*, vol. 147, pp. 569-580.

Tikhonov, A. N.; Goncharsky, A. V.; Stepanov, V. V.; Yagola, A. G. (1990): Numerical Methods for the Solution of Ill-Posed Problems. Kluwer Academic, London.