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Algebraic Formulation of Elastostatics: the Cell Method

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Abstract: The theory of elasticity is usually formulated using differential calculus. We will show that it is possible to give an algebraic or discrete or finite formulation, by starting directly from experimental laws, i.e. by avoiding any discretization process of the differential equations. This direct formulation can be immediately used for numerical solution in elasticity problems and, from a theoretical point of view, it shows some interesting features which are hidden in the differential formulation or are not considered at all.

Keywords: Cell Method, Discrete, Finite, Algebraic, Elastostatics.

1 Introduction

1.1 Differential formulation

Since the birth of the differential calculus three centuries ago physical laws have been expressed by means of differential equations. This is the case of the equations of Laplace, Poisson, D'Alembert, Navier, Navier-Stokes, Fourier and Maxwell right up to the equations of the twentieth century dealing with quantum mechanics, such as those of Schrödinger, Dirac, Klein-Gordon and the Einstein equation dealing with the relativistic gravitational theory.

The reason why we express physical laws in mathematical terms is twofold: one is the need to have general *information* about the properties of solutions and the other is to get the *solution* of specific problems.

Regarding the solution of specific problems, the differential formulation does not generally allow us to draw closed form solution, eccept in very special cases, with simple geometries, homogeneous and isotropic materials, simple boundary conditions. This requires us to make use of approximate methods, when possible, or numerical methods as a last resort. Since the computer is indispensable for the latter, this has opened up the era of computational physics.

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Computational physics, however, requires an *algebraic* formulation which is usually obtained by discretizing the differential equations: all the existing numerical methods used to solve physical and engineering problems, such as FEM, BEM, FDM, FVM, etc., do so, with the exception of cellular automata [Leamy (2008)]. By doing this, we transform a differential equation in a system of algebraic equations using purely mathematical notions, such as the orthogonality of the residuals to the shape functions, the least square method, the collocation method, etc.. In this way we have abandoned the physical content of the phenomenon described. It seems reasonable to ask to ourselves: is the differential formulation the only possible starting point to give a mathematical description of physical fields?

1.2 Algebraic formulation

In this paper we want to show that *we can obtain an algebraic formulation of elastostatics by starting directly from experimental facts, which avoids going through the differential formulation.* By doing this we remain closely linked to the physical and geometrical content of the problem. The direct algebraic formulation thus obtained can be immediately used for the numerical solution. The notions involved in this formulation are very simple, i.e. we avoid purely mathematical manipulations of the equations. Hence we avoid the weak formulation, the forming of the residual, the orthogonality of the residual to the shape functions, etc.

A direct algebraic formulation, using ÒglobalÓ variables, involves some advantages, which include:

- **a global variable is continuous** across the separation surface of two materials, e.g. displacements and surface forces;
- **we do not need jump conditions**. The jump conditions regard the field functions and they are derived from the continuity of global variables, e.g. the strains;
- **singularities do not arise.** In fact, singularities come from the ratio between a finite quantity and an infinitesimal extension, typically an area or a volume. Since an algebraic formulation does not perform the limit, it is free of singularities. Hence in the apex of a fracture and at the point of application of a concentrated force the stress remains finite.
- global variables are, in general, **quantities measured in laboratory**, while the corresponding densities are deduced from the global quantities.

The novelty of a direct algebraic formulation is that it highlights properties which the differential formulation keeps hidden or does not take in consideration at all.

It is surprising to note that for elastostatics, both in 2D and in 3D, *the direct algebraic formulation leads to the **same** stiffness matrix as FEM*, but it is obtained in a simpler way.

We will see that, by performing a direct algebraic formulation, nothing is lost and much is gained.

1.3 Existing literature

In previous papers it was shown how to apply this philosophy to scalar field theories, such as thermal conduction and diffusion [Tonti (2001a)], acoustics [Tonti (2001b)] and electromagnetism [Tonti (2001c)]. In [Tonti (2001a)] it was shown that with a quadratic interpolation on simplicial meshes and dealing with *scalar* problems, one obtains a *fourth* order of convergence.

A number of papers about the Cell Method have been published for computational electromagnetism, see <http://discretephysics.dica.units.it>. In particular the Cell Method has been used for scalar problems in combination with the *meshless* method: in 2D and in 3D with a second order convergence [Zovatto and Nicolini (2003), (2006)], in 2D even with a *fourth* order convergence [Zovatto and Nicolini (2007)].

The Cell Method for elastostatics, presented in details in this paper, has already been applied to elastostatic problems in 2D by [Cosmi (2001), (2002), (2005)]; to bone tissue [Cosmi and Dreossi (2007), (2007b), (2008b)]; to bone tissue with the meshless method by [Taddei, Pani, Zovatto, Tonti, Viceconti (2008)]; to fatigue [Cosmi and Hoglievina (2008c)] and to elastodynamic problems [Cosmi (2005b), (2008)]. Moreover with a quadratic interpolation one obtains a convergence of order 3.5, while FEM obtains the third order [Cosmi (2001)]. The method has been applied to elasto-plasticity and crack problems [Nappi, Rajgelj, Zaccaria (1999), (2000)] and to masonry [Nappi, and Tin-Loi (2001)], [Ferretti, Casadio, Ricci, Di Leo (2006), (2006b), (2008)]. The method has also been applied to fracture mechanics [Ferretti, Viola, Di Leo (2002), (2002b)], [Ferretti (2003), (2004), (2004b), (2004c), (2005), (2005b)].

1.4 The main notions used

To obtain a direct algebraic formulation, we must pay the price of acquainting ourselves with some new notions. These notions have been presented in [Tonti (2001a)] and we recommend reading at least pages pp.237-248 of that paper. The new notions are listed in the following

1. the division of physical variables into three classes: **configuration**, **source** and **energy** variables;

2. the notion of **global variables**;

3. the notion of **space elements** endowed with *inner* or *outer* orientation and their classification;

4. the notion of **primal and dual cell complexes** endowed with *inner* and *outer* orientation respectively;

5. the **association** of global variables with oriented space elements and the corresponding classification.

Let us summarize these notions.

1.4.1 Classification of physical variables

As is well known, physical quantities can be divided into two classes, physical *parameters* and physical *variables*. But it is not well known that physical variables, in turn, can be classified into three classes, according to the *role* they play in a theory. In fact it is possible to divide them in *configuration* variables, *source* variables and *energy* variables. This classification was introduced in electromagnetism by Penfield and Haus [(1967), p.155] but, unfortunately, after forty years still it has not been used in literature.

1.4.2 Global variables

The expression *global variable* is seldom used in physics, while the expression *integral variable* is more common. Integral variables are those variables obtained by performing a line or surface or volume integration of field functions. For example, mass is the volume integral of mass density; vortex flux is the surface integral of the vorticity vector; work along a line is the line integral of a force.

In [Tonti (2001a)] the term *global* variable was considered synonimous with *integral* variable. A subsequent experience has suggested the following definition.

DEF. *With the term* ***global variable in space*** *we mean a physical variable which is not a line or area or volume density of another variable.*

From this definition it follows that all integral variables are global variables but there are other variables which are global but not integral. For example, displacement and temperature do not arise from the integration of field variables, hence they are not integral variables. On the other hand, these variables are not densities of other variables, hence they are global variables in space.

1.4.3 Global variables and space elements

A fundamental property of global physical variables is that they are associated with the space elements, i.e. points (**P**), lines (**L**), surfaces (**S**) and volumes (**V**). This association takes into account the orientation of the space element in the sense that the global variable changes sign when the orientation of the corresponding space element is inverted.

By inspection, the following important property can be seen: *configuration variables are associated with space elements endowed with inner orientation, while source variables are associated with space elements endowed with outer orientation.*

To make clear this association, we must clarify what we mean by orientation of a space element.

1.4.4 Oriented space elements

A space element can be endowed with two different kinds of orientation: the **inner** and the **outer** orientation.

With reference to Fig.(1), the *inner* orientation of a line is a direction of motion along the line, while the *outer* orientation is a direction of rotation around the line. For example, consider the right-handed or left-handed rotation of a polarized light beam.

The *inner* orientation of a surface is the inner orientation of its bounding line, while the *outer* orientation is a sense of crossing the surface, from one side to another.

The *inner* orientation of a polyhedron, for example of a parallelepiped, is assigned when we define an inner orientation to one of its faces and we propagate it to the remaining faces in a compatible manner. The term ÒcompatibleÓ means that the orientation on two adjacent faces induces opposite orientations on the common edge. The outer orientation of a parallelepiped means that the normals to the boundaries are directed outwards or inwards.

The notion of orientation of a point is less evident. We will say that a point has an inner orientation when the lines which end at the point are directed towards the point, like a sink, or outwards from the point, like a source. A point has an outer orientation when it is associated with a left or right handed screw.

Space elements endowed with inner orientation will be denoted by placing a bar over their letters, i.e. $\overline{\mathbf{P}}, \overline{\mathbf{L}}, \overline{\mathbf{S}}, \overline{\mathbf{V}}$, while space elements endowed with outer orientation will be denoted by placing a tilde over their letters, i.e. $\widetilde{\mathbf{P}}, \widetilde{\mathbf{L}}, \widetilde{\mathbf{S}}, \widetilde{\mathbf{V}}$. To simplify, we will refer to a straight line element, to a plane surface element and to a parallelepiped. It is convenient to consider the geometrical object formed by a line

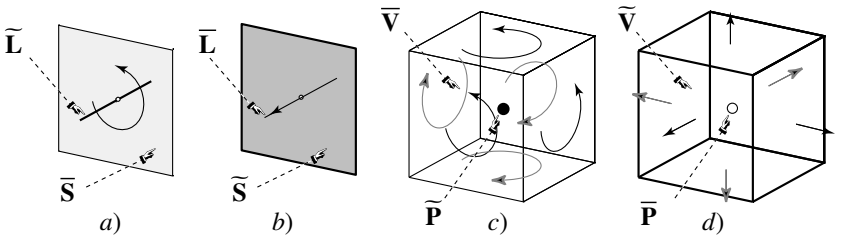


Figure 1: The oriented space elements.

segment and a rectangle which crosses it at the midpoint and which is orthogonal to it. Another geometrical object is the one formed by a parallelepiped and its centre of gravity. The plane surface is a rectangle which intersects the straight line element to its midpoint and which is orthogonal to it.

Fig.(1) shows how the two types of orientation are linked to each other: Fig.(1*a*) shows that the outer orientation of the line segment corresponds to an inner orientation of the rectangle. Fig.(1*b*) shows that the outer orientation of a surface corresponds to the inner orientation of a segment which intersects it. Fig.(1*c*) shows that the outer orientation of a point corresponds to the inner orientation of a volume which contains it and Fig.(1*d*) shows that the inner orientation of a point corresponds to the outer orientation of a volume which contains it.

The correspondence between an element of dimension p and an element of dimension $n-p$, where n is the dimension of the embedding space (in this case $n=3$), is called **duality** and the corresponding elements are called **dual**. Fig.(1) also shows that we can move from an inner orientation of a space element to the outer orientation of its dual using the screw rule.

1.4.5 Primal and dual cell complexes

The shape of the geometrical objects of Fig.(1) and their orientation suggest assembling them in a complex of cells as shown in Fig.(2). We can see how, composing the geometrical objects, we generate two sets of cells, which are staggered. The complex, whose elements have an inner orientation, is called **primal cell complex** while the cell complex, whose elements are endowed with outer orientation, is called **dual cell complex**.

If all elements of a cell complex, i.e. vertices, edges, faces and cells, are endowed with inner orientation, the corresponding elements of the dual complex, i.e. cells, faces, edges and vertices, automatically acquire an outer orientation, as shown in Fig.(3).

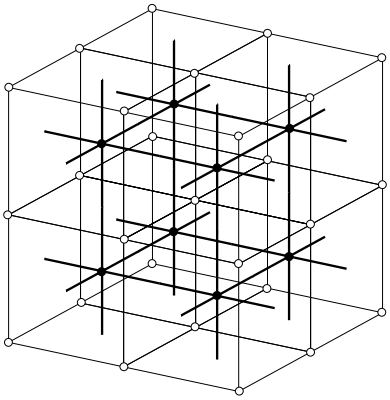


Figure 2: The primal cell complex (thin lines) and its dual (thick lines).

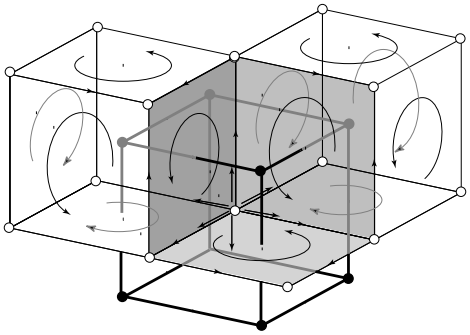


Figure 3: The inner orientations of the elements of the primal complex induce the outer orientations on the elements of the dual complex.

Since global physical quantities have a natural association with the oriented space elements, the two cell complexes constitute a geometrical framework for the algebraic formulation of every physical theory. This framework plays an analogous role to the role of the coordinate system for the differential formulation. The systematic use of a pair of cell complexes, a primal and a dual one, endowed with inner and outer orientation respectively, suggests the name of *Cell Method*.

An essential point, which characterizes the direct algebraic formulation, is the importance of the concept of dual space elements. Indeed, in the differential formulation we refer to an infinitesimal parallelepiped of sides $\mathrm{d}x, \mathrm{d}y, \mathrm{d}z$ both to deal with equilibrium and with the analysis of deformations. In contrast to this practice, with reference to Fig.(4), let us consider a monodimensional element, subjected to its weight, like a stalactite. We see that, to perform the analysis of deformation, next

to a first subdivision (primal) it is opportune to consider a second subdivision (dual) on which to impose the equilibrium. Indeed, given the displacement of the faces of the primal cell, we can evaluate the strain and hence the stress in every primal cell: with these stresses we can impose the equilibrium on the dual cells. *Hence*

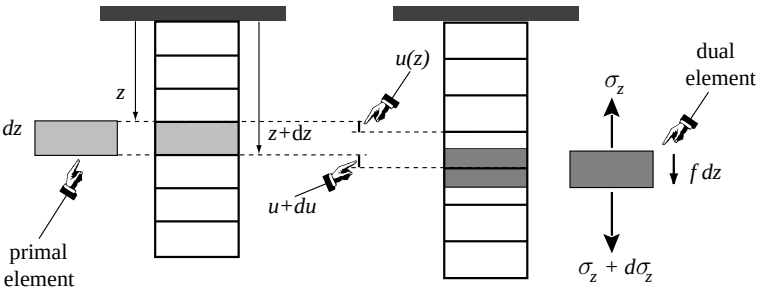


Figure 4: Deformation of a monodimensional element.

it is inappropriate to use the same subdivision for the analysis of deformations and stresses. In fact, the equilibrium "at a point", which is a typical expression of the differential formulation, must be understood as the equilibrium of a portion of material which contains the point.

2 Elastostatics in 2D

2.1 Fundamental problem

The fundamental problem of elastostatics can be stated as follows: *given an elastic solid in an assigned reference configuration, given the volume forces, the external surfaces forces, the material and the constraints, find the deformed configuration and the stress distribution within the solid.*

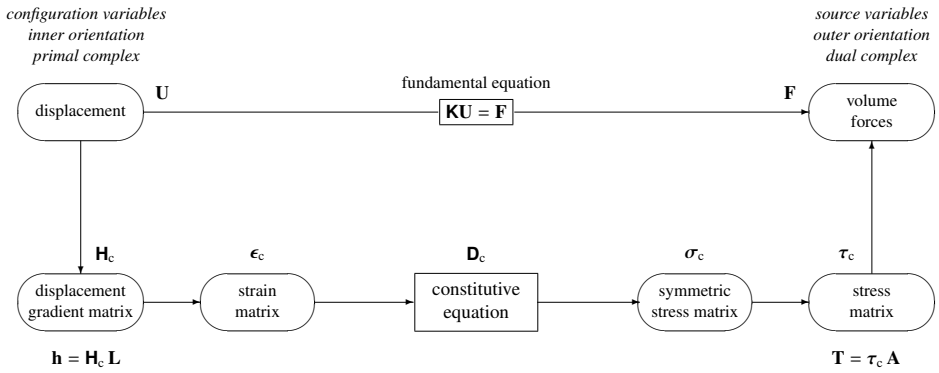
The main unknowns of the problem are the displacements in each point of the domain from their reference configuration, i.e. the displacement vector $\vec{u}(\mathbf{P})$.

REMARKS ON NOTATION. The term ÒvectorÓ is used here with two different meaning, corresponding to two successive generalizations. The first is the **geometric** vector, commonly used in physics and conceived as a direct segment with the well known operations defined on it. It will be denoted by putting an arrow over the letter, i.e. $\vec{v}$. The second generalization is the **algebraic** vector considered as an ordered set of n numbers, subject to the well known operations. It will be denoted in bold. i.e. $\mathbf{v}$. When a geometrical vector is involved in matrix calculations we can use the bold character as well. Hence, the two notations $\vec{v} = \vec{\omega} \times \vec{r}$ and $\mathbf{v} = \Omega \mathbf{r}$ are equivalent.

The main global variables of deformable solids are presented in Table 1.

Let us consider a cell complex in the region occupied by the solid in its reference configuration, as usual. We take as unknowns the displacement vectors at every node of the primal complex. Since every node of the primal complex lies inside a dual cell, we consider this dual cell as a *tributary region* of the node and we impose the equilibrium condition on it. In this way we write as many vectorial equations as the numbers of nodes in the region, i.e. the number of the displacement vectors. This is what is implicitly done in the differential formulation when we impose the equilibrium to every infinitesimal volume enclosing a point. In fact the partial derivatives involve the neighbouring of a point. This is similar to FVM using the vertex-centred method, while FEM imposes the equilibrium to nodes.

Table 1: Classification diagram of elastostatics.



2.2 Analysis of deformation

An elastic body has a unique natural state to which the body returns when all external loads are removed. Nodal displacements, strains and stresses are measured from this natural state: their values are counted as zero in that state [Fung (1964, p.154)]. We will denote by $\vec{u}$ the *displacement* of a point $\mathbf{P}$ from its position $\mathbf{P}_0$ in the natural state of the body taken as its reference configuration.

By definition, a solid is said to be deformable when the distance between its points may change. The most common device to measure this variation is the *strain gauge*. Using this tool we can see that the increase in length for unit length, at least in elastic phase, is of the order of 0.000001, for common materials such as still, concrete, wood, etc. This is a very small number, but with a finite value, not an *infinitesimal* value! We will start the analysis of deformation taking into account this fact.

Strain is usually defined starting from an *infinitesimal* deformation of an *infinitesimal* rectangle. Since an algebraic formulation of mechanics of deformable solids does not require partial derivatives, geometrical aspects become overriding on the analytical ones.

With reference to Fig.(5), let **P** and **Q** be two points of the solid body in a reference configuration. Let $\vec{L}$ be the vector connecting these two points, defined as the *relative position vector*. Let $\vec{u}(\mathbf{P}), \vec{u}(\mathbf{Q})$ be the displacements of the points **P** and **Q** from the reference configuration. Let $\mathbf{P}'$ and $\mathbf{Q}'$ be the new position of the points **P** and **Q** respectively. Let us consider the difference between the displacements of the two points **P** and **Q** and denote it by $\vec{h}$. We define

$$\begin{array}{ll} \vec{L} \stackrel{\text{def}}{=} \vec{r}(\mathbf{Q}) - \vec{r}(\mathbf{P}) & \textit{relative position} \\ \vec{h} \stackrel{\text{def}}{=} \vec{u}(\mathbf{Q}) - \vec{u}(\mathbf{P}) & \textit{relative displacement} \end{array} \tag{1}$$

Since the relative displacement vector $\vec{h}$ refers to the couple $(\mathbf{P}, \mathbf{Q})$, it refers to the relative position vector $\vec{L}$, hence we will write $\vec{h}[\overline{\mathbf{L}}]$. The relative displacement vector $\vec{h}$ [Love (1944, p.37)] is the dual of the internal surface force $\vec{T}$, as Table 1 shows.

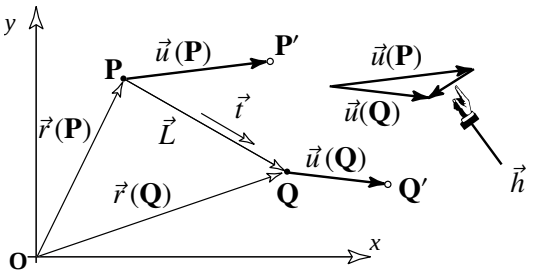


Figure 5: The relative position vector $\vec{L}$ and the relative displacement vector $\vec{h}$ of two points **P** and **Q** .

2.3 Displacement gradient

A region is said to be of *regularity* if the displacement is continuous, and has a *continuous variation* inside the region. Since on a separation surface of two materials the displacement is continuous, but it has discontinuous variations, a region, which contains a separation surface, cannot be a region of regularity. The reason for decomposing a region, occupied by a *homogeneous* material, into regions of regularity is that the displacement field can be locally considered as having a linear

behaviour, i.e. the components u_x, u_y can be considered as *affine* functions of the cartesian coordinates

$$\begin{cases} u(x,y) &= a + H_{xx}\,x + H_{xy}\,y \\ v(x,y) &= b + H_{yx}\,x + H_{yy}\,y\ . \end{cases} \tag{2}$$

where $a, b, H_{xx}, H_{xy}, H_{yx}, H_{yy}$ are unknown coefficients. Fig.(6) shows this behaviour.

REMARK. Remind that an affine function has a linear behaviour, but this does not necessarily mean that the function will assume the zero value at the origin of the coordinate system.

A region is characterized by a *uniform* deformation when the linear strain of a line segment is the same for every parallel segments in the region. *In a region where the deformation is uniform, the displacement is affine*.

Let us consider a triangular cell with vertices h, i, j.

REMARK. The triangle is the simplest polygon and the tetrahedron is the simplest polihedron. For this reason the triangle and the tetrahedron are the **simplex** of the plane and the space respectively. A cell complex made of simplexes is called **simplicial complex**.

By writing Eq.(2) for every node, we can find the six unknown coefficient by solving the resulting system. The fact that in an affine displacement the number of the unknown coefficients is equal to the number of the components of the displacement at the vertices is a peculiar property of simplexes in spaces of whatever dimension. Now we are interested in finding the four constants $H_{xx}, H_{xy}, H_{yx}, H_{yy}$. To this end

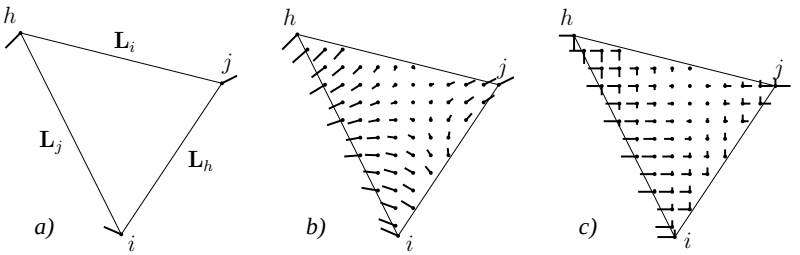


Figure 6: *a)* Given the displacements of the three vertices of a simplex, *b)* an affine displacement inside a simplex is defined. *c)* The horizontal and vertical components clearly show their linear variation.

we apply Eq.(2) to the two vertices, for example i and j, of the edge hi of the triangle. We obtain

$$\begin{cases} u_i - u_h &= H_{xx}(x_i - x_h) + H_{xy}(y_i - y_h) \\ u_i - u_h &= H_{yx}(x_i - x_h) + H_{yy}(y_i - y_h)\ . \end{cases} \tag{3}$$

Since we use a simplex, it is possible (and convenient) to denote the edge *hi* by L_j, i.e. with the label of the opposite vertex j. Hence, the relative position vector, connecting the vertex h with the vertex i, will be denoted by $\vec{L}_j$ and the relative displacement vector will be denoted by $\vec{h}_j \stackrel{\text{def}}{=} \vec{u}_i - \vec{u}_h$. We see that the differences, which appear in Eq.(3), in matrix notation can be written as follows

$$\begin{bmatrix} h_{jx} \\ h_{jy} \end{bmatrix} = \begin{bmatrix} H_{xx} & H_{xy} \\ H_{yx} & H_{yy} \end{bmatrix}_{\mathrm{c}} \begin{bmatrix} L_{jx} \\ L_{jy} \end{bmatrix} \tag{4}$$

or

$$\mathbf{h}_j = \mathbf{H}_{\mathrm{c}}\, \mathbf{L}_j \quad \text{and} \quad \mathbf{h}_i = \mathbf{H}_{\mathrm{c}}\, \mathbf{L}_i \quad \mathbf{h}_h = \mathbf{H}_{\mathrm{c}}\, \mathbf{L}_h \ . \tag{5}$$

The four coefficients $H_{xx}, H_{xy}, H_{yx}, H_{yy}$ are the elements of a matrix called the *displacement gradient matrix*, we denote $\mathbf{H}_{\mathrm{c}}$. This matrix corresponds to the tensor $\nabla\vec{u}$ in the differential formulation.

Eq.(4) shows that the role of the displacement gradient matrix is to link the relative displacement vectors $\vec{h}_j, \vec{h}_i, \vec{h}_h$ with the relative position vector $\vec{L}_j, \vec{L}_i, \vec{L}_h$ of the corresponding side.

To find the four coefficients in terms of the three nodal displacements of the vertices, we will make reference to Fig.(7). We consider the three vertices in the same order of the inner orientation of the simplex. To find the first two coefficients,

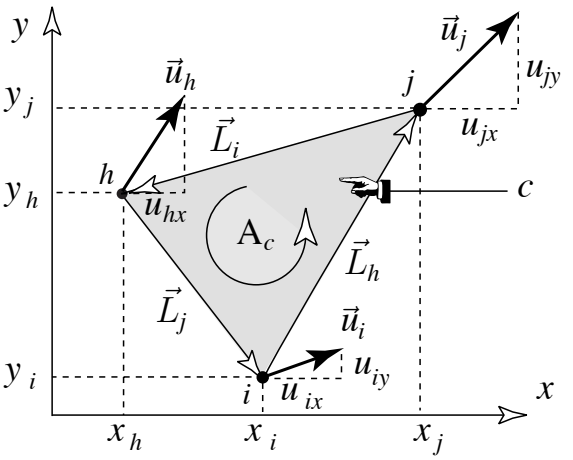


Figure 7: The three nodal displacements relative to the primal cell c.

H_{xx}, H_{xy}, we apply the first equation of the system, Eq.(4), relative to the x compo-

nents, to the two edges jh and hi, obtaining

$$\begin{cases} h_{ix} &= H_{xx} L_{ix} + H_{xy} L_{iy} \\ h_{jx} &= H_{xx} L_{jx} + H_{xy} L_{jy} \ . \end{cases} \tag{6}$$

We can write Eq.(6) as follows

$$\begin{bmatrix} L_{ix} & L_{iy} \\ L_{jx} & L_{jy} \end{bmatrix} \begin{bmatrix} H_{xx} \\ H_{xy} \end{bmatrix}_{\mathrm{c}} = \begin{bmatrix} h_{ix} \\ h_{jx} \end{bmatrix}. \tag{7}$$

Since the matrix $\mathbf{L}$ is known, we can invert it obtaining

$$\begin{bmatrix} H_{xx} \\ H_{xy} \end{bmatrix}_{\mathrm{c}} = \frac{1}{2A_{\mathrm{c}}} \begin{bmatrix} L_{jy} & -L_{iy} \\ -L_{jx} & L_{ix} \end{bmatrix}_{\mathrm{c}} \begin{bmatrix} h_{ix} \\ h_{jx} \end{bmatrix} \tag{8}$$

where the coefficient $1/2A_{\mathrm{c}}$ comes from the inversion of the matrix. In fact, it can be easily seen that the determinant of the matrix is double the area A_{c} of the simplex.

It can be noted that the three edge vectors form a closed polygon the sum of which is zero, i.e.

$$\vec{L}_h + \vec{L}_i + \vec{L}_j = 0 \longrightarrow \begin{cases} (L_{ix} + L_{jx}) = -L_{hx} \\ (L_{iy} + L_{jy}) = -L_{hy} \end{cases} . \tag{9}$$

If we write the x components of the relative displacements, respectively h_{ix} and h_{jx} in terms of the x components of the nodal displacements, Eq.(8) can be written as follows

$$\begin{bmatrix} H_{xx} \\ H_{xy} \end{bmatrix}_{\mathrm{c}} = \frac{1}{2A_{\mathrm{c}}} \begin{bmatrix} -L_{hy} & -L_{iy} & -L_{jy} \\ +L_{hx} & +L_{ix} & +L_{jx} \end{bmatrix} \begin{bmatrix} u_h \\ u_i \\ u_j \end{bmatrix} . \tag{10}$$

It should be noted that in the mechanics of solids a plane cell complex makes sense when the solid is considered as a layer of uniform thickness. In fact, we must distinguish between the case of plane stress from that of plane strain. With plane stresses, strains do not lie in a plane due to the lateral contraction. From this point of view the sides of the simplexes are the traces in the plane of the faces of the prism, as shown in Fig.(8). This remark enables to write the last equation in a more elegant form, introducing the area vectors $\vec{A}_h, \vec{A}_i, \vec{A}_j$ of the faces of the prisms. The introduction of the area vectors will be very useful in the $3D$ case. The x and y components of the three area vectors can be expressed as follows

$$\begin{cases} A_{hx} = +t\, L_{hy} & A_{ix} = +t\, L_{iy} & A_{jx} = +t\, L_{jy} \\ A_{hy} = -t\, L_{hx} & A_{iy} = -t\, L_{ix} & A_{jy} = -t\, L_{jx} \end{cases} . \tag{11}$$

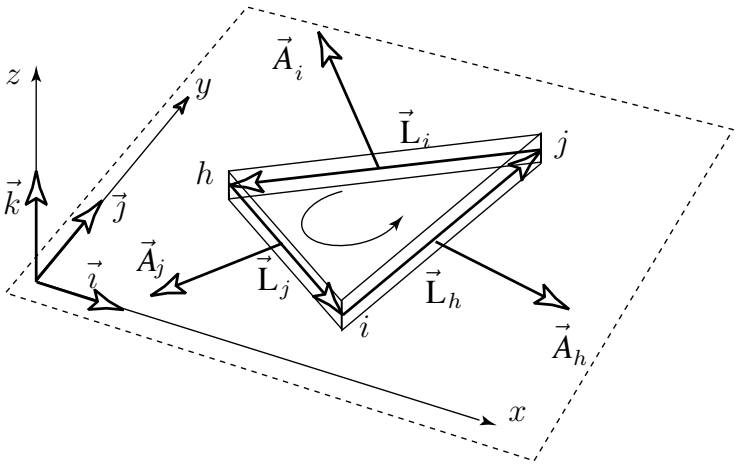


Figure 8: The triangular prism of thickness t.

Hence, after some manipulation, we can rewrite Eq.(10)

$$\left[\begin{array}{c} H_{xx} \\ H_{xy} \end{array}\right]_{\mathrm{c}} = -\frac{1}{2tA_{\mathrm{c}}} \left[\begin{array}{ccc} A_{hx} & A_{ix} & A_{jx} \\ A_{hy} & A_{iy} & A_{jy} \end{array}\right]_{\mathrm{c}} \left[\begin{array}{c} u_h \\ u_i \\ u_j \end{array}\right]. \tag{12}$$

In an analogous way we can find a similar expression for the coefficients H_{yx} and H_{yy}, obtaining

$$\left[\begin{array}{c} H_{yx} \\ H_{yy} \end{array}\right]_{\mathrm{c}} = -\frac{1}{2tA_{\mathrm{c}}} \left[\begin{array}{ccc} A_{hx} & A_{ix} & A_{jx} \\ A_{hy} & A_{iy} & A_{jy} \end{array}\right]_{\mathrm{c}} \left[\begin{array}{c} v_h \\ v_i \\ v_j \end{array}\right]. \tag{13}$$

Combining the latter formulae into one, we have

$$\left[\begin{array}{cc} H_{xx} & H_{xy} \\ H_{yx} & H_{yy} \end{array}\right]_{\mathrm{c}} = -\frac{1}{2tA_{\mathrm{c}}} \left[\begin{array}{ccc} A_{hx} & A_{ix} & A_{jx} \\ A_{hy} & A_{iy} & A_{jy} \end{array}\right]_{\mathrm{c}} \left[\begin{array}{cc} u_h & v_h \\ u_i & v_i \\ u_j & v_j \end{array}\right]. \tag{14}$$

This formula gives the displacement gradient matrix $\mathbf{H}_{\mathrm{c}}$ in term of the displacements of the three vertices of the cell c.

The analysis of deformation is based on the analysis of the variation in length of a line segment L connecting two points $\mathbf{P}$ and $\mathbf{Q}$, shown in Fig.(5). Considering the vector $\vec{L} = (\mathbf{Q} - \mathbf{P})$ before deformation and the vector $\vec{L}' = (\mathbf{Q}' - \mathbf{P}')$ after the

deformation, we can write

$$\begin{aligned} (L+\Delta L)^2 &= (\vec{L}+\vec{h})\cdot(\vec{L}+\vec{h}) \\ L^2+2L\Delta L+(\Delta L)^2 &= L^2+2\vec{L}\cdot\vec{h}+h^2\ . \end{aligned} \tag{15}$$

If we consider small displacements, the vector $\vec{h}$ is small and h^2 and $(\Delta L)^2$ will be very small respect to L^2. Hence, Eq.(15) becomes

$$L\Delta L \approx \vec{L}\cdot\vec{h}\ . \tag{16}$$

Dividing both members by L^2, introducing the unit vector $\vec{t}=\vec{L}/L$ and remembering the definition of linear strain, we obtain

$$\epsilon(\vec{t}) \stackrel{\text{def}}{=} \frac{\Delta L}{L} \approx \vec{t}\cdot\frac{\vec{h}}{L} \tag{17}$$

or in matrix notation

$$\epsilon \approx \mathbf{t}^{\mathsf{T}}\frac{\mathbf{H}_{\mathrm{c}}\mathbf{L}}{L} = \mathbf{t}^{\mathsf{T}}\mathbf{H}_{\mathrm{c}}\mathbf{t}\ . \tag{18}$$

Making explicit the compact notation we obtain

$$\epsilon(\mathbf{t}) \approx \mathbf{t}^{\mathsf{T}}\mathbf{H}_{\mathrm{c}}\mathbf{t} = \begin{bmatrix} t_x & t_y \end{bmatrix}\begin{bmatrix} H_{xx} & H_{xy} \\ H_{yx} & H_{yy} \end{bmatrix}_{\mathrm{c}}\begin{bmatrix} t_x \\ t_y \end{bmatrix}\ . \tag{19}$$

Hence, the linear strain in a direction is given by a quadratic form formed by the displacement gradient matrix and the unit vector in that direction.

Since, a priori, the matrix $\mathbf{H}_{\mathrm{c}}$ is not symmetric, it can be decomposed in the sum of a symmetric and a skew symmetrix part, according to the relations

$$\begin{aligned} \epsilon_{\mathrm{c}} &\stackrel{\text{def}}{=} \frac{1}{2}[\mathbf{H}_{\mathrm{c}}+\mathbf{H}_{\mathrm{c}}^{\mathsf{T}}] \qquad \textit{strain matrix} \\ \Omega_{\mathrm{c}} &\stackrel{\text{def}}{=} \frac{1}{2}[\mathbf{H}_{\mathrm{c}}-\mathbf{H}_{\mathrm{c}}^{\mathsf{T}}] \qquad \textit{rotation matrix} \end{aligned} \tag{20}$$

and since it can be easily proved that a quadratic form of a skew symmetric matrix vanishes, we see that the same result of Eq.(19) can be obtained using only the symmetric part, i.e.

$$\epsilon(\mathbf{t}) \approx \mathbf{t}^{\mathsf{T}}\epsilon_{\mathrm{c}}\mathbf{t} = \begin{bmatrix} t_x & t_y \end{bmatrix}\begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} \\ \epsilon_{xy} & \epsilon_{yy} \end{bmatrix}_{\mathrm{c}}\begin{bmatrix} t_x \\ t_y \end{bmatrix}\ . \tag{21}$$

Hence, to evaluate the linear strain along a direction, it is not necessary to use the whole displacement gradient $\mathbf{H}_{\mathrm{c}}$, but it is enough to use the strain matrix ϵ.

The components of the strain matrix, given by Eq.(21), are

$$\epsilon_{xx} = H_{xx} \qquad \epsilon_{xy} = \epsilon_{yx} = \frac{1}{2}(H_{xy} + H_{yx}) \qquad \epsilon_{yy} = H_{yy} \tag{22}$$

or, in general

$$\epsilon_{hh} \stackrel{\text{def}}{=} H_{hh} \qquad \gamma_{hk} \stackrel{\text{def}}{=} H_{hk} + H_{kh} = 2\epsilon_{hk}. \tag{23}$$

Combining Eq.(14) and Eq.(22) we obtain

$$\begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{bmatrix}_{\mathrm{c}} = -\frac{1}{2tA_{\mathrm{c}}} \begin{bmatrix} A_{hx} & 0 & A_{ix} & 0 & A_{jx} & 0 \\ 0 & A_{hy} & 0 & A_{iy} & 0 & A_{jy} \\ A_{hy} & A_{hx} & A_{iy} & A_{ix} & A_{jy} & A_{jx} \end{bmatrix}_{\mathrm{c}} \begin{bmatrix} u_h \\ v_h \\ u_i \\ v_i \\ u_j \\ v_j \end{bmatrix}_{\mathrm{c}}. \tag{24}$$

Assuming

$$\begin{aligned} \mathbf{u}_{\mathrm{c}} &\stackrel{\text{def}}{=} [u_h \quad v_h \quad u_i \quad v_i \quad u_j \quad v_j]^{\mathsf{T}} \\ \epsilon_{\mathrm{c}} &\stackrel{\text{def}}{=} [\epsilon_{xx} \quad \epsilon_{yy} \quad \gamma_{xy}]^{\mathsf{T}} \end{aligned} \tag{25}$$

we can write Eq.(24) in the short notation

$$\epsilon_{\mathrm{c}} = \mathbf{B}_{\mathrm{c}} \mathbf{u}_{\mathrm{c}} \ . \tag{26}$$

This equation coincides with the FEM one [Zienkiewicz (1971, p.94)] ; [Huebner (1971, p.234)]. It should be noted that, using the area vectors $\vec{A}_i$, we obtain a more expressive formula than the one typically used in FEM.

2.4 Analysis of stress

When we perform the stress analysis, we explore how the internal surface forces depend on the orientation of the plane surface to which they refer. This analysis is usually performed in the neighbourhood of a point by imposing the equilibrium of surface and volume forces on an *infinitesimal* tetrahedron. The tetrahedron is chosen with three faces parallel to the coordinate planes of a Cartesian coordinate system and a face with arbitary space orientation. The tetrahedron is assumed infinitesimal because this makes possible to ignore volume and inertia forces as *infinitesimal of higher order* with respect to the surface forces. This is known as the analysis of Cauchy.

REMARK. In the following we will denote by $\vec{T}$ the *surface force* and by $\vec{F}$ the *volume force*: both are **global** variables, but $\vec{T}$ is associated with a plane surface $\widetilde{\mathbf{S}}$ endowed with outer orientation and of area A, while $\vec{F}$ is associated with a volume $\widetilde{\mathbf{V}}$ endowed with outer orientation and volume V. Hence we can write $\vec{T}[\widetilde{\mathbf{S}}]$ and $\vec{F}[\widetilde{\mathbf{V}}]$ respectively. The *stress vector* $\vec{t}$ and the *body force* $\vec{f}$ are densitary variables and are defined by

$$\vec{t} \stackrel{\text{def}}{=} \frac{\vec{T}}{A} \qquad \vec{f} \stackrel{\text{def}}{=} \frac{\vec{F}}{V}. \tag{27}$$

The vectors $\vec{t}$ and $\vec{f}$ inherit the association with surface and volume, hence we can write $\vec{t}[\widetilde{\mathbf{S}}]$ and $\vec{f}[\widetilde{\mathbf{V}}]$ respectively.

In the following we will use only the global variables $\vec{T}$ and $\vec{F}$.

To prove that volume forces can be neglected with respect to surface forces, let us imagine a small cubic portion of water inside a lake. Let us denote by d the length of the cube edge and with H the depth of the small cube. The ratio between the volume forces and the surface forces $(\rho d^3)/(\rho d^2 H)$ decreases linearly with d when the dimension of the cube decreases. From this evidence we are strongly oriented to consider an *infinitesimal* volume element in order to eliminate both volume and inertia forces. Since an algebraic formulation ignores the notion of limit, we will not consider an infinitesimal volume element: hence an algebraic formulation is necessarily approximate. We must be satisfied to know that surface forces prevail over volume forces when the cell size decreases.

A uniform stress arises in a region when the internal surface force $\vec{T}$, acting on a plane surface, is invariant under translation of the plane surface. This implies that volume forces are absent. This approximation is done also in FEM when we assumes linear shape functions. Uniformity implies that the surface force $\vec{T}$ acting on a surface described by the area-vector $\vec{d}$ is a linear function of the area-vector $\vec{d}$. This can be written

$$\begin{bmatrix} T_x \\ T_y \end{bmatrix} = \begin{bmatrix} \tau_{xx} & \tau_{xy} \\ \tau_{yx} & \tau_{yy} \end{bmatrix} \begin{bmatrix} a_x \\ a_y \end{bmatrix}. \tag{28}$$

This is the Cauchy equation which is usually deduced from the equilibrium equation of deformable solids. In brief

- in the differential formulation equilibrium is applied to an *infinitesimal* tetrahedron, so that volume forces, at the limit, disappear: *the relation is exact;*
- in the algebraic formulation the equilibrium is applied to a small, but *finite*, tetrahedron and volume forces are neglected: *the relation is approximate.*

The stress matrix τ can be decomposed into its symmetric and skew-symmetric part

as follows

$$\sigma \stackrel{\text{def}}{=} \frac{1}{2}(\tau+\tau^{\mathsf{T}}) \qquad \mu \stackrel{\text{def}}{=} \frac{1}{2}(\tau-\tau^{\mathsf{T}}) \,. \tag{29}$$

The equilibrium to rotation imposes that $\mu = 0$, hence only the symmetric stress tensor σ remains. It follows that the stress inside every primal cell c is described by only three variables, i.e. $\sigma_{xx}, \sigma_{yy}, \sigma_{xy}$. Eq.(28) can be rewritten in the alternative form, called Voigt notation [see Belytschko, Liu, Moran (2000, p.615)]

$$\begin{bmatrix} T_x \\ T_y \end{bmatrix}_c = \begin{bmatrix} a_x & 0 & a_y \\ 0 & a_y & a_x \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix}_c . \tag{30}$$

Eq.(30) displays the components of the stress matrix σ by an *algebraic* vector which is preferable for numerical computation because Hooke's law, usually expressed by a fourth order tensor, reduces to a 3×3 matrix in 2D.

Let us return to the surface forces acting on the faces of the dual cell, contained in a primal cell c, as shown in Fig.(9). If we use the Voronoi dual, the point C is the circumcentre, while, if we use the barycentric dual, the point C is the centre of mass. In both cases the faces of the dual cell meet the edges of the primal cell c at their midpoints. With reference to Fig.(9), let us denote by $(\vec{T}^c(h))'$ and $(\vec{T}^c(h))''$ the surface forces acting on the two faces AC and BC respectively. Since we must sum these two forces for every primal cell c, which has a common node in h, their sum is equal to the force $\vec{T}^c(h)$ acting on the face AB connecting the midpoints of the faces hi and hj. The reason is that, assuming uniformity of stresses, we have excluded volume forces on the triangle ABC, hence $(\vec{T}^c(h))' + (\vec{T}^c(h))'' = \vec{T}^c(h)$.

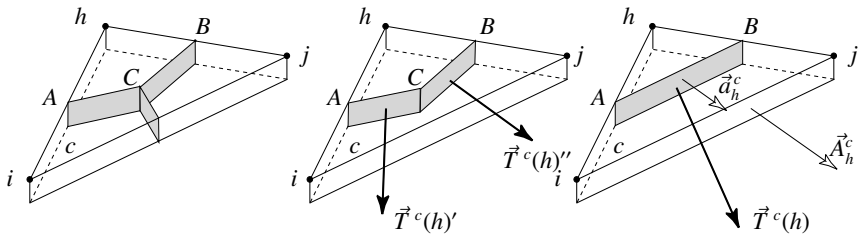


Figure 9: The surface force on the cell c relative to the node h.

The area-vector $\vec{a}_h$ of the face AB in Fig.(9) is half the area-vector of the face h of the primal cell c, opposite to the vertex h, i.e. $\vec{a}_h = \vec{A}_h^c/2$.

To evaluate the force $\vec{T}^c(h)$ we use Eq.(30)

$$\left[\begin{array}{c} T_x(h) \\ T_y(h) \end{array}\right]_c = \frac{1}{2}\left[\begin{array}{ccc} A_{hx} & 0 & A_{hy} \\ 0 & A_{hy} & A_{hx} \end{array}\right]_c \left[\begin{array}{c} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{array}\right]_c . \tag{31}$$

2.5 Constitutive equations

The constitutive relations of solid mechanics are tested on specimens, made of **homogeneous** material, subjected to a system of forces which cause a **uniform** stress and strain distribution over the thickness of solid, **ignoring** volume forces.

As it is known, a state of stress is said *plane* when the surface forces act in a plane: in this case $\sigma_z = 0$. This implies that an element of a solid can expand or contract in the third dimension and that, consequently, the state of deformation is not plane.

Let us denote by E the elastic modulus, G the shear modulus, ν the Poisson ratio, σ the axial stress, ϵ the axial strain and γ the shear strain. Referring to (Fig.10) and using the principle of the superposition of effects we obtain

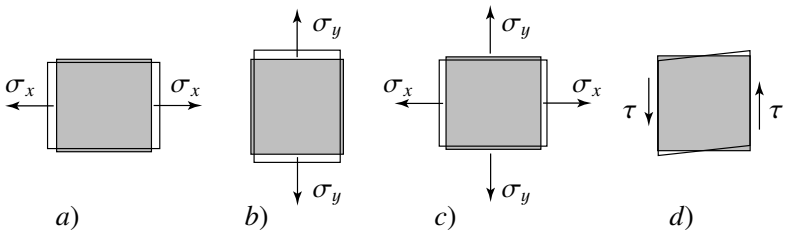


Figure 10: Experiments which permit to infer the Hooke law.

$$\left\{\begin{array}{lcll} \epsilon_x & = & \dfrac{1}{E}\sigma_x - \dfrac{\nu}{E}\sigma_y & \\ \epsilon_y & = & \dfrac{1}{E}\sigma_y - \dfrac{\nu}{E}\sigma_x & G = \dfrac{E}{2(1+\nu)}. \\ \gamma & = & \dfrac{1}{G}\tau & \end{array}\right. \tag{32}$$

These equations can be written in matrix notation ([Zienkiewicz (1971, p.53)]; [Burnett (1987, p.746)]) as follows

$$\left\{\begin{array}{c} \epsilon_x \\ \epsilon_y \\ \gamma \end{array}\right\}_c = \frac{1}{E}\left[\begin{array}{ccc} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{array}\right]_c \left\{\begin{array}{c} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{array}\right\}_c . \tag{33}$$

Inverting this equation we obtain

$$\left\{ \begin{array}{c} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{array} \right\}_{\mathrm{c}} = \frac{E}{1-\nu^2} \left[\begin{array}{ccc} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{array} \right]_{\mathrm{c}} \left\{ \begin{array}{c} \epsilon_x \\ \epsilon_y \\ \gamma \end{array} \right\}_{\mathrm{c}}. \tag{34}$$

An analogous relation is valid for plane deformation, i.e. $\epsilon_z = 0$

$$\left\{ \begin{array}{c} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{array} \right\}_{\mathrm{c}} = \frac{E}{(1+\nu)(1-2\nu)} \left[\begin{array}{ccc} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \dfrac{1-2\nu}{2} \end{array} \right]_{\mathrm{c}} \left\{ \begin{array}{c} \epsilon_x \\ \epsilon_y \\ \gamma \end{array} \right\}_{\mathrm{c}}. \tag{35}$$

For the cases of plane stress, plane strain and the three-dimensional one, the stress-strain law can be written in the form

$$\sigma_{\mathrm{c}} = \mathbf{D}_{\mathrm{c}}\, \epsilon_{\mathrm{c}} \tag{36}$$

where $\mathbf{D}_c$ is a symmetric matrix, like the matrices of Eq.(34) and Eq.(35). Note that the constitutive relations *are applied in a **finite region** (not in an infinitesimal region!) and are valid under conditions of uniformity of strain and stress and material homogeneity.* Let us remark that

- the equilibrium equation is valid whatever the size of the body to which it is applied: it is valid both for a ship at rest on the sea and for an ion in a crystal, hence it is valid for any dimension. It can be expressed in algebraic terms using only the notion of sum of vectors;
- the link between the stress vector and the area vector is linear in a region of **uniform stress**.
- the link between the relative displacement vector $\vec{h}$ and the relative position vector $\vec{L}$ is linear in a region in which the displacement can be considered an affine function of the coordinates, hence in a region of **uniform strain**;
- the constitutive equations are valid in small regions of uniform stress and strain, hence in a region in which the material is **homogeneous**.

These four properties make possible a purely algebraic treatment of elasticity.

2.6 Equilibrium equation

With reference to Fig.(11d), let us consider a node h in which the displacement vector $\vec{u}(h)$ is unknown. Let us denote with the same label h its dual cell and by

$\mathcal{E}(h)$ the set of primal cells which have the node h as common vertex. Let $\vec{F}^c(h)$ be the volume force acting on that part of the dual cell h which is contained in the primal cell c and let $\vec{T}^c(h)$ be the surface force acting from outside on that part of the boundary of the dual cell h which is contained in the primal cell c. Moreover, if the node h lies on the boundary, its dual cell is incomplete, as shown in Fig.(12) and an external surface force $\vec{B}(h)$ must be added to the internal forces.

In the Cell Method *the equilibrium condition is imposed on every dual cell,* in contrast with FEM where the equilibrium is imposed on nodes. With reference to Figs. (11) and (12), assuming

$$\vec{T}(h) \stackrel{\text{def}}{=} \sum_{c \in \mathcal{E}(h)} \vec{T}^c(h) \qquad \vec{F}(h) \stackrel{\text{def}}{=} \sum_{c \in \mathcal{E}(h)} \vec{F}^c(h) \tag{37}$$

$$\vec{B}(h) \stackrel{\text{def}}{=} \vec{B}'(h) + \vec{B}''(h) \tag{38}$$

we can write the equilibrium for every dual cell h by the equation

$$\boxed{\vec{T}(h) + \vec{F}(h) + \vec{B}(h) = 0 \qquad \text{with } h = 1, 2, \ldots N} \tag{39}$$

where N is the number of dual cells, i.e. the number of vertices of the primal cell complex. or, in matrix notation

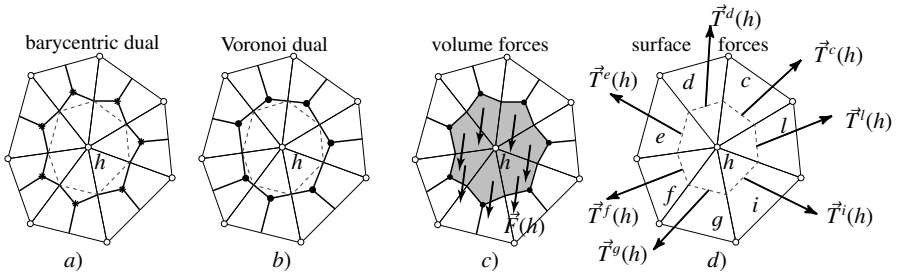


Figure 11: The auxiliary polygon is the same for barycentric and Voronoi dual cells.

$$\sum_{c \in \mathcal{E}(h)} \left[\begin{array}{c} T_x(h) \\ T_y(h) \end{array} \right]_{\text{c}} + \left[\begin{array}{c} F_x(h) + B_x(h) \\ F_y(h) + B_y(h) \end{array} \right] = 0 \,. \tag{40}$$

This equation expresses the equilibrium by means of simple geometrical and physical considerations. Note that, unlike what happens in the differential formulation, we should not write boundary conditions separately, because boundary forces,

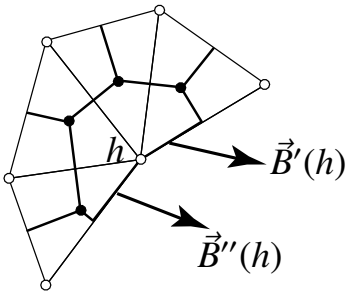


Figure 12: Equilibrium applied to the incomplete dual cell of a boundary node h.

which are generally assigned like volume forces, can be simply added to volume forces of the corresponding dual cell. Moreover a concentrated force need not be divided into three equivalent nodal forces according to the *lever rule* as in FEM. These are significant differences compared to the differential formulation.

2.7 Fundamental equation

We call *fundamental equation* the relation which expresses the fundamental problem. This equation links the nodal displacements with the volume forces of the dual cells.

The internal surface forces depend on the deformation, hence on displacements. In general volume forces are independent of them. When the internal surface forces $\vec{T}^c(h)$ are expressed in terms of the displacements of the node h and its neighbouring nodes, via stresses and strains, we obtain a set of algebraic equations which contain the nodal displacements. Taking into account Eq.(31), we can write

$$\sum_{c\in\mathcal{E}(h)} \frac{1}{2}\begin{bmatrix} A_{sx} & 0 & A_{sy} \\ 0 & A_{sy} & A_{sx} \end{bmatrix}_{\mathrm{c}} \mathbf{D}_{\mathrm{c}}\mathbf{B}_{\mathrm{c}}\mathbf{u}_{\mathrm{c}} + \begin{bmatrix} F_x(h)+B_x(h) \\ F_y(h)+B_y(h) \end{bmatrix} = 0. \tag{41}$$

Eq.(41) is the fundamental equation of elastostatics obtained with the Cell Method and it is the algebraic correspondent of the Navier differential equation.

Since the fundamental equation must be written only for those nodes where the displacements are unknown, the number of equations is equal to the number of unknowns.

2.8 An alternative

A straightforward way to derive the system (39) is to compose the system by assembling the coefficients. To this aim we will consider an arbitrary primal cell c and

the three regions of the dual cells relative to the three nodes h, i, j. Fig.(13) shows

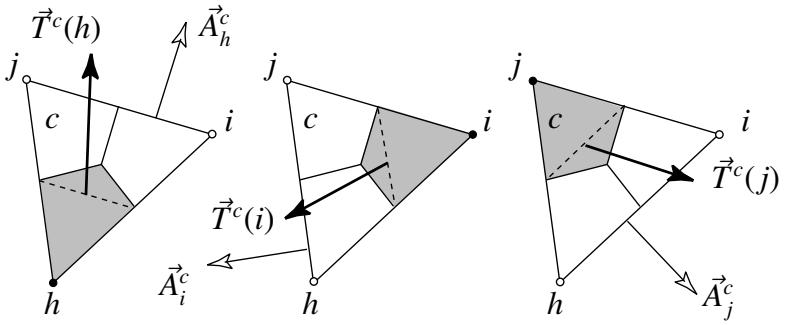


Figure 13: The three surface forces acting on those faces of the dual polygons which enter the cell c.

the three surface forces $\vec{T}^c(h)$, $\vec{T}^c(i)$, $\vec{T}^c(j)$, which are relative to the three regions of the three different dual cells. If we apply Cauchy equation in the form of Eq.(31) to the three faces depicted with dashed lines and combine the three formulae into one, we obtain

$$\begin{bmatrix} T_x(h) \\ T_y(h) \\ T_x(i) \\ T_y(i) \\ T_x(j) \\ T_y(j) \end{bmatrix}_{\mathrm{c}} = \frac{1}{2} \begin{bmatrix} A_{hx} & 0 & A_{hy} \\ 0 & A_{hy} & A_{hx} \\ A_{ix} & 0 & A_{iy} \\ 0 & A_{iy} & A_{ix} \\ A_{jx} & 0 & A_{jy} \\ 0 & A_{jy} & A_{jx} \end{bmatrix}_{\mathrm{c}} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix}_{\mathrm{c}} . \tag{42}$$

Introducing the algebraic vectors

$$\begin{aligned} \mathbf{T}_{\mathrm{c}} &\stackrel{\mathrm{def}}{=} [T_x(h) \quad T_y(h) \quad T_x(i) \quad T_y(i) \quad T_x(j) \quad T_y(j)]^{\mathsf{T}}_{\mathrm{c}} \\ \sigma_{\mathrm{c}} &\stackrel{\mathrm{def}}{=} [\sigma_{xx} \quad \sigma_{yy} \quad \sigma_{xy}]^{\mathsf{T}}_{\mathrm{c}} \end{aligned} \tag{43}$$

we can write

$$\mathbf{T}_{\mathrm{c}} = (-tA_{\mathrm{c}})\,\mathbf{B}^{\mathsf{T}}_{\mathrm{c}}\sigma_{\mathrm{c}} \; . \tag{44}$$

We see that the matrix of Eq.(42), apart from the factor $-tA_{\mathrm{c}}$, is the transpose of that of Eq.(26). We note that the product tA_{c} is the volume of the triangular prism of area A_{c} and thickness t.

Composing Eq.(44) with Eq.(34) and with Eq.(26) and assuming

$$\mathbf{k}_c \stackrel{\text{def}}{=} (tA_c)\,\mathbf{B}_c^{\mathsf{T}}\,\mathbf{D}_c\,\mathbf{B}_c \tag{45}$$

we can write

$$\mathbf{T}_c = -\mathbf{k}_c\mathbf{u}_c\ . \tag{46}$$

The composition of this formula is illustrated in the upper part of Table 3. Since the matrix $\mathbf{B}_c$ coincides with the matrix of FEM, it follows that *the local stiffness matrix* $\mathbf{k}_c$ *also coincides with the local stiffness matrix of FEM*.

Table 2: Links between variables of elastostatics in a $2D$ space.

configuration variables		*source variables*
displacement		volume force
$[u_{hx}\ u_{hy}\ u_{ix}\ u_{iy}\ u_{jx}\ vu_{jy}]_c^{\mathsf{T}}$		$[f_{hx}\ f_{hy}\ f_{ix}\ f_{iy}\ f_{jx}\ f_{jy}]_c^{\mathsf{T}}$
$\downarrow$		$\uparrow$
relative displacement		surface force
$[h_{hx}\ h_{hy}\ h_{ix}\ h_{iy}\ h_{jx}\ h_{jy}]_c^{\mathsf{T}}$		$[t_{hx}\ t_{hy}\ t_{ix}\ t_{iy}\ t_{jx}\ t_{jy}]_c^{\mathsf{T}}$
$\downarrow$		$\uparrow$
displacement gradient $\mathbf{H}_c$		general stress matrix τ_c
$\downarrow$		$\uparrow$
symm. strain matrix ϵ_c		symm. stress matrix σ_c
$\downarrow$		$\uparrow$
$[\epsilon_{xx}\ \epsilon_{yy}\ \gamma]_c^{\mathsf{T}}$	$\longrightarrow$ Hooke $\longrightarrow$	$[\sigma_{xx}\ \sigma_{yy}\ \sigma_{xy}]_c^{\mathsf{T}}$

2.9 Global stiffness matrix

Denoting with N the total number of nodes of the region, let us introduce the following four global algebraic vectors

$$\begin{aligned}
\mathbf{U} &\stackrel{\text{def}}{=} \left[\ u_x(1)\ \ u_y(1)\ \cdots\ u_x(N)\ \ u_y(N)\ \right]^{\mathsf{T}}\\
\mathbf{T} &\stackrel{\text{def}}{=} \left[\ T_x(1)\ \ T_y(1)\ \cdots\ T_x(N)\ \ T_y(N)\ \right]^{\mathsf{T}}\\
\mathbf{F} &\stackrel{\text{def}}{=} \left[\ F_x(1)\ \ F_y(1)\ \cdots\ F_x(N)\ \ F_y(N)\ \right]^{\mathsf{T}}\\
\mathbf{B} &\stackrel{\text{def}}{=} \left[\ B_x(1)\ \ B_y(1)\ \cdots\ B_x(N)\ \ B_y(N)\ \right]^{\mathsf{T}}\ .
\end{aligned} \tag{47}$$

Let us proceed considering one cell at a time. By doing this, we do not build one equation at a time, but we build the individual coefficients which must be properly

located to form the final system. This operation, common to finite element method, is called *assembly*.

The nodal vector $\mathbf{u}_c$ is composed of the Cartesian components of the displacements of the simplex, referred to a local numbering h, i, j. Such a vector can be obtained by extracting the corresponding terms from the global vector $\mathbf{U}$. Introducing the *location matrix* $\mathbf{L}_c$, [Hughes (2000), p.42], [Belytschko; Liu; Moran (2000), p.39] we can write

$$\mathbf{u}_c = \mathbf{L}_c \mathbf{U} \ . \tag{48}$$

The global force vector $\mathbf{T}$ is formed by combining the vectors $\mathbf{T}_c$. This is accomplished using the transposed matrix $\mathbf{L}_c^\top$ as shown in the formula

$$\mathbf{T} = \sum_c \mathbf{L}_c^\top \mathbf{T}_c \ . \tag{49}$$

Hence, composing Eq.(46) with Eq.(48) we obtain

$$\mathbf{T} = -\sum_c \mathbf{L}_c^\top \mathbf{k}_c \mathbf{L}_c \ \mathbf{U} \ . \tag{50}$$

Introducing the *global* stiffness matrix $\mathbf{K}$

$$\mathbf{K} \stackrel{\text{def}}{=} \sum_c \mathbf{L}_c^\top \mathbf{k}_c \mathbf{L}_c \tag{51}$$

we can write Eq.(50)

$$\mathbf{T} = -\mathbf{K}\mathbf{U} \tag{52}$$

as shown in Table 3. The system (39), which express the equilibrium equations, can than be written in the compact form

$$\boxed{-\mathbf{K}\mathbf{U} + \mathbf{F} + \mathbf{B} = \mathbf{0} \ .} \tag{53}$$

This is the algebraic equivalent of Navier differential equation of elastostatics.

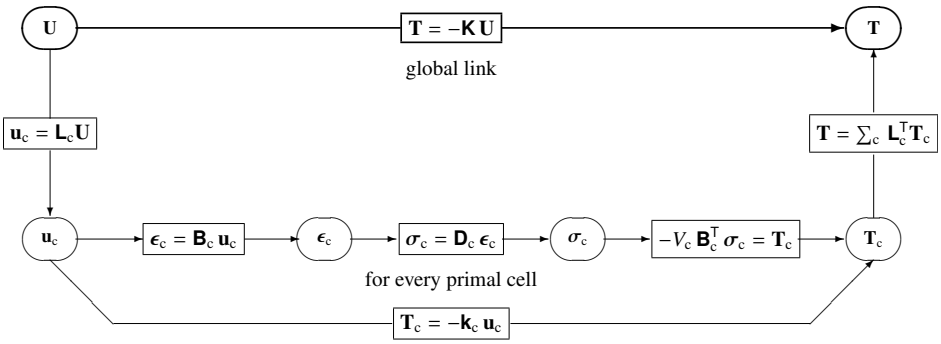
3 Elastostatics in 3D

3.1 Analysis of deformation

Let us consider a simplex c in the tridimensional space, i.e. a tetrahedron as shown in Fig.(14). If the deformation is uniform inside the tetrahedron, the displacement is an affine function, i.e.

$$\left\{ \begin{array}{lcl} u(x,y,z) & = & a + H_{xx}x + H_{xy}y + H_{xz}z \\ v(x,y,z) & = & b + H_{yx}x + H_{yy}y + H_{yz}z \\ w(x,y,z) & = & c + H_{zx}x + H_{zy}y + H_{zz}z \end{array} \right. \ . \tag{54}$$

Table 3: Derivation of local and global stiffness matrices.



We want to construct the displacement gradient matrix $\mathbf{H}_c$ inside the tetrahedron c, like we have done for a triangle in the 2D space. Its vertices are denoted with h,i,j,k and the corresponding nodal displacements with $\vec{u}_h$, $\vec{u}_i$, $\vec{u}_j$, $\vec{u}_k$. The $4\times 3 = 12$ constants of the Eq.(54) can be expressed in terms of the $4\times 3=12$ components of the displacements of the vertices. Let us write the first equation of the system Eq.(54), for all the vertices h,i,j,k of the cell c. Performing a linear combination of the four equations so obtained, we have

$$\begin{bmatrix} u_i-u_h \\ u_j-u_h \\ u_k-u_h \end{bmatrix} = \begin{bmatrix} x_i-x_h & y_i-y_h & z_i-z_h \\ x_j-x_h & y_j-y_h & z_j-z_h \\ x_k-x_h & y_k-y_h & z_k-z_h \end{bmatrix} \begin{bmatrix} H_{xx} \\ H_{xy} \\ H_{xz} \end{bmatrix}_c . \tag{55}$$

With reference to Fig.(14), we will start considering the vertex h and the three edges $\vec{L}_\alpha, \vec{L}_\beta, \vec{L}_\gamma$, which have the vertex h in common. Putting

$$\begin{array}{lll} h_{\alpha x} \stackrel{\text{def}}{=} u_i-u_h & h_{\beta x} \stackrel{\text{def}}{=} u_j-u_h & h_{\gamma x} \stackrel{\text{def}}{=} u_k-u_h \\ L_{\alpha x} \stackrel{\text{def}}{=} x_i-x_h & L_{\beta x} \stackrel{\text{def}}{=} x_j-x_h & L_{\gamma x} \stackrel{\text{def}}{=} x_k-x_h \\ etc. & & \end{array} \tag{56}$$

we obtain

$$\begin{bmatrix} L_{\alpha x} & L_{\alpha y} & L_{\alpha z} \\ L_{\beta x} & L_{\beta y} & L_{\beta z} \\ L_{\gamma x} & L_{\gamma y} & L_{\gamma z} \end{bmatrix}_c \begin{bmatrix} H_{xx} \\ H_{xy} \\ H_{xz} \end{bmatrix}_c = \begin{bmatrix} h_{\alpha x} \\ h_{\beta x} \\ h_{\gamma x} \end{bmatrix}_c . \tag{57}$$

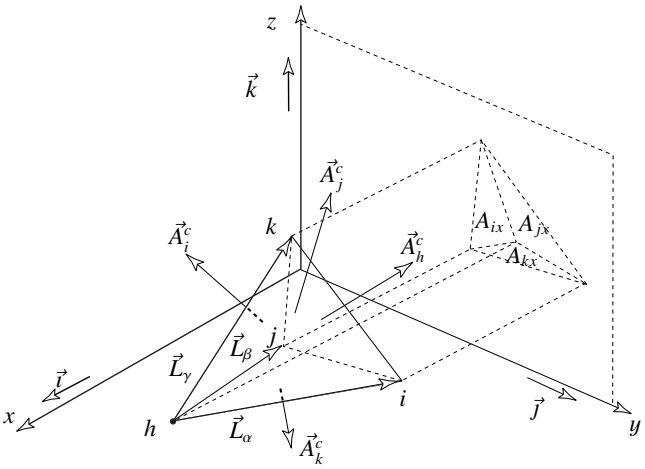


Figure 14: The projections of the tetrahedra faces on the yz plane.

With reference to Fig.(14), we see that the face A_i is described by the area-vector

$$\vec{A}_i = \frac{1}{2}\vec{L}_\beta \times \vec{L}_\gamma = \frac{1}{2}\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ L_{\beta x} & L_{\beta y} & L_{\beta z} \\ L_{\gamma x} & L_{\gamma y} & L_{\gamma z}\ . \end{vmatrix}. \tag{58}$$

In an analogous way, for the remaining faces, we have

$$\vec{A}_j = \frac{1}{2}\vec{L}_\alpha \times \vec{L}_\gamma \qquad \vec{A}_k = \frac{1}{2}\vec{L}_\beta \times \vec{L}_\alpha. \tag{59}$$

From these formulae we deduce the components of the area-vectors $\vec{A}_i, \vec{A}_j, \vec{A}_k$

$$\begin{aligned} A_{ix} &= \frac{1}{2}\begin{vmatrix} L_{\beta y} & L_{\beta z} \\ L_{\gamma y} & L_{\gamma z} \end{vmatrix} \quad A_{iy} = -\frac{1}{2}\begin{vmatrix} L_{\beta x} & L_{\beta z} \\ L_{\gamma x} & L_{\gamma z} \end{vmatrix} \\ A_{iz} &= \frac{1}{2}\begin{vmatrix} L_{\beta x} & L_{\beta y} \\ L_{\gamma x} & L_{\gamma y} \end{vmatrix} \end{aligned} \tag{60}$$

$$\begin{aligned} A_{jx} &= \frac{1}{2}\begin{vmatrix} L_{\alpha y} & L_{\alpha z} \\ L_{\gamma y} & L_{\gamma z} \end{vmatrix} \quad A_{jy} = -\frac{1}{2}\begin{vmatrix} L_{\alpha x} & L_{\alpha z} \\ L_{\gamma x} & L_{\gamma z} \end{vmatrix} \\ A_{jz} &= \frac{1}{2}\begin{vmatrix} L_{\alpha x} & L_{\alpha y} \\ L_{\gamma x} & L_{\gamma y} \end{vmatrix} \end{aligned} \tag{61}$$

$$\begin{aligned} A_{kx} &= \frac{1}{2}\begin{vmatrix} L_{\beta y} & L_{\beta z} \\ L_{\alpha y} & L_{\alpha z} \end{vmatrix} \quad A_{ky} = -\frac{1}{2}\begin{vmatrix} L_{\beta x} & L_{\beta z} \\ L_{\alpha x} & L_{\alpha z} \end{vmatrix} \\ A_{kz} &= \frac{1}{2}\begin{vmatrix} L_{\beta x} & L_{\beta y} \\ L_{\alpha x} & L_{\alpha y} \end{vmatrix}. \end{aligned} \tag{62}$$

Moreover, since the four triangles are faces of a polyhedron, the sum of the area-vectors of the faces vanishes. Hence

$$\vec{A}_h = -(\vec{A}_i + \vec{A}_j + \vec{A}_k)\ . \tag{63}$$

The x, y, z components of $\vec{A}_h$ are given by

$$\begin{cases} A_{hx} = -A_{ix} - A_{jx} - A_{kx} \\ A_{hy} = -A_{iy} - A_{jy} - A_{ky} \\ A_{hz} = -A_{iz} - A_{jz} - A_{kz}. \end{cases} \tag{64}$$

The oriented volume of the tetrahedron is given by

$$V_{\mathrm{c}} = \frac{1}{6}(\vec{L}_\alpha \times \vec{L}_\beta)\cdot \vec{L}_\gamma = \frac{1}{6}\begin{vmatrix} L_{\alpha x} & L_{\alpha y} & L_{\alpha z} \\ L_{\beta x} & L_{\beta y} & L_{\beta z} \\ L_{\gamma x} & L_{\gamma y} & L_{\gamma z} \end{vmatrix}. \tag{65}$$

The matrix, which corresponds to this determinant, coincides with the matrix $\mathbf{L}$ given by Eq.(57). The determinant is equal to $6V_{\mathrm{c}}$. To invert the matrix $\mathbf{L}$, let us observe that the minors of the matrix have a simple geometrical meaning, as the equations (60), (61), (62) show: each one is the double of the (oriented) area of the projections of the tetrahedral faces on the three coordinate planes, with opposite sign. Using Cramer's rule, we can write

$$\begin{cases} H_{xx} = \dfrac{1}{6V_{\mathrm{c}}}\begin{vmatrix} h_{\alpha x} & L_{\alpha y} & L_{\alpha z} \\ h_{\beta x} & L_{\beta y} & L_{\beta z} \\ h_{\gamma x} & L_{\gamma y} & L_{\gamma z} \end{vmatrix} \\ = \dfrac{1}{6V_{\mathrm{c}}}\left[\begin{vmatrix} L_{\beta y} & L_{\beta z} \\ L_{\gamma y} & L_{\gamma z} \end{vmatrix} h_{\alpha x} - \begin{vmatrix} L_{\gamma y} & L_{\gamma z} \\ L_{\alpha y} & L_{\alpha z} \end{vmatrix} h_{\beta x} + \begin{vmatrix} L_{\alpha y} & L_{\alpha z} \\ L_{\beta y} & L_{\beta z} \end{vmatrix} h_{\gamma x}\right] \\ = -\dfrac{1}{3V_{\mathrm{c}}}\left[A_{ix}h_{\alpha x} + A_{jx}h_{\beta x} + A_{kx}h_{\gamma x}\right] \\ = -\dfrac{1}{3V_{\mathrm{c}}}\left[A_{hx}u_h + A_{ix}u_i + A_{jx}u_j + A_{kx}u_k\right] \end{cases} \tag{66}$$

Doing the same operation for H_{xy} and H_{xz}, we obtain

$$\begin{cases} H_{xx} &= -\dfrac{1}{3V_c}\Big[A_{hx}u_h + A_{ix}u_i + A_{jx}u_j + A_{kx}u_k\Big] \\ H_{xy} &= -\dfrac{1}{3V_c}\Big[A_{hy}v_h + A_{iy}v_i + A_{jy}v_j + A_{ky}v_k\Big] \\ H_{xz} &= -\dfrac{1}{3V_c}\Big[A_{hz}w_h + A_{iz}w_i + A_{jz}w_j + A_{kz}w_k\Big]. \end{cases} \tag{67}$$

An analogous equation can be obtained by performing the same process on the second and third row of the system (54). In this way we can find the whole matrix $\mathbf{H}_c$.

Introducing the normal strain ϵ_{hk} and the shear strain γ_{hk}

$$\epsilon_{hh} \stackrel{\text{def}}{=} H_{hh} \qquad \gamma_{hk} \stackrel{\text{def}}{=} H_{hk} + H_{kh} = 2\epsilon_{hk} \tag{68}$$

we can write

$$\begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{bmatrix}_c =$$

$$-\frac{1}{3V_c}\left[\begin{array}{ccc|ccc|ccc|ccc} A_{hx} & 0 & 0 & A_{ix} & 0 & 0 & A_{jx} & 0 & 0 & A_{kx} & 0 & 0 \\ 0 & A_{hy} & 0 & 0 & A_{iy} & 0 & 0 & A_{jy} & 0 & 0 & A_{ky} & 0 \\ 0 & 0 & A_{hz} & 0 & 0 & A_{iz} & 0 & 0 & A_{jz} & 0 & 0 & A_{kz} \\ A_{hy} & A_{hx} & 0 & A_{iy} & A_{ix} & 0 & A_{jy} & A_{jx} & 0 & A_{ky} & A_{kx} & 0 \\ A_{hz} & 0 & A_{hx} & A_{iz} & 0 & A_{ix} & A_{jz} & 0 & A_{jx} & A_{kz} & 0 & A_{kx} \\ 0 & A_{hz} & A_{hy} & 0 & A_{iz} & A_{iy} & 0 & A_{jz} & A_{jy} & 0 & A_{kz} & A_{ky} \end{array}\right]_c \begin{bmatrix} u_h \\ v_h \\ w_h \\ \dots \\ ... \\ u_k \\ v_k \\ w_k \end{bmatrix}_c . \tag{69}$$

Making the assumptions

$$\begin{aligned} \mathbf{u}_c &\stackrel{\text{def}}{=} [u_h\ v_h\ w_h\ u_i\ v_i\ w_i\ u_j\ v_j\ w_j\ u_k\ v_k\ w_k]_c^{\mathsf{T}} \\ \epsilon_c &\stackrel{\text{def}}{=} [\epsilon_{xx}\ \epsilon_{yy}\ \epsilon_{zz}\ \gamma_{xy}\ \gamma_{yz}\ \gamma_{zx}]_c^{\mathsf{T}} \end{aligned} \tag{70}$$

we can write

$$\epsilon_c = \mathbf{B}_c\,\mathbf{u}_c \tag{71}$$

which has the same form as Eq.(26).

3.2 Analysis of stress

We will consider a simplicial complex with its barycentric subdivision. Every dual polyhedron is conceived as a tributary region of the corresponding vertex. Given a plane surface, contained in the cell c, described by an area-vector $\vec{A}_c$, we can evaluate the surface force $\vec{T}_c$ acting on it, using Eq.(73).

With reference to Fig.(15d), we consider the shaded surfaces 1, 2, 3 of the dual polyhedron h contained in the cell c.

The sum of $\vec{A}_h, \vec{A}_i, \vec{A}_j, \vec{A}_k$ vanishes and also the sum of the area-vectors $\vec{A}_1, \vec{A}_2, \vec{A}_3, \vec{A}_4, \vec{A}_5, \vec{A}_6$ vanishes. Since the area-vectors $\vec{A}_4, \vec{A}_5, \vec{A}_6$ are one third of the corresponding area-vectors of the faces of the primal cell c, we obtain

$$\vec{A}_1+\vec{A}_2+\vec{A}_3 = -(\vec{A}_4+\vec{A}_5+\vec{A}_6) = -\frac{1}{3}(\vec{A}_k+\vec{A}_i+\vec{A}_j) = \frac{1}{3}\vec{A}_h \ . \tag{72}$$

Hence, the sum of the vectors $\vec{A}_1, \vec{A}_2, \vec{A}_3$ is one third of the area-vector $\vec{A}_h$. This surface is shown in Fig.(15f). We assume that inside every primal cell c the ma-

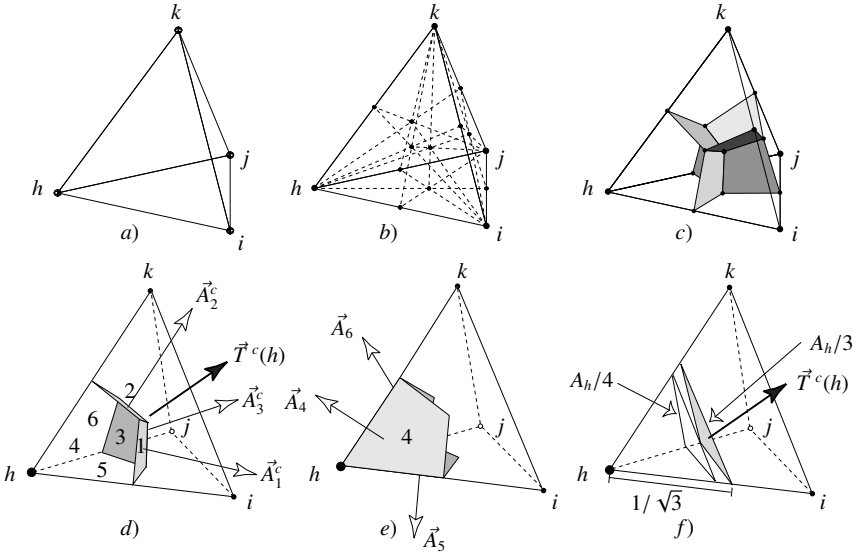


Figure 15: The construction of the six faces of the dual polyhedron contained in the tetrahedron.

terial is homogeneous and the strain is uniform. Hence, also the stress is uniform. By denoting with $\vec{T}^c(h)$ the sum of the forces acting on the faces 1, 2, 3, due to the supposed uniformity of the stress, this force is equal to the force associated with

the shaded plane area of Fig.(15f): this face is parallel to the face A_h and has area $A_h/3$. If we apply the three dimensional analogous of the Cauchy formula, Eq.(30), to the shaded area, we obtain

$$\begin{bmatrix} T_x(h) \\ T_y(h) \\ T_z(h) \end{bmatrix}_{\mathrm{c}} = \frac{1}{3} \begin{bmatrix} A_{hx} & 0 & 0 & A_{hy} & 0 & A_{hz} \\ 0 & A_{hy} & 0 & A_{hx} & A_{hz} & 0 \\ 0 & 0 & A_{hz} & 0 & A_{hy} & A_{hx} \end{bmatrix}_{\mathrm{c}} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{bmatrix}_{\mathrm{c}} \tag{73}$$

which is the homologous of Eq.(31). Moreover

$$\begin{bmatrix} T_x(h) \\ T_y(h) \\ T_z(h) \\ T_x(i) \\ T_y(i) \\ T_z(i) \\ T_x(j) \\ T_y(j) \\ T_z(j) \\ T_x(k) \\ T_y(k) \\ T_z(k) \end{bmatrix}_{\mathrm{c}} = \frac{1}{3} \begin{bmatrix} A_{hx} & 0 & 0 & A_{hy} & 0 & A_{hz} \\ 0 & A_{hy} & 0 & A_{hx} & A_{hz} & 0 \\ 0 & 0 & A_{hz} & 0 & A_{hy} & A_{hx} \\ A_{ix} & 0 & 0 & A_{iy} & 0 & A_{iz} \\ 0 & A_{iy} & 0 & A_{ix} & A_{iz} & 0 \\ 0 & 0 & A_{iz} & 0 & A_{iy} & A_{ix} \\ A_{jx} & 0 & 0 & A_{jy} & 0 & A_{jz} \\ 0 & A_{jy} & 0 & A_{jx} & A_{jz} & 0 \\ 0 & 0 & A_{jz} & 0 & A_{jy} & A_{jx} \\ A_{kx} & 0 & 0 & A_{ky} & 0 & A_{kz} \\ 0 & A_{ky} & 0 & A_{kx} & A_{kz} & 0 \\ 0 & 0 & A_{kz} & 0 & A_{ky} & A_{kx} \end{bmatrix}_{\mathrm{c}} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{bmatrix}_{\mathrm{c}} \tag{74}$$

which is the homologous of Eq.(42). The matrix which appears in this equation, apart from the factor $-V_{\mathrm{c}}$, is the transpose of the matrix we found in Eq.(69)

Introducing the local algebraic vectors

$$\begin{aligned} \mathbf{T}_{\mathrm{c}} &\stackrel{\mathrm{def}}{=} [T_x(h)\ T_y(h)\ T_z(h)\ \cdots\ T_x(k)\ T_y(k)\ T_z(k)]_{\mathrm{c}}^{\mathsf{T}} \\ \sigma_{\mathrm{c}} &\stackrel{\mathrm{def}}{=} [\sigma_{xx}\ \sigma_{yy}\ \sigma_z\ \sigma_{yz}\ \sigma_{xz}\ \sigma_{xy}]_{\mathrm{c}}^{\mathsf{T}} \end{aligned} \tag{75}$$

we can write

$$\mathbf{T}_{\mathrm{c}} = -V_{\mathrm{c}}\, \mathbf{B}_{\mathrm{c}}^{\mathsf{T}} \sigma_{\mathrm{c}} \ . \tag{76}$$

This equation is the same as Eq.(44), which we found in the two-dimensional case. Moreover, introducing the global algebraic vectors, which are analogous to the ones

in Eq.(47)

$$\begin{aligned}
\mathbf{U} &\stackrel{\text{def}}{=} \left[\begin{array}{cccccc} u_x(1) & u_y(1) & u_z(1) & \cdots & u_x(N) & u_y(N) \; u_z(N) \end{array} \right]^{\mathsf{T}} \\
\mathbf{T} &\stackrel{\text{def}}{=} \left[\begin{array}{cccccc} T_x(1) & T_y(1) & T_z(1) & \cdots & T_x(N) & T_y(N) \; T_z(N) \end{array} \right]^{\mathsf{T}} \\
\mathbf{F} &\stackrel{\text{def}}{=} \left[\begin{array}{cccccc} F_x(1) & F_y(1) & F_z(1) & \cdots & F_x(N) & F_y(N) \; F_z(N) \end{array} \right]^{\mathsf{T}} \\
\mathbf{B} &\stackrel{\text{def}}{=} \left[\begin{array}{cccccc} B_x(1) & B_y(1) & B_z(1) & \cdots & B_x(N) & B_y(N) \; B_z(N) \end{array} \right]^{\mathsf{T}}
\end{aligned} \tag{77}$$

we can write the same equations of Section 2.9, thus obtaining the fundamental equation in the same form as Eq.(53).

4 Conclusion

The five notions listed a the beginning of this paper derive from a detailed analysis of physical variables which is usually disregarded. These notions derive from the close link between global variables and space elements. The differential formulation ignores this link because it does not make use of global variables. The main reason for this loss is the use of field functions used to perform the derivatives, which are the fundamental ingredients of the differential formulation. In the differential formulation all variables, being described by field functions, are functions of point. No other space elements have place in the differential formulation. In this sense *the differential formulation has lost its link with geometry, a link that is indispensable for an algebraic formulation.*

If we look at the definition of field functions, we see that most of them are obtained from *global* variables by calculating their densities and rates. All of these are mean densities. Later, by performing the limit, we obtain the field functions and, by performing the partial derivatives on them, we develop the differential formulation. We can see that if we avoid the limit process, we can perform a decription of physics using global variables, which not only refer to points, but to the other space elements too. This is the reason why in the direct algebraic formulation we need to use global variables, rather than field functions. By using global variables the link with geometry is mantained.

Based on this observation, we obtain a direct algebraic formulation. But why is it so important to re-establish this link with geometry? Because, *by paying attention to the geometrical origin of variables, new interesting properties do appear and, from a theoretical point of view, we obtain* ***in a simpler way*** *the same results obtained by the differential formulation.*

How can we highlight the link between physical variables and geometry? The answer is: by analyzing the process of creation of physical variables. Let us note that balance laws, circuital laws and constitutive relations, usually written in differen-

tial terms, are also valid in global terms. For example, the equilibrium condition is valid for a finite portion of a body not only for an infinitesimal portion. The application of the balance on infinitesimal portions is a practice justified only by our habit of using the differential method. Furthermore, the constitutive relations are tested in laboratory under conditions of uniformity of stress on finite samples. *If the physical laws and the constitutive relations can be written directly in an algebraic form, why not use global variables from the very beginning, thus avoiding the use of the field functions?*

In this direct formulation we have obtained the same fundamental equation of FEM without going through the weak formulation of the differential equations of elasticity. The spirit of the present formulation is quite different from FEM approach because the surface forces are calculated with direct geometrical considerations, imposing the equilibrium on the dual cells of every node.

The method differs also from the Finite Volume Method, even though it is similar to the node-centred case, because we do not perform an integration of field variables, but we use global variables directly.

We want to stress that ***physical laws can be written as relations between global variables directly***: so equilibrium states that the sum of forces, i.e. global variables in space, vanishes.

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References

Belytschko, T.; Kam Liu, W.; Moran, B. (2000): *Nonlinear Finite Elements for Continua and Structures*, Wiley.

Burnett, D. S. (1987): *Finite Element Analysis*, Addison Wesley.

Cosmi, F. (2001): Numerical Solution of Plane Elasticity Problems with the Cell Method. *CMES: Computer Modeling in Engineering & Sciences*, vol. 2, n. 3, pp.365-372.

Cosmi, F. (2002): *Application of the Cell Method in Elasticity Problems*. WCCM V, Vienna, Austria, Editors: Mang, H.A.; Rammerstorfer, F.G.; Eberhardsteiner, J., Publisher: Vienna University of Technology, Austria, ISBN 3-9501554-0-6, July 7-12.

Cosmi, F. (2005): Two-dimension estimate of effective properties of solid with random voids, *Theoretical and Applied Fracture Mechanics, Elsevier Science*, vol. 42, n. 2, pp.183-186.

Cosmi, F. (2005b): Elastodynamics wih the Cell Method. *CMES: Computer Modeling in Engineering & Sciences*, vol. 8, n. 3, pp.191-200.

Cosmi, F.; Dreossi, D. (2007): Numerical and experimental structural analysis of trabecular architectures. *Meccanica*, Springer, Dordrecht, Netherlands, vol. 42, n. 1, pp.85-93.

Cosmi, F.; Dreossi, D. (2007b): The Application of the Cell Method for the Clinical Assessment of Bone Fracture Risk. *Acta of Bioengineering and Biomechanics*, Wroclaw University of Technology, Wroclaw, Polonia, vol. 9, n. 2, pp.35-39.

Cosmi, F. (2008): Dynamical Analysis of Mechanical Components: A Discrete Model for Damping. *CMES: Computer Modeling in Engineering & Sciences*, vol. 27, n. 3, pp.187-195.

Cosmi, F.; Steimberg, N.; Dreossi, D.; Mazzoleni, G. (2008b): Structural analysis of rat bone explants kept in vitro in simulated microgravity conditions. *Journal of the Mechanical Behavior of Biomedical Materials*, Elsevier Ltd., The Netherlands, DOI 10.1016/j.jmbbm.

Cosmi, F.; Hoglievina, M. (2008c): An Application of the Cell Method to Multiaxial Fatigue Assessment of a Test Component under Different Criteria. *Strain*, Blackwell Publishing, 2008, DOI: 10.1111/j.1475-1305.2008.00555.x

Ferretti, E.; Viola, E.; Di Leo, A. (2002): Crack Trajectory in Tensioned Concrete Plates using the Cell Method. WCCM V, Vienna, Austria, Editors: Mang, H.A.; Rammerstorfer, F.G.; Eberhardsteiner, J., Publisher: Vienna University of Technology, Austria, ISBN 3-9501554-0-6, July 7-12.

Ferretti, E.; Viola, E.; Di Leo, A. (2002): Limiting Load in Concrete Plates with Cracks: a Cell Method (CM)-Based Calculation. ASEM, Pusan (Korea), 21-23 August 2002.

Ferretti, E. (2003): Crack Propagation Modeling by Remeshing using the Cell Method (CM). *CMES: Computer Modeling in Engineering & Science*, vol. 4, n. 1, pp.51-72.

Ferretti, E. (2004): A Cell Method (CM) Code for Modeling the Pullout Text Stepwise. *CMES: Computer Modeling in Engineering & Science*, vol. 6, n. 5, pp.453-476.

Ferretti, E. (2004): Crack-Path Analysis for Brittle and Non-Brittle Cracks: A Cell Method Approach. *CMES: Computer Modeling in Engineering & Science*, vol. 6, n. 3, pp.227-244.

Ferretti, E. (2004): Modeling of the Pullout Test Through the Cell Method. RRRTEA, International Conference of Restoration, Recycling and Rejuvenation Technology for Engineering and Architecture Application, Cesena (Italy), 7-11 June, pp.180-192.

Ferretti, E. (2005): A Local Strictly Nondecreasing Material Law for Modeling Softening and Size-Effect: a Discrete Approach. *CMES: Computer Modeling in Engineering & Science*, vol. 9, n. 1, pp.19-48.

Ferretti, E. (2005): On Nonlocality and Locality: Differential and Discrete Formulations. ICF XI, 11^{th}International Conference on Fracture, Torino (Italy), 20-25 March.

Ferretti, E.; Casadio, E.; Ricci, P.; Di Leo, A. (2006): A CM Modeling of the Shear Test on Mansory. WCCM VII, Los Angeles (California), 15-22 July.

Ferretti, E.; Casadio, E.; Ricci, P.; Di Leo, A. (2006): A new CM Code for Modeling Cracks in Mansory Walls. GIMC XVI, Bologna (Italy), 26-28 June.

Ferretti, E.; Casadio, E.; if all elements Di Leo, A. (2008): Masonry Walls under Shear Test: a CM Modeling. *CMES: Computer Modeling in Engineering & Science*, vol. 30, n. 3, pp.163-190.

Fung, Y. C. (1965): *Foundation of Solid Mechanics*, Prentice Hall.

Huebner, K. H. (1975): *The Finite Element Method for Engineers*, Wiley & Sons.

Hughes T. J. R . (2000): *The Finite Element Method*, Dover.

Leamy, M. J. (2008): Application of cellular automata modelling seismic elastodynamics. *International Journal of Solids and Structures*, vol. 45, pp.4835-4849.

Love, A. E. H. (1944): *A treatise on the mathematical theory of elasticity*, Dover.

Nappi A., Rajgelj S. and Zaccaria D. (1999): Application of the Cell Method to Elastic-Plastic Analysis, *Physics and Mechanics of Finite Plastic & Viscoplastic Deformation* edited by Akhtar S. Khan NEAT PRESS, Fulton, Maryland.

Nappi, A.; Rajgelj, S.; Zaccaria, D. (2000): A Discrete Formulation Applied to Crack Growth Problems, *Role of Mechanics for Development of Science and*

Technology - Proc. of an Internetional Conference of Role of Mechanics for Development of Science and Tecnology, Xi'an Jiaotong University, Xi'an, China, vol. 1, pp.395-406.

Nappi, A; Tin-Loi, F. (2001): A Numerical Model for Masonry Implemented in the Framework of a Discrete Formulation. *Structural Engineering and Mechanics*, vol. 11, n. 2, pp.171-184.

Penfield, P; Haus, H. (1967): *Electrodynamics of Moving Media*, M. I. T. Press

Taddei, F.; Pani, M.; Zovatto, L.; Tonti, E.; Viceconti, M. (2008): A new meshless approach for subject-specific strain prediction in long bones: Evaluation of accuracy. *Clinical Biomechanics*, vol. 23, pp.1192-1199.

Tonti, E. (1972): A Mathematical Model for Physical Theories. *Rend. Acc. Lincei*, vol. LII, pp.175-181, 350-356.

Tonti, E. (2001c): Finite Formulation of the Electromagnetic Field. *Progress in Electromagnetic Research, PIER 32*, pp.1-42.

Tonti, E. (2001b): A Finite Formulation for the Wave Equation. *Journal of Computational Acoustics*, vol. 9, n. 4, pp.1355-1382.

Tonti, E. (2001a): A Direct Discrete Formulation of Field Laws: The Cell Method. *CMES: Computer Modeling in Engineering and Science*, vol. 2, n. 2, pp.237-258.

Zienkiewicz, O. C. (1971): *The Finite Element Method in Engineering Science*, McGraw-Hill.

Zovatto, L.; Nicolini, M. (2003): The meshless approach for the cell method: a new way for the numerical solution of discrete conservation laws. *International Journal of Computational Engineering Science*, vol. 4, pp.869-880.

Zovatto, L.; Nicolini, M. (2006): Extension of the Meshless Approach for the Cell Method to Three-Dimensional Numerical Integration of Discrete Conservation Laws. *International Journal for Computational Methods in Engineering Science and Mechanics*, vol. 7, n. 2, pp.69-79.

Zovatto, L.; Nicolini, M. (2007): Improving the Convergence Order of the Meshless Approach for the Cell Method for Numerical Integration of Discrete Conservation Laws. *International Journal for Computational Methods in Engineering Science and Mechanics*, vol. 8, n. 5, pp.273Ð281.