# Evaluation of Elastic-Plastic Crack Tip Parameters using Partition of Unity Finite Element Method and Pseudo Elastic Analysis

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Abstract: This paper presents a methodology based on Partition of Unity Finite Element Method (PUFEM) and Pseudo Elastic Analysis for solving material non-linear fracture problems within the scope of total deformation theory of plasticity. Local enrichment base functions are used to represent the asymptotic field near the crack tip and discontinuous field across the crack faces. An iterative linear elastic analysis using PUFEM is carried out for the determination of elasticplastic crack tip stress fields by treating effective material properties as spatial field variables. The effective material parameters are defined using deformation theory and are updated in an iterative manner based on strain controlled projection method using experimental uniaxial tensile test curve. Discrete system of linear equations for the elastic-plastic analysis is obtained from the weak form of the equilibrium equation using the enriched trial function. Application of the present methodology has been illustrated considering non-linear fracture problems for the evaluation of elastic-plastic crack-tip parameters using boundary layer analysis. Ramberg-Osgood model with different hardening exponents is used to characterize the material behavior. Results of the present study for J-dominant HRR field and K dominant elastic field are compared with both the analytical and non-linear finite element solutions and found to be in very good agreement.

**Keywords:** Partition of Unity Finite Element Method, Pseudo Elastic Analysis, Elastic-plastic, HRR singularity, J-integral, Effective material parameters.

## Nomenclature

a crack length

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$a_i$	coefficient associated with local patch influence function $\psi x_i$
$b_i$	body force vector component
B	strain displacement matrix
D	constitutive matrix
Ε	Young's modulus
$E_{eff}$	effective Young's modulus
F	force vector
$I_n$	dimensionless integration constant
$J_I, J_{II}$	mode I and mode II J-integrals
$J_{x_1}, J_{x_2}$	sum and product J-integrals
$J_{Sx_1}, J_{Sx_2}$	symmetric sum and product J-integrals
$J_{ASx_1}, J_{ASx_2}$	anti symmetric sum and product J-integrals
Κ	stress intensity factor
K	stiffness matrix
q	function defined in Eq. 42
m	hardening exponent
n <sub>i</sub>	unit outward normal component
$N_i$	nodal shape function
$N_{en}$	enriched shape function matrix
$S_{ij}$	deviatoric stress tensor component
t <sub>i</sub>	surface traction vector component
u <sup>i</sup>	displacement vector component
W	strain energy density
$x_i$	Cartesian co-ordinates
α	yield offset in Ramberg-Osgood model
$\delta_{ij}$	Kronecker delta
ε	experimental uniaxial total strain
$\epsilon^p_{eq}$	equivalent plastic strain
$\epsilon_{ij}$	strain tensor component
ν	Poisson's ratio
$v_{eff}$	effective Poisson's ratio
σ	experimental uniaxial stress
$\sigma_0$	yield stress
$\epsilon_0$	yield strain
$\sigma_{ij}$	stress tensor component
$\sigma_{eq}$	equivalent stress
au	shear traction
Φ	Hencky's scalar valued function
$\phi$	partition of unity function
Ψ	local patch influence function

Ω	domain
Γ	surface
$\Gamma_t$	traction prescribed surface
$\Gamma_c$	crack surface
$\partial \Omega^{el}$	element boundary

## Superscript

e	elastic
р	plastic
el	element

# Subscript

eq	equivalent
eff	effective
en	enriched

## Abbreviations

CCT	Centre Crack Tension specimen	
DECT	Double Edge Crack Tension specimen	
EPFM	Elastic Plastic Fracture Mechanics	
EPRI	Electrical Power Research Institute	
FEM	Finite Element Method	
GFEM	Generalized Finite Element Method	
LEFM	Linear Elastic Fracture Mechanics	
PEM	Pseudo Elastic Method	
PU	Partition of Unity	
PUFEM	Partition of Unity Finite Element Method	
PUM	Partition of Unity Method	
SECT	Single Edge Crack Tension specimen	
SIF	Stress Intensity Factor	
XFEM	Extended Finite Element Method	

## 1 Introduction

Most of the modern engineering structural materials undergo large scale inelastic deformation at the crack tip and conventional SIF based fracture design is no longer

valid. Instead, Elastic-plastic fracture characterization is required for life prediction and damage tolerant design of engineering structures. Rice (1968) proposed a path independent line integral called J-integral encircling the crack tip to characterize the elastic-plastic fracture field. Using the line integral, Hutchinson (1968a, 1968b) and Rice and Rosengren (1968) studied stress/strain singularities dominant at the crack tip region of an elastic-plastic material. Their study showed the existence of stress/strain singularity, known as HRR singularity, well within the plastic zone surrounding the crack tip. Results also showed that the intensity coefficient of this singularity, J-integral, uniquely characterize the elastic-plastic crack tip field

Work of Shih and German (1981) using finite element analysis under small scale yielding condition for mode-I problems showed that though J-integral characterizes the elastic-plastic crack tip field, the region of influence of HRR singularity varies from geometry to geometry and also depend on the material hardening parameter. Bradford (1984) analyzed the crack tip stress and strain field in an edge cracked square plate under mode-II loading in the elastic-plastic regime with power law hardening material model assumption. He has shown that the HRR singularity is applicable for a wide range of loading conditions, right from small scale yielding to general yielding conditions. Atluri et al. (1984) proposed T\*-integral as an alternate elastic-plastic fracture parameter during stable crack growth since J-integral is not valid in this case due to unloading at the crack tip.

In recent years, considerable research is being carried out in meshless methods, a new class of numerical method, for solving partial differential equations associated with boundary value and initial value problems. These methods are highly suitable for crack propagation problems, large deformation studies and complex geometries where remeshing is time consuming and expensive. Various commonly used meshless methods are Smooth Particle Hydrodynamics (SPH), Diffuse Element Method (DEM), Element Free Galerkin method (EFGM), H-P cloud method, Reproducing Kernel Particle Method (RKPM), Meshless Local Petrov-Galerkin Method (MLPG) [Atluri and Zhu (1998)], Partition of Unity Finite Element Method (PUFEM) [Melenk and Babuska (1996)], Generalized Finite Element Method (GFEM) [Strouboulis, Babuska and Copps (2000)], Extended Finite Element Method (XFEM) [Sukumar and Prévost (2003)] and etc.

MLPGM originated by Atluri (2004) is considered as a true mesh free method since does not require any mesh either for the interpolation of the solution variables or for the integration of the weak form as compared to EFGM which requires background mesh for integration. The MLPG method has been further enhanced to solve 3D elasto-static problems [Han and Atluti (2004)], non linear static and dynamic problems [Han, Rajendran, Atluri (2005)] and 2D stationary and transient problems in piezoelectric and magneto-electric-elastic material models with continuously varying material properties [Sladek.S, Sladek.V, Solek, Atluri (2008)]. Enriched meshless methods based on EFGM has been developed for fracture analysis in the three dimensional domain by Chen et al. (2005) and in functionally graded materials by Wen, Aliabadi and Liu (2008). Li, Liu, Wang (2008) studied crack propagation under ductile fracture using RKPM and Gurson-Tvergaard-Needleman constitutive model.

Basic theory and applications of Partition of Unity Finite Element Method are explained in detail with numerical examples in the report of Melenk and Babuska (1996). PUFEM uses patch based interpolation approximations compared to element based interpolation approximations of standard FEM. Treating the entire domain as group of overlapping patches with local enrichment functions defined over each patch, PUFEM enforces continuity over the entire domain using a set of  $C^0$  functions called PU functions. Moes, John Dolbow and Belytschko (1999) proposed PUFEM based enrichment technique using Heaviside and trigonometric functions for modeling the cracks. The accuracy of the method was demonstrated by predicting crack growth with relatively coarse mesh. Strouboulis, Babuska and Copps (2000, 2001) discussed in detail the design and implementation of Generalized Finite Element Method as a direct extension of standard finite element method. Discontinuities and singularities were modeled using special enrichment functions under the framework of partition of unity. Sukumar and Prèvost (2003) have studied in detail the numerical implementation of Extended Finite Element Method in linear elastic materials. Quasi-static crack growth in mixed and pure mode cases is studied in their work. The versatility of PUFEM was demonstrated by Fan, Liu and Lee (2004) by directly extracting pure mode stress intensity factors in a mixed mode problem. They used enrichment functions only at the crack tip corresponding to truncated asymptotic series of linear elastic displacement field along with p-version FEM and used coincident nodes to simulate rest of the crack surface. Cai and Zhu (2008) proposed a new Local Meshless Shepard and Least Square (LMSLS) method based on the local Petrov-Galerkin weak form in which Shepard least square interpolation (SLS) offer much needed Kronecker-delta property for imposing the essential boundary conditions.

Though substantial research has been carried out in elastic regime using partition of unity concept, study in elastic-plastic regime is still in the developing stage. Rao and Rahman (2004) used elastic-plastic crack tip enrichment functions to capture the HRR singularity using element free Galerkin method. Elguedj et al. (2006) studied about various EPFM enrichment functions for XFEM modeling and used incremental non-linear solver for the elastic-plastic analysis. Elastic-plastic fracture solutions of various crack problems are compared with linear solutions in their work. Hagihara, et al. (2007) used element-free Galerkin method to calculate

elastic-plastic fracture mechanics parameters, J-integral and T\* -integral for both stationary crack problem and stable crack growth problem in 2D domain. Long, Liu and Li (2008) studied elastic-plastic fracture problems using meshless Petrov-Galerkin method and incremental plasticity analysis.

Pseudo elastic method, initially developed by Desikan and Sethuraman (2000) is an iterative method based on the Hencky's total deformation theory of plasticity, where material properties are treated as spatial field variables. Same methodology is used in elastic-plastic fracture study along with element free Galerkin method by Sethuraman and Reddy (2004, 2008) and in the inelastic analysis of 2D problems using Radial Point Interpolation Method by Dai, Liu, Han and Li (2006).

In the present paper, pseudo elastic methodology is coupled along with partition of unity finite element method for elastic-plastic fracture characterization and simulation of HRR singular field at the crack tip. Effectiveness of the proposed method has been demonstrated by the elastic-plastic analysis of various fracture problems using different material hardening models.

# 2 Methodology

The present work is based on the two numerical methods namely Pseudo Elastic Method (PEM) and Partition of Unity Finite Element Method (PUFEM). This section outlines PUFEM followed by PEM.

For static elastic-plastic problems in a continuous medium, the weak form of the equilibrium equation (principle of virtual work) is written as

$$\iiint_{\Omega} \sigma_{ij} \delta \varepsilon_{ij} d\Omega - \iiint_{\Omega} b_i \delta u_i d\Omega - \iint_{\Gamma_t} t_i \delta u_i d\Gamma = 0$$
<sup>(1)</sup>

## 2.1 Partition of unity finite element method approximation

Explicit form of partition of unity frame work is used to generalize the standard FEM approximations which result in node based conforming approximation compared to element based approximation in FEM. Unlike FEM, PUM treats the domain as an assemblage of overlapping node centered sub domains called covers or patches. Approximations representing local behavior are defined for the patches and are called patch functions. Solution for the entire domain is developed by multiplying patch functions with corresponding partition of unity function.

For the given domain of patches  $\{\Omega_i\}$  with each patch having a set of local patch influence functions,  $\psi_i$ , the partition of unity approximation for displacement at

any arbitrary location  $\boldsymbol{x}$  in the domain  $\Omega$  is of the form

$$\boldsymbol{u}^{h}(\boldsymbol{x}) = \sum_{i \in N} \phi_{i} \left( \sum_{j=1}^{M} a_{j}^{(i)} \boldsymbol{\psi}_{j}^{(i)} \right)$$
(2)

where M is the number of local functions associated with each patch, N is set of nodes in the domain and  $a_j^{(i)}$  are constant coefficients associated with  $\psi_j^{(i)}$ . Set of functions  $\phi_i$  (one for each patch) are  $C^0$  partition of unity on  $\Omega$  with the following properties

$$0 \le \phi_i(\boldsymbol{x}) \le 1 \; \forall \boldsymbol{x} \in \Omega_i \quad \sum_i^N \phi_i(\boldsymbol{x}) = 1 \; \forall \boldsymbol{x} \in \Omega$$
(3)

In the present study, set of elements that share a common node is used to form a patch with vertex at that node. For four node quadrilateral elements, maximum number of overlapping patches at any point x in the domain is 4. Shape functions of the vertex node corresponding to each element of the patch are used to form the partition of unity function for that patch.

Using finite element shape functions as patch cover functions and considering the standard displacement finite element degrees of freedom, PUFEM displacement approximation in Eq. 2 can be written as

$$\boldsymbol{u}^{h}(\boldsymbol{x}) = \sum_{i \in N} N_{i}(\boldsymbol{x}) \left( \boldsymbol{u}_{i} + \sum_{j=1}^{M} a_{j}^{(i)} \boldsymbol{\psi}_{j}^{(i)}(\boldsymbol{x}) \right)$$
(4)

Hierarchical enrichments are used in the present study for two dimensional crack modeling and is of the form [Moes, John Dolbow and Belytschko (1999)]

$$\boldsymbol{u}^{h}(\boldsymbol{x}) = \left\{ \begin{matrix} \boldsymbol{u}^{1} \\ \boldsymbol{u}^{2} \end{matrix} \right\} = \sum_{i \in N} N_{i}(\boldsymbol{x}) \left[ \underbrace{\left\{ \begin{matrix} \boldsymbol{u}_{i}^{1} \\ \boldsymbol{u}_{i}^{2} \end{matrix} \right\}}_{i \in N} + \underbrace{H(\boldsymbol{x}) \left\{ \begin{matrix} \boldsymbol{a}_{i}^{1} \\ \boldsymbol{a}_{i}^{2} \end{matrix} \right\}}_{i \in N_{f}} + \underbrace{\sum_{\alpha = 1}^{4} \Phi_{\alpha}(\boldsymbol{x}) \left\{ \begin{matrix} \boldsymbol{b}_{i\alpha}^{1} \\ \boldsymbol{b}_{i\alpha}^{2} \end{matrix} \right\}}_{i \in N_{c}} \right]$$
(5)

where superscripts 1, 2 represent the components in  $x_1$  and  $x_2$  directions respectively.  $N_c$  is the set of nodes in the crack tip region and  $N_f$  is the set of crack face nodes other than those in crack tip region,  $H(\mathbf{x})$  is Heaviside function,  $\Phi_{\alpha}(\mathbf{x})$  are crack tip enrichment functions,  $u_i$ 's are the nodal displacement degrees of freedom,  $a_i$ 's are enriched degrees of freedom associated with Heaviside function and  $b_i$ 's are enriched degrees of freedom associated with crack tip enrichment functions.

Crack discontinuity other than crack tip region is modeled using Heaviside function,  $H(\mathbf{x})$  which takes value +1 for points above the crack and -1 for points below the crack.

Heaviside function used in the present study is of the form

$$H(f(\mathbf{x})) = \begin{cases} 1 & f(\mathbf{x}) > 0 \\ -1 & f(\mathbf{x}) < 0 \end{cases}$$
(6)

where x is any point in the domain and f(x) is signed area of a triangle formed by crack end points and x as vertices.

For isotropic materials, the asymptotic displacement field near the crack tip region is represented by the following four enrichment functions [Fleming, Chu, Moran, Belytschko, Lu and Gu (1997)].

$$\Phi_{\alpha} = \left\{ \sqrt{r} \sin(\theta/2), \sqrt{r} \cos(\theta/2), \sqrt{r} \sin(\theta) \sin(\theta/2), \sqrt{r} \sin(\theta) \cos(\theta/2) \right\}$$

$$(\text{LEFM})$$

$$\left\{ r^{1/(m+1)} \sin(\theta/2), r^{1/(m+1)} \cos(\theta/2), r^{1/(m+1)} \sin(\theta) \sin(\theta/2), r^{1/(m+1)} \sin(\theta) \cos(\theta/2) \right\}$$

$$(7)$$

$$r^{1/(m+1)} \sin(\theta) \cos(\theta/2) \right\} (\text{EPFM})$$

where  $r, \theta$  are crack tip polar co-ordinates and m is the strain hardening exponent. The above functions are used to represent the crack tip region with the first term,  $\sqrt{r}\sin(\theta/2)$  modeling the discontinuity across the crack face.

Positive semi definite system resulting from the linear dependence of finite element shape functions and patch functions is solved using the perturbation approach [Duarte, Babuska and Oden (2000)].

#### 2.2 Discrete Equations

In the absence of body forces, the weak form of equilibrium equation in Eq. 1 can be written in the following matrix form

$$\int_{\Omega} \delta \boldsymbol{\varepsilon}^{T} \boldsymbol{D} \boldsymbol{\varepsilon} d\Omega - \int_{\Gamma} \delta \boldsymbol{u}^{T} \boldsymbol{t} d\Gamma = 0$$
(8)

Trial function (Eq. 5), can be written in the form

$$\begin{cases}
 u^1 \\
 u^2
 \end{cases} = N_{en} u_{en}$$
(9)

where

$$\boldsymbol{N}_{en} = [\boldsymbol{N}_1 \, \boldsymbol{N}_2 \, \dots \, \boldsymbol{N}_N] \tag{10}$$

and

$$\boldsymbol{u}_{en} = \left[\underbrace{\left\{\begin{array}{c}u_1^1\\u_1^2\right\}\left\{a_1^1\right\}\left\{b_{1\alpha}^1\\b_{1\alpha}^2\right\}_{\alpha=1\dots4}}_{node\,1}\cdots\underbrace{\left\{\begin{array}{c}u_N^1\\u_N^2\right\}\left\{a_N^1\right\}\left\{b_{N\alpha}^1\\a_N^2\right\}\left\{b_{N\alpha}^2\right\}_{\alpha=1\dots4}}_{node\,N}\right]^T \tag{11}$$

Components of  $N_{en}$  are given by

$$\boldsymbol{N}_{i} = \left[\underbrace{\begin{cases} N_{i} & 0\\ 0 & N_{i} \end{cases}}_{\text{conventional}} \underbrace{\begin{cases} N_{i}H & 0\\ 0 & N_{i}H \end{cases}}_{\text{Heaviside enrichement}} \underbrace{\begin{cases} N_{i}\Phi_{\alpha} & 0\\ 0 & N_{i}\Phi_{\alpha} \end{cases}}_{\text{asymptotic enrichment}} \right]$$
(12)

For conventional nodes (not enriched), components of  $N_{en}$  and  $u_{en}$  are

$$\boldsymbol{N}_{i} = \begin{bmatrix} N_{i} & 0\\ 0 & N_{i} \end{bmatrix}, \quad \boldsymbol{u}_{i} = \begin{pmatrix} u_{i}^{1}\\ u_{i}^{2} \end{pmatrix}$$
(13)

For nodes 1, 2, 5, 6 which are enriched with degrees of freedom associated with Heaviside function (refer Fig. 1), components of  $pmbN_{en}$  and  $u_{en}$  are

$$\boldsymbol{N}_{i} = \begin{bmatrix} N_{i} & 0 & N_{i}H & 0\\ 0 & N_{i} & 0 & N_{i}H\\ \text{conventional Heaviside enrichement} \end{bmatrix} \quad \boldsymbol{u}_{i} = \left\{ u_{i}^{1} u_{i}^{2} a_{i}^{1} a_{i}^{2} \right\}^{T}$$
(14)

For nodes 3, 4, 7, 8 which are enriched with degrees of freedom associated with asymptotic functions (refer Fig. 1), components of  $N_{en}$  and  $u_{en}$  are

$$\boldsymbol{N}_{i} = \begin{bmatrix} N_{i} & 0\\ 0 & N_{i}\\ \text{conventional} & \underbrace{\left\{N_{i}\Phi_{\alpha} & 0\\ 0 & N_{i}\Phi_{\alpha}\right\}_{\alpha=1...4}}_{\text{asymptotic enrichment}} \end{bmatrix} \quad \boldsymbol{u}_{i} = \left\{u_{i}^{1} u_{i}^{2} \left\{b_{i\alpha}^{1} b_{i\alpha}^{2}\right\}_{\alpha=1...4}\right\}^{T} \quad (15)$$

Substituting  $\delta \boldsymbol{\varepsilon} = \boldsymbol{B}_{en} \delta \boldsymbol{u}_{en}$ ,  $\delta \boldsymbol{u} = \boldsymbol{N}_{en} \delta \boldsymbol{u}_{en}$  in Eq. 8

$$\int_{\Omega} [\boldsymbol{B}_{en} \delta \boldsymbol{u}_{en}]^T \boldsymbol{D} \boldsymbol{B}_{en} \boldsymbol{u}_{en} d\Omega - \int_{\Gamma} [\boldsymbol{N}_{en} \delta \boldsymbol{u}_{en}]^T \boldsymbol{t} d\Gamma = 0$$
(16)

where

$$\boldsymbol{B}_{en} = \left[ \boldsymbol{B}_{1}^{u} \boldsymbol{B}_{1}^{a} \left\{ \boldsymbol{B}_{1\alpha}^{b} \right\}_{\alpha=1...4} \dots \boldsymbol{B}_{N}^{u} \boldsymbol{B}_{N}^{a} \left\{ \boldsymbol{B}_{N\alpha}^{b} \right\}_{\alpha=1...4} \right]$$
(17)

Components of  $\boldsymbol{B}_{en}$  are given by

$$\boldsymbol{B}_{i}^{u} = \begin{bmatrix} N_{i,1} & 0\\ 0 & N_{i,2}\\ N_{i,2} & N_{i,1} \end{bmatrix}, \quad \boldsymbol{B}_{i}^{a} = \begin{bmatrix} (N_{i}H)_{,1} & 0\\ 0 & (N_{i}H)_{,2}\\ (N_{i}H)_{,2} & (N_{i}H)_{,1} \end{bmatrix},$$

$$\boldsymbol{B}_{i\alpha}^{b} = \begin{bmatrix} (N_{i}\Phi_{\alpha})_{,1} & 0\\ 0 & (N_{i}\Phi_{\alpha})_{,2}\\ (N_{i}\Phi_{\alpha})_{,2} & (N_{i}\Phi_{\alpha})_{,1}\alpha = 1...4 \end{bmatrix}$$
(18)

( ),1, ( ),2 represent partial derivatives with respect to  $x_1, x_2$  co-ordinates. Eq. 16 reduces to

$$\boldsymbol{K}_{en}\boldsymbol{u}_{en} = \boldsymbol{F}_{en} \tag{19}$$

where  $K_{en}$  and  $F_{en}$  are the stiffness matrix and force vector defined by

$$\boldsymbol{K}_{en} = \int_{\Omega} [\boldsymbol{B}_{en}]^T \boldsymbol{D} \boldsymbol{B}_{en} d\Omega = \int_{\Omega} \begin{bmatrix} [\boldsymbol{B}^{u}]^T \\ [\boldsymbol{B}^{a}]^T \\ \{\boldsymbol{B}^{b}_{\alpha}\}_{\alpha=1..4}^T \end{bmatrix} \boldsymbol{D} \begin{bmatrix} \boldsymbol{B}^{u} \boldsymbol{B}^{a} \left\{ \boldsymbol{B}^{b}_{\alpha} \right\}_{\alpha=1..4}^{\alpha} \end{bmatrix} d\Omega$$
(20)

$$\boldsymbol{K}_{en} = \int_{\Omega} \begin{bmatrix} \boldsymbol{B}^{u} \boldsymbol{B}^{T} \boldsymbol{D} \boldsymbol{B}^{u} & \boldsymbol{B}^{u} \boldsymbol{B}^{T} \boldsymbol{D} \boldsymbol{B}^{a} & \boldsymbol{B}^{u} \boldsymbol{B}^{T} \boldsymbol{D} \left\{ \boldsymbol{B}^{b}_{\alpha} \right\}_{\alpha=1..4} \\ \boldsymbol{B}^{a} \boldsymbol{B}^{T} \boldsymbol{D} \boldsymbol{B}^{u} & \boldsymbol{B}^{a} \boldsymbol{B}^{T} \boldsymbol{D} \boldsymbol{B}^{a} & \boldsymbol{B}^{a} \boldsymbol{B}^{T} \boldsymbol{D} \left\{ \boldsymbol{B}^{b}_{\alpha} \right\}_{\alpha=1..4} \\ \left\{ \boldsymbol{B}^{b}_{\alpha} \right\}_{\alpha=1..4}^{T} \boldsymbol{D} \boldsymbol{B}^{u} & \left\{ \boldsymbol{B}^{b}_{\alpha} \right\}_{\alpha=1..4}^{T} \boldsymbol{D} \boldsymbol{B}^{a} & \left\{ \boldsymbol{B}^{b}_{\alpha} \right\}_{\alpha=1..4}^{T} \boldsymbol{D} \left\{ \boldsymbol{B}^{b}_{\alpha} \right\}_{\alpha=1..4} \end{bmatrix} d\Omega$$

$$(21)$$

$$\boldsymbol{F}_{en} = \int_{\Gamma_t} \left[ \boldsymbol{N}_{en} \right]^T \boldsymbol{t} d\Gamma$$
(22)

Shape functions of the vertex node, corresponding to its support elements in the patch, are used to construct the partition of unity function for that patch. These shape functions have value unity at the vertex and zero on the boundary of the patch. Together these shape functions form  $C^0$  partition of unity over the entire domain ensuring inter element continuity. Therefore trial function at any interior point depends on the non zero shape functions of four surrounding nodes and associated degrees of freedom. Thus construction of stiffness matrix and force vector can be done element wise and assembled in the usual finite element procedure.

Element stiffness matrix is given by

$$\boldsymbol{K}_{en}^{el} = \int_{\Omega^{el}} \left[ \boldsymbol{B}_{en}^{el} \right]^T \boldsymbol{D} \boldsymbol{B}_{en}^{el} d\Omega$$
(23)

where

$$\boldsymbol{B}_{en} = \begin{bmatrix} \boldsymbol{B}_{1}^{u} \, \boldsymbol{B}_{1}^{a} \, \left\{ \boldsymbol{B}_{1\alpha}^{b} \right\}_{\alpha=1...4} \, \boldsymbol{B}_{2}^{u} \, \boldsymbol{B}_{2}^{a} \, \left\{ \boldsymbol{B}_{2\alpha}^{b} \right\}_{\alpha=1...4} \cdots \\ \boldsymbol{B}_{3}^{u} \, \boldsymbol{B}_{3}^{a} \, \left\{ \boldsymbol{B}_{3\alpha}^{b} \right\}_{\alpha=1...4} \, \boldsymbol{B}_{4}^{u} \, \boldsymbol{B}_{4}^{a} \, \left\{ \boldsymbol{B}_{4\alpha}^{b} \right\}_{\alpha=1...4} \end{bmatrix} \quad (24)$$

Size of element stiffness matrix varies with its associated nodal enriched degrees of freedom. For an element with no enriched nodes, size of stiffness matrix will be same as that of conventional element i.e. 8x8. If all four nodes of an element are enriched with Heaviside function, size of stiffness matrix will be 16x16 and for elements with all four nodes asymptotically enriched, size of stiffness matrix will be 40x40.

Elemental load vector is given by

$$\boldsymbol{F}_{en} = \left\{ F_1^u F_1^a \left\{ F_{1\alpha}^b \right\}_{\alpha=1\dots4} F_2^u F_2^a \left\{ F_{2\alpha}^b \right\}_{\alpha=1\dots4} F_3^u F_3^a \left\{ F_{3\alpha}^b \right\}_{\alpha=1\dots4} F_4^u F_4^a \left\{ F_{4\alpha}^b \right\}_{\alpha=1\dots4} \right\}^T \quad (25)$$

with components

$$F_{i}^{u} = \int_{\Gamma_{i} \cap \partial \Omega^{el}} \mathbf{N}_{i}^{T} t d\Gamma,$$

$$F_{i}^{a} = \int_{\Gamma_{i} \cap \partial \Omega^{el}} [\mathbf{N}_{i} H]^{T} t d\Gamma,$$

$$F_{i\alpha}^{b} = \int_{\Gamma_{i} \cap \partial \Omega^{el}} [\mathbf{N}_{i} \Phi_{\alpha}]^{T} t d\Gamma$$
(26)

For plane stress, constitutive matrix **D** is given by

$$\boldsymbol{D} = \frac{E}{1 - v^2} \begin{bmatrix} 1 & v & 0\\ v & 1 & 0\\ 0 & 0 & (1 - v)/2 \end{bmatrix}$$
(27)

E is replaced by E/  $(1-v^2)$ , and v by v/ (1-v) for plane strain.

For elastic-plastic problems, in evaluating the stiffness matrix K, the effective Poisson's ratio and effective Young's modulus are used appropriately. In stiffness matrix and subsequent stress evaluation, the constitutive matrix is changed from D to

 $D_{eff}$  using the material parameters  $E_{eff}$  and  $v_{eff}$ . The element stiffness matrix for elastic-plastic analysis can be written as

$$\boldsymbol{K}_{en}^{el} = \int_{\Omega^{el}} \left[ \boldsymbol{B}_{en}^{el} \right]^T \boldsymbol{D}_{eff} \, \boldsymbol{B}_{en}^{el} d\Omega \tag{28}$$

The construction of constitutive matrix  $D_{eff}$  is detailed in the next section.

#### 2.3 Pseudo Elastic Methodology

The stress-total strain relationship for materials under Hencky's total deformation theory of plasticity is

$$\varepsilon_{ij} = \left(\frac{1+\nu}{E} + \Phi\right)\sigma_{ij} - \left(\frac{\nu}{E} + \frac{\Phi}{3}\right)\sigma_{kk}\delta_{ij}$$
<sup>(29)</sup>

where  $\Phi$  is a scalar valued function defined in deformation theory and is given in terms of equivalent stress and equivalent plastic strain as

$$\Phi = \frac{3}{2} \frac{\varepsilon_{eq}^p}{\sigma_{eq}} \tag{30}$$

The equivalent plastic strain and equivalent stress are given by

$$\varepsilon_{eq}^{p} = \sqrt{\frac{2}{3}}\varepsilon_{ij}^{p}\varepsilon_{ij}^{p}, \sigma_{eq} = \sqrt{\frac{3}{2}}S_{ij}S_{ij}$$
(31)

The deviatoric stress tensor is defined as

$$S_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij} \tag{32}$$

Eq. 29 can rewritten in the following form

$$\varepsilon_{ij} = \left(\frac{1 + v_{eff}}{E_{eff}}\right) \sigma_{ij} - \left(\frac{v_{eff}}{E_{eff}}\right) \sigma_{kk} \delta_{ij}$$
(33)

where  $v_{eff}$  and  $E_{eff}$  are the effective material parameters and are defined as

$$\frac{1}{E_{eff}} = \frac{1}{E} + \frac{2}{3}\Phi, \quad v_{eff} = E_{eff}\left(\frac{\nu}{E} + \frac{\Phi}{3}\right)$$
(34)

These effective material parameters are functions of the final state of stress. Since the final state of stress at every point is unique,  $v_{eff}$  and  $E_{eff}$  can be treated as field variables at every spatial point. Elastic-plastic stress-strain relation in Eq. 33 describes the elastic behavior of the continuum for constant values of  $v_{eff}$  and  $E_{eff}$ . Geometrically  $E_{eff}$  is nothing but the secant modulus defined on the experimental uniaxial material curve. Thus effective material parameters in all iterations are calculated using experimental uniaxial curve and Eq. 34. Though strain controlled projection method, arc method and Neuber's method [Desikan and Sethuraman (2000)] can be used to calculate  $E_{eff}$  in successive iterations, only strain controlled projection method is used in the present study. The constitutive matrix  $D_{eff}$  for elastic-plastic regime is evaluated by substituting these modified material parameters  $E_{eff}$  and  $v_{eff}$  as E and v in Eq. 27.

## **3** Numerical Implementation

#### 3.1 PUFEM Implementation

PUFEM implementation presented in the current study is based on the work of Sukumar and Prévost (2003). Domain is discretized with mesh without considering crack geometry. Crack geometry is considered as a virtual segment across interiors of a group of elements and is assumed to be terminating in the interior of an element. The crack face discontinuity and crack tip singular field are characterized using Heaviside and asymptotic enrichment functions. A sample mesh is shown in Fig. 1 with an edge crack terminating at interior of an element. 4 nodes around the crack tip usually form the node set  $N_c$  for asymptotic enrichment. Alternatively a fixed circular domain around the crack tip can also be used to identify nodes for asymptotic enrichments. Nodes that belong to elements that are split by the crack segment, but not selected for asymptotic enrichment, are included in the set  $N_f$  for Heaviside enrichment. Nodes that are enriched with asymptotic functions have 10 degrees of freedom per node (two conventional degrees of freedom and eight degrees of freedom associated with four enrichment base functions). Nodes with Heaviside enrichment have 4 degrees of freedom per node (two conventional degrees of freedom and two degrees of freedom associated with enrichment base function).

Nodes 1, 2, 5, 6 of the above mesh belong to node set  $N_c$  and nodes 3, 4, 7, 8 belong to node set  $N_f$ . Signed area of the triangle formed with material point (either gauss point /node) and end points of the crack segment as vertices is used to assign the value of Heaviside function. Construction of element stiffness matrices and element force vectors for partitioned elements include integration of discontinuous functions. The partitioned elements are ones whose all nodes are enriched, thus encompassing the discontinuity (crack). The elements 'A', 'B', 'C' are the partitioned elements for the mesh shown in Fig. 1.



Figure 1: Nodal enrichment scheme and crack representation in a regular mesh



Figure 2: Partitioning of element intersected by crack segment

Integration of stiffness matrix in case of elements with crack segment in the interior requires special treatment because of the discontinuous enrichment functions involved. Element is partitioned into smaller subdomains conforming to the crack segment.

Stiffness evaluation requires looping over smaller partitioned subdomains as shown in Fig. 2. Integration over domain 'abcd' requires evaluation of shape functions, which are defined only for the parent element, at local gauss points. This is done by mapping physical co-ordinates of gauss points in the subdomain 'abcd' to mapped quadrilateral of the parent element i.e.  $(\xi'_1, \xi'_2) \rightarrow \mathbf{x} \rightarrow (\xi_1, \xi_2)$ . Newton-Raphson iterative method is used for this purpose.

Contribution of subdomain 'abcd' to elemental stiffness matrix is given by

$$\boldsymbol{K}_{en}^{abcd} = \int_{\Omega^{abcd}} \left[ \boldsymbol{B}_{en}(\boldsymbol{x}(\boldsymbol{\xi}(\boldsymbol{\xi}'))) \right]^T \boldsymbol{D}_{eff} \boldsymbol{B}_{en}(\boldsymbol{x}(\boldsymbol{\xi}(\boldsymbol{\xi}'))) \det(J(\boldsymbol{\xi}(\boldsymbol{\xi}'))) \det(J(\boldsymbol{\xi}')) d\xi_1' d\xi_2'$$
(35)

where  $\boldsymbol{\xi}(\xi_1, \xi_2)$  is the parent element co-ordinate system and  $\boldsymbol{\xi}'(\xi_1', \xi_2')$  is the subdomain co-ordinate system.

Global co-ordinate system is aligned with crack tip co-ordinate system for calculating of asymptotic enrichment functions and post processing of results. If any one node of an element is enriched, its contribution to element stiffness matrix is to be added in addition to contribution associated with conventional degrees of freedom. Strain-displacement matrices of partitioned elements 'A', 'B', 'C' are

$$\mathbf{B}_{en}^{A} = [\mathbf{B}_{1}^{u} \, \mathbf{B}_{1}^{a} \, \mathbf{B}_{2}^{u} \, \mathbf{B}_{2}^{a} \, \mathbf{B}_{6}^{u} \, \mathbf{B}_{6}^{a} \, \mathbf{B}_{5}^{u} \, \mathbf{B}_{5}^{a}] 
\mathbf{B}_{en}^{B} = \left[\mathbf{B}_{2}^{u} \, \mathbf{B}_{2}^{a} \, \mathbf{B}_{3}^{u} \, \mathbf{B}_{3\alpha}^{b} \, \mathbf{B}_{7}^{u} \, \mathbf{B}_{7}^{a} \, \mathbf{B}_{6}^{a} \, \mathbf{B}_{6\alpha}^{b}\right] 
\mathbf{B}_{en}^{C} = \left[\mathbf{B}_{3}^{u} \, \mathbf{B}_{3\alpha}^{b} \, \mathbf{B}_{4}^{u} \, \mathbf{B}_{4\alpha}^{b} \, \mathbf{B}_{8}^{u} \, \mathbf{B}_{8\alpha}^{b} \, \mathbf{B}_{7}^{u} \, \mathbf{B}_{7\alpha}^{b}\right]$$
(36)

Global stiffness matrix and force vector are constructed similar to standard finite element method. Global nodal degrees of freedom array is created first, using all nodal degrees of freedom including both conventional and enriched degrees of freedom. Element stiffness matrices evaluated using Eq. 28 are assembled in the global stiffness matrix according to the global degrees of freedom of its member nodes.

#### 3.2 Pseudo Elastic Implementation

Pseudo elastic method is an iterative procedure based on the elastic-plastic stress strain relation given in Eq. 33. All material points are assigned uniaxial material



Figure 3: Projection method to evaluate effective material parameters using experimental uniaxial tensile curve

properties, E and v in the initial iteration. After the first iteration, effective modulus,  $E_{eff}$ , at all material points are calculated using projection method as illustrated in Fig. 3. Point A corresponds to equivalent stress at a material point after the first iteration and point B is the corresponding point on experimental uniaxial tensile curve.

The effective modulus,  $E_{eff}$  (secant modulus of experimental uniaxial curve) is the slope of the line OB. Effective Poisson ratio,  $v_{eff}$  is calculated using Eq. 34. Values of  $E_{eff}$  and  $v_{eff}$  are used as modified material parameters in the next analysis. Iterations are repeated until the state of stress and strain of all material points follow the uniaxial material curve. Points D, E, F represent corresponding points on the material curve in subsequent iterations.

### 4 Elastic Plastic Singular Crack Tip Field and J integral

Hutchinson (1968a, 1968b), Rice and Rosengren (1968) showed that in elasticplastic materials within the plastic zone, elastic strains are very small and stress strain behavior reduces to pure power law even for general Ramberg-Osgood material model. Based on deformation theory of plasticity, they proposed path independent J-integral which characterizes the stress/strain field (HRR singularity) in non-linear materials as given below

$$\sigma_{ij} = \sigma_0 \left(\frac{J}{\alpha \sigma_0 \varepsilon_0 I_n r}\right)^{\frac{1}{\mathbf{m}+1}} \tilde{\sigma}_{ij}(\theta, \mathbf{m})$$

$$\varepsilon_{ij} = \alpha \varepsilon_0 \left(\frac{J}{\alpha \sigma_0 \varepsilon_0 I_n r}\right)^{\frac{\mathbf{m}}{\mathbf{m}+1}} \tilde{\varepsilon}_{ij}(\theta, \mathbf{m})$$

$$u^i = \alpha \varepsilon_0 r \left(\frac{J}{\alpha \sigma_0 \varepsilon_0 I_n r}\right)^{\frac{\mathbf{m}}{\mathbf{m}+1}} \tilde{u}^i(\theta, \mathbf{m})$$
(37)

where r and  $\theta$  are the polar coordinates centered at the crack tip,  $I_n$  is a dimensionless integration constant which is a function of strain hardening exponent, m and  $\tilde{\sigma}_{ij}$ ,  $\tilde{\epsilon}_{ij}$  and  $\tilde{u}^i$  are dimensional functions which depend on  $\theta$  and m [Shih (1983)]. J is the amplitude of the crack tip singular field. The above relations are based on the power law hardening material response given by,

$$\frac{\varepsilon}{\varepsilon_0} = \alpha \left(\frac{\sigma}{\sigma_0}\right)^{\rm m} \tag{38}$$

where  $\alpha$  is the material constant,  $\sigma_0$  is the reference yield stress,  $\varepsilon_0 = \sigma_0/E$  is the reference yield strain. For linear elastic materials, m=1 and for perfectly plastic response,  $m = \infty$ . The J-integral defines the severity of the crack tip stress/strain field both in the elastic and elastic-plastic regime. For a general cracked body subjected to remote loading, J-integral in the absence of body forces is given by

$$J = \int_{\Gamma} W dx_2 - t_i \frac{\partial u^i}{\partial x_1} d\Gamma$$
(39)

where  $\Gamma$  is any arbitrary closed contour around the crack tip starting from one crack face to the other. *W* is the strain energy density defined by,

$$W = \int \sigma_{ij} d\varepsilon_{ij} \tag{40}$$

while t is the traction vector and u is the displacement vector on path  $\Gamma$ .

### 4.1 Numerical Evaluation of J-integral

Indicial contour form of J-integral [Raju and Shiva Kumar (1990)] is given by

$$J_{x_k} = \int_{\Gamma} \left[ W n_k - \sigma_{ij} \frac{\partial u^i}{\partial x_k} n_j \right] d\Gamma \quad k, j = 1, 2$$
(41)



Figure 4: A symmetric mesh around crack tip for J-integral evaluation using decomposition method

 $n_i$  is component of unit normal vector to the path  $\Gamma$ .  $J_{x_1}$ , component of J evaluated in the x direction, is the sum of  $J_I$  and  $J_{II}$  (J-integrals associated with mode I and mode II deformation) in mixed mode cases and  $J_{x_2}$  is the product integral.

A ring of elements around the crack tip, used for J-integral evaluation with boundaries  $\Gamma_1$  and  $\Gamma_2$  is shown in Fig. 4. Applying divergence theorem and using an arbitrary function  $q(\mathbf{x})$  such that

$$q(\mathbf{x}) = 1 \text{ on } \Gamma_1$$

$$q(\mathbf{x}) = 0 \text{ on } \Gamma_2$$
(42)

contour integral is converted to an equivalent domain integral given by,

$$J_{x_k} = -\int\limits_{A} \left[ W \frac{\partial q}{\partial x_k} - \sigma_{ij} \frac{\partial u^i}{\partial x_k} \frac{\partial q}{\partial x_j} \right] dA + \int\limits_{\Gamma c^+ \cup \Gamma c^-} \left[ W n_k - \sigma_{ij} \frac{\partial u^i}{\partial x_k} n_j \right] q d\Gamma$$
(43)

Above equation contains a domain integral term defined over an area enclosed by  $\Gamma_1$ ,  $\Gamma_2$  and crack faces  $\Gamma c^+$ ,  $\Gamma c^-$  and line integral term defined over crack faces  $\Gamma c^+$  and  $\Gamma c^-$ .

Decomposition method [Raju and Shiva Kumar (1990)] has been adopted in the present study for domain integral evaluation of J in both pure and mixed mode cases. In decomposition method, non zero line integral  $J_{x_2}$  along crack faces are made to vanish by splitting stress field into symmetric and anti symmetric components. This requires symmetric mesh about the crack axis. Displacement and stress/strain fields are decomposed into symmetric and anti-symmetric components. The displacement of symmetric node sets P and P' in the domain are decomposed into symmetric and anti symmetric and anti symmetric symmetric symmetric and anti symmetric symmetric symmetric and anti symmetric symmetric and anti symmetric components using Eq. 44. Nodes from elements H, I, J, K, L, M and N form the set P and symmetric nodes from elements H', I', J',

K', L', M' and N' form the set P' (refer Fig. 4).

$$\begin{cases} u^{1} \\ u^{2} \\ s \end{cases}_{S} = \frac{1}{2} \begin{cases} u^{1P} + u^{1P'} \\ u^{2P} - u^{2P'} \end{cases}, \quad \begin{cases} u^{1} \\ u^{2} \\ s \end{cases}_{AS} = \frac{1}{2} \begin{cases} u^{1P} - u^{1P'} \\ u^{2P} + u^{2P'} \end{cases}$$
(44)

Resulting displacement field gives rise to symmetric and anti symmetric stress/strain fields. J-integral is evaluated by looping over the ring of elements.

$$J_{x_k} = \sum_{i}^{N} \left( J_{x_k} \right)_i \tag{45}$$

where N is number of elements in the domain of evaluation.

Symmetric stress field is responsible for mode I deformation and anti-symmetric stress field is responsible for mode II deformation. Evaluation of first term of Eq. 43 using symmetric field gives  $J_I = J_{Sx_1}$  and using anti symmetric field, gives  $J_{II} = J_{ASx_1}$ . Because of symmetric and anti symmetric nature of stress and displacement fields,  $J_{Sx_2}$ ,  $J_{ASx_2}$  become zero.

#### **5** Numerical Examples

Initially, the effectiveness of the present method is demonstrated using a boundary layer analysis for an edge crack problem with its outer boundary subjected to mode-I elastic displacement field. Next, various mode-I and mode-II problems are considered for the study. J-integral and stress/strain fields are evaluated using the present pseudo-elastic method and compared with the results available in literature and also with the solution obtained from non-linear finite element analysis [ANSYS (2000)].

## 5.1 Boundary layer analysis

Boundary layer approach is adopted here to predict the J-integral dominant HRR field and K dominant elastic field around the crack tip under plane stress conditions. In this method, displacement field corresponding to mode-I elastic solution is imposed on the outer boundary of the domain [Rao and Rahman (2004)]. Physical domain of the problem consists of a square plate, size 100 mm, with an edge crack, length 50 mm, terminating at centre of the plate.

Physical domain and corresponding mesh used are shown in Figs. 5(a) and 5(b). Quadrilateral mesh of 405 elements and 416 nodes are used to discretize the domain. Crack tip co-ordinate system is considered to be coincident with global co-ordinate system. Finer mesh is used around the crack tip region. EPFM asymptotic enrichment functions are used for modeling crack in the domain.



Figure 5: Geometrical details (a) Physical domain (b) Finite element model

Ramberg-Osgood material model

$$\frac{\varepsilon}{\varepsilon_0} = \frac{\sigma}{\sigma_0} + \alpha \left[\frac{\sigma}{\sigma_0}\right]^m \tag{46}$$

where  $\varepsilon_0 = \sigma_0/E$  is considered for the analysis. The values of material parameters used are Young's modulus (*E*) =200 GPa, Poisson's ratio ( $\nu$ ) =0.3, yield stress ( $\sigma_0$ ) =700 MPa, yield offset  $\alpha$  =0.1 and hardening exponent m =3.

Each element cut by the crack is partitioned into 16 smaller quadrilateral sub domains, conforming to the crack edges and a higher order 6x6 Gauss quadrature is used for the numerical integration. Penalty method is used to enforce the displacement field along the outer periphery of the domain. Results obtained from the present study are compared with both HRR singularity solution and linear elastic crack tip field.

Computed elastic-plastic stress field from the present study under plane stress condition along the radial lines at  $\theta = 0^\circ$ ,  $\theta = 45^\circ$  and  $\theta = 90^\circ$  ahead of the crack tip are presented below. The variation of normalized radial stress ( $\sigma_{rr}/\sigma_0$ ) and tangential stress ( $\sigma_{\theta\theta}/\sigma_0$ ) with respect to a normalized radial distance ( $r\sigma_0/J$ ) ahead of the crack tip for  $\theta=0^\circ$ ,  $\theta=45^\circ$  and  $\theta=90^\circ$  under plane stress condition are shown in Figs. 6, 7 and 8 respectively.

The distribution of normalized shear stress  $\sigma_{r\theta}/\sigma_0$  along radial lines  $\theta=45^\circ$  and  $\theta=90^\circ$  are also given in Figs. 9(a) and 9(b). The above graphs clearly show the



Figure 6: Variation of normalized stress ahead of crack tip along  $\theta = 0^{\circ}$  (a) radial stress (b) tangential stress



Figure 7: Variation of normalized stress ahead of crack tip along  $\theta$ =45° (a) radial stress (b) tangential stress

existence of J dominated crack tip zone as well as K dominated elastic outer zone with smooth transition between the two zones.

### 5.2 Mode I Problems

Three mode I problems, (Fig. 10) Centre Crack Tension specimen (CCT), Single Edge Crack Tension specimen (SECT) and Double Edge Crack Tension specimen (DECT), subjected to remote tension in both plane stress and plane strain conditions are considered for the pseudo-elastic analysis and fracture characterization.



Figure 8: Variation of normalized stress ahead of crack tip along  $\theta$ =90° (a) radial stress (b) tangential stress



Figure 9: Variation of normalized shear stress ahead of crack tip along (a)  $\theta$ =45° (b) 90°

Crack length (a) =50 mm, width (W) =100 mm and L/W=10 are used as geometrical dimensions for modeling the specimens. Ramberg-Osgood material model with Young's modulus (*E*) = 200 GPa, Poisson's ratio (v) = 0.3, yield stress ( $\sigma_0$ ) = 200 MPa, yield offset  $\alpha$  = 3/7 and hardening exponent m = 3, 5 and 10 is considered for the present study. Half models of the physical domain are used for analysis in case of CCT and DECT specimens, making use of symmetry and full model of the domain is used in case of SECT specimen. Remote load corresponding to 80% of the yield stress is applied. Domain is discretized with 345 elements and 372 nodes. In present study, all nodes in the rectangular region of size 100x100 mm around the



Figure 10: Geometrical configuration (a) Center crack tension specimen (b) Single edge crack tension specimen (c) Double edge crack tension specimen

crack tip are enriched with asymptotic enrichment functions given in Eq. 7.

Penalty method is adopted for enforcing symmetric boundary condition in centre crack and double edge crack problems and also for enforcing constraint equations connecting enriched degrees of freedom of crack face nodes lying on the plane of symmetry. Various domains, as shown in Fig. 11(b) are used for J-integral evaluation to check the domain independence. J-integral values, calculated for the normalized far field stress  $\sigma/\sigma_0=0.5$  and hardening exponent m=3 are compared with results evaluated from EPRI estimation scheme [Anderson (2000)] in Tab. 1 for plane strain condition. Variation of J-integral values for different domains is found to be less than 1%.

All three specimens are analyzed for various normalized boundary stresses/tractions  $(\sigma/\sigma_0)$  and for different strain hardening exponents. Normalized stresses are varied from 0.1 to 0.8 for all cases with hardening exponent, m = 3, 5 and 10.

J-integral values for different normalized far field stresses with m=3 for centre crack problem under plane strain and plane stress conditions are presented in Figs. 12(a) and 12(b). J-integral values from the present analysis are found to be in good agreement with results of EPRI estimation scheme over the considered range of loading.



Figure 11: (a) Domain discretization (b) Domains around crack tip for J-integral evaluation

Domain no.	J kJ/m <sup>2</sup> $J_{EPRI}$ =12.877	% Deviation from EPRI scheme
1	12.8654	0.09
2	12.9181	0.32
3	12.9080	0.24
4	12.8800	0.02
5	12.8365	0.31

Table 1: Variation of J-integral for different domains

J-integral variation with different strain hardening exponents for same crack configuration and loading, under plane strain and plane stress loading are presented in Figs. 13(a) and 13(b) respectively. The effect of m on J is pronounced when  $\sigma/\sigma_0$  is more than 0.6.

Next, J-integral variation with the applied load for single edge crack problem for m=3 under both plane cases are given next in Figs. 14(a) and 14(b).

Double edge crack specimen is also analyzed under identical conditions and the results under plane strain and plane stress loading are presented in Figs. 15(a) and 15(b) respectively.

Stress and strain field evaluated ahead of the crack tip using the present method are compared with HRR solution and non-linear finite element solution in Figs. 16 and 17. It is observed that the normalized stress fields  $\sigma_{22}/\sigma_0$  and  $\sigma_{11}/\sigma_0$  ob-



Figure 12: J-integral variation with the applied load for centre crack problem, m=3 (a) Plane strain (b) Plane stress



Figure 13: J-integral variation with the applied load for centre crack problem for various strain hardening exponents (a) Plane strain (b) Plane stress

tained from the present method matches well with the HRR solution closer to crack tip ( $r\sigma_0/J < 5$ ) but slightly under predict when compared with the HRR solution at larger distances from the crack tip ( $r\sigma_0/J > 5$ ). But better agreement between results of present study and non-linear finite element solution is observed at larger distances compared to HRR results at the crack tip region. Prediction of normal strains ( $\varepsilon_{11}$  and  $\varepsilon_{22}$ ) based on the present method matches well with the HRR and FEA solutions.

The present study is extended to study the effect of normalized far field stresses



Figure 14: J-integral variation with the applied load for single edge crack problem a) Plane strain b) Plane stress



Figure 15: J-integral variation with the applied load for double edge crack problem under a) Plane strain b) Plane stress

on crack tip field. The normal stress distribution  $\sigma_{22}/\sigma_0$ , ahead of the crack tip for three different loadings with m=3 for centre crack problem are plotted in Fig. 18. Invariance of the stress field with normalized far field stresses for a particular hardening exponent and  $r\sigma_0/J$  support the fact that the single parameter J can characterize the entire crack tip field in non-linear materials. It can also be observed that stress field from the present study is in close agreement with HRR solution near the crack tip region ( $r\sigma_0/J < 5$ ) and tend to deviate more from HRR solution at higher values of  $r\sigma_0/J$ .



Figure 16: Stress variation ahead of the crack tip for centre crack problem under plane stress condition with m=3 and stress ratio=0.5; a)  $\sigma_{22}/\sigma_0$  vs.  $r\sigma_0/J$ , b)  $\sigma_{11}/\sigma_0$  vs.  $r\sigma_0/J$ 



Figure 17: Strain variation ahead of the crack tip for centre crack problem under plane stress condition; a)  $\varepsilon_{22}$  vs.  $r\sigma_0/J$ , b)  $\varepsilon_{11}$  vs.  $r\sigma_0/J$ 

#### 5.3 Edge crack under shear

Next, an edge cracked specimen under mode II loading is considered [Bradford (1984)] for the elastic-plastic analysis. Physical domain consists of a square plate, size 100 mm, with an edge crack, 50 mm long, terminating at centre of the plate. Loading of the plate is varied from small scale yielding to general yielding condition. Equal and opposite traction is applied on the faces AB and CD while the  $x_2$ -displacement of the nodes on the edges AD and BC are fixed to simulate shear



Figure 18:  $\sigma_{22}/\sigma_0$  variation ahead of the crack tip for centre crack problem under plane stress condition with m=3

as shown in Fig. 19(b). One of the nodes near to middle of BC is fixed to avoid rigid body mode.



Figure 19: Edge cracked square plate subjected to shear traction. a) Physical configuration b) Mesh with applied boundary condition

Physical domain and mesh with applied boundary condition are shown in Fig. 19. Power law hardening material model with m=3 is assumed.

$$\varepsilon = \frac{\sigma}{E} \text{ for } \varepsilon \le \varepsilon_0,$$

$$\varepsilon = \varepsilon_0 \left(\frac{\sigma}{\sigma_0}\right)^m \text{ for } \varepsilon > \varepsilon_0$$
(47)

The following material property values are used for the analysis: Young's modulus (E) = 210 GPa, yield stress  $(\sigma_0) = 210$  MPa and Poisson's ratio (v) = 0.3. Von-Mises yield criteria and plane strain state are assumed.

The magnitude of applied shear traction is varied from 10.2 to 104 MPa. J-integral values, calculated using decomposition method, are presented in Fig. 20 along with reference solution and found to be in close agreement over the considered range of load application.



Figure 20: Variation of J-integral with applied shear traction

Shear stress and shear strain ahead of the crack tip for external traction of 70 MPa are compared with HRR and non-linear finite element solution and are presented in Figs. 21(a) and 21(b) respectively. Here again, stress and strain field obtained from the present method compare well with the J controlled HRR field and non-linear finite element solution thereby illustrating the capability of the method in predicting the stress-strain field for material non-linear problems within the scope of deformation theory.



Figure 21: a) Variation of shear stress  $\sigma_{12}$  vs.  $r\sigma_0/J$  ahead of crack tip b) Variation of shear strain,  $\gamma_{12}$  vs.  $r\sigma_0/J$  ahead of crack tip

## 6 Conclusions

A partition of unity finite element method coupled with pseudo elastic analysis is presented for the elastic-plastic fracture characterization of material non-linear problems. In the partition of unity finite element method enrichment functions are used to model the crack face discontinuity and crack tip asymptotic field. The applicability of the pseudo elastic method has been exploited within the partition of unity finite element method framework for solving material non-linear problems in a linear fashion.

Effectiveness of the method is illustrated considering various case studies. The resulting stress field of the boundary layer approach, along various radial lines starting from the crack tip, matched well with J dominated near tip field and K dominated far field. Fracture parameter evaluated for case of CCT, SECT, and DECT specimens showed close agreement with results of EPRI estimation scheme. Stress and strain field obtained for these cases from the present study matched well with HRR singular field for normalized distance,  $(r\sigma_0/J)$  less than 5 and tend to deviate more at larger normalized distances, but matches well with the non-linear finite element solution.

J-integral values, estimated using the decomposition method, for edge crack problem subjected to remote shear traction are found to be in good agreement with the available literature results. The asymptotic crack tip stress fields obtained by the present method for the shear are compared with both HRR singular stress fields and also with non-linear finite element solution and are found to be in good agreement.

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