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A rotation free formulation for static and free vibration analysis of thin beams using gradient smoothing technique

X.Y. Cui[1,2], G. R. Liu[2,3], G. Y. Li[1,4] and G. Zheng[1]

Abstract

In this paper, a gradient smoothed formulation is proposed to deal with a fourth-order differential equation of Bernoulli-Euler beam problems for static and dynamic analysis. Through the smoothing operation, the C^1 continuity requirement for fourth-order boundary value and initial value problems can be easily relaxed, and C^0 interpolating function can be employed to solve C^1 problems. In present thin beam problems, linear shape functions are employed to approximate the displacement field, and smoothing domains are further formed for computing the smoothed curvature and bending moment field. Numerical examples indicate that very accurate results can be yielded when a reasonable number of nodes are used.

Keywords: Numerical methods, Meshfree; Smoothed Galerkin weak form; Gradient field smoothing; Beam element

1 Introduction

In the past decades, meshfree methods have been proposed and applied in more and more fields of particular engineering and scientific problems [Liu (2002); Atluri (2004); Atluri (2005)]. Thin beam problem is a typical example of four-order differential equations where C^1 continuity is required under the Galerkin framework. A number of works for solving four-order differential equations have been investigated based on meshfree methods. An element-free Galerkin (EFG) method was proposed using moving least-squares (MLS) approximation [Belytschko et al. (1994)]. Krysl and Belytschko (1995) presented a thin plate formulation using the

[1] State Key Laboratory of Advanced Design and Manufacturing for Vehicle Body, Hunan University, Changsha, 410082, PR China

[2] Centre for Advanced Computations in Engineering Science (ACES), Department of Mechanical Engineering, National University of Singapore, 9 Engineering Drive 1, 117576 Singapore

[3] Singapore-MIT Alliance (SMA), E4-04-10, 4 Engineering Drive 3, 117576, Singapore

[4] Corresponding author. Tel: 86-731-8821717; Fax: 86-731-8822051; Email: gyli@hnu.cn

EFG method. Liu and Chen (2001) developed the EFG method for static and free vibration analyses of thin plates of complicated shape. As a truly-meshless method, the meshless local Petrov–Galerkin (MLPG) approach has been proposed as a fundamentally new concept [Atluri and Zhu (1998); Atluri and Shen (2002)]. Atluri *et al.* (1999) analyzed the thin beam problem using the MLPG method with a generalized moving least squares (GMLS) approximation. Gu and Liu (2001a) developed the MLPG formulation for static and free vibration analyses of thin plates. Long and Atluri (2002) used the MLPG to for solving the thin plate bending problems. Atluri and Shen (2005) presented both the primal and mixed MLPG methods for 4^{th} order ordinary differential equations (ODE). Andreaus et al. (2005) used the MLPG method for vibration analysis of cracked Euler-Bernoulli beams. A local point interpolation method (LPIM) is proposed for static and dynamic analysis of thin beams [Gu and Liu (2001b)]. Raju *et al.* (2004) utilized a radial basis function (RBF) approach in the MLPG method for analyzing the Euler-Bernoulli beam problems. The other works of thin beam and plate analysis include those given by Raju and Phillips (2003), Lai *et al.* (2008).

A strain smoothing stabilization procedure [Chen *et al.* (2001)] was proposed recently to compute the nodal strain by applying a divergence theorem. Using strain smoothing technique, Liu *et al.* (2005) formulated the linear conforming point interpolation method (LC-PIM) using PIM shape functions created by simple point interpolations based on a set of local nodes. As the smoothed operation in the LC-PIM based on the nodes of the mesh, it is also called node-based smoothed PIM (NS-PIM). Liu and Zhang (2008) found that NS-PIM is variationally consistent, can provide much better stress results, and more importantly can provide upper bound solution in energy norm. Recently, Liu (2008) presented a generalized gradient smoothing technique, the corresponding smoothed bilinear forms, and the smoothed Galerkin weakform. He found that the methods using gradient smoothing technique have bound properties, further relate the requirement of the assumed solution space, and can tune the stiffness of the model effectively. However, the current publications of these methods are limited to 2nd order boundary value problems, and no research for the 4th order boundary value problems, such as Bernoulli-Euler beam problems, has been reported.

In early work of the 4th order boundary value problems, displacement and slope boundary conditions were imposed at the same point. Therefore, it is natural to introduce the slope as another independent variable in the interpolation schemes in the 4th order problem. In present work, we present a way to solve the 4th order boundary value problems using simple linear point interpolation method, and a rotation-free Euler-Bernoulli beam element is proposed. The independent variables of the 4th order problems only consider displacements which are interpolated using linear interpolation function. The C^1 continuity requirement can be relaxed through a gradient smoothing technique. The problem domain can be discretized into linear elements, and a linear finite element approximation of the displacement field within each element. Based on the nodes of the elements, non-overlapping smoothing domains are formed as the integration domain, and the curvature and bending moment fields are computed in it. To show the performance of the proposed method, a series of benchmark examples have been presented and comparisons are made with analytical solutions. The excellent results have been obtained illustrating the efficiency and accuracy of the present method.

2 Elasto-static analysis

2.1 Basic Theory of Euler-Bernoulli Beam

For the given bending stiffness *EI*, the governing equation of an Euler-Bernoulli beam is expressed as fourth-order differential equation:

$$EI\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} = f \quad \text{in domain } \Omega \tag{1}$$

where w is transverse deflection and f is the distributed load over the beam.

The boundary conditions are given as follows:

$$w = w_\Gamma \text{ on } \Gamma_w, \quad -\frac{\mathrm{d}w}{\mathrm{d}x} = \theta_\Gamma \text{ on } \Gamma_\theta \tag{2}$$

$$Q = -EI\frac{\mathrm{d}^3 w}{\mathrm{d}x^3} = Q_\Gamma \text{ on } \Gamma_Q, \quad M = EI\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = M_\Gamma \text{ on } \Gamma_M, \tag{3}$$

where Q and M denote the shear force and the bending moment, respectively. Γ_w and Γ_θ are essential boundaries where deflection and slope are specified, respectively. Γ_Q and Γ_M are natural boundaries where shear force and bending moment are specified, respectively.

By multiplying with a test function δw, the weak form of Eq. (1) can be obtained

$$\int_\Omega \delta w \left(EI\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} - f \right) \mathrm{d}x = 0 \tag{4}$$

Applying Green divergence theorem, Eq. (4) can be given by

$$\int_\Omega EI\frac{\mathrm{d}^2(\delta w)}{\mathrm{d}x^2}\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\mathrm{d}x - \int_\Omega \delta w f \mathrm{d}x + nEI\delta w\frac{\mathrm{d}^3 w}{\mathrm{d}x^3}\bigg|_\Gamma - nEI\frac{\mathrm{d}(\delta w)}{\mathrm{d}x}\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\bigg|_\Gamma = 0 \tag{5}$$

where n is the unit outward normal to domain Ω, and Γ is the boundary of the domain Ω including Γ_w, Γ_θ, Γ_Q and Γ_M.

As the test function δw vanishes on the prescribed essential boundary, only the natural boundary conditions are effective and Eq. (5) can be rewritten as

$$\int_{\Omega} EI \frac{\mathrm{d}^2(\delta w)}{\mathrm{d}x^2}\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\mathrm{d}x = \int_{\Omega} \delta w f \mathrm{d}x + \tilde{Q}\delta w\big|_{\Gamma_Q} - \tilde{M}\frac{\mathrm{d}(\delta w)}{\mathrm{d}x}\bigg|_{\Gamma_M} \tag{6}$$

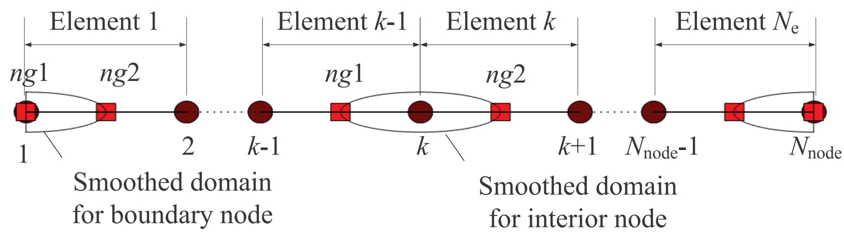


Figure 1: Problem domain discretization and the smoothing domains associated with nodes.

2.2 *Smoothed Galerkin weak form*

As shown in Fig.1, the problem domain is divided into N_e elements with a total of N_{node} nodes. For each node, a smoothing domain is formed by the nearer halves of the two neighboring elements, such that $\Omega = \Omega_1 \cup \Omega_2 \cup \ldots \cup \Omega_{N_{\mathrm{node}}}$ and $\Omega_i \cap \Omega_j = \emptyset$, $(i \neq j,\ i = 1,\ldots,N_{\mathrm{node}},\ j = 1,\ldots,N_{\mathrm{node}})$. An interior node k is sandwiched in the smoothing domain Ω_k bounded by Γ_k, which contains two points $ng1$ (the center of element k-1) and $ng2$ (the center of element k). The smoothing domain Ω_1 for the boundary node 1 is formed only by one half of element 1, and the similar is for $\Omega_{N_{\mathrm{node}}}$. In the present formulation, the displacement interpolation is element based, but the integration is based on the smoothing domains associated with the nodes. The smoothing domain Ω_k is influenced by N_k nodes that are the nodes of the elements contributing to Ω_k. For domain associated with a boundary node N_k=2; for example, nodes 1 and 2 influence Ω_1. For domain associated with an interior node N_k=3; for example, nodes k-1, k and k+1 influence Ω_k.

Using the gradient smoothing operation and Green divergence theorem, the smoothed gradient fields in the smoothing domain Ω_k are given by

$$\overline{\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}} = \frac{1}{l_k}\int_{\Omega_k}\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\mathrm{d}x = \frac{1}{l_k}\int_{\Gamma_k}\left(n(x)\frac{\mathrm{d}w}{\mathrm{d}x}\right)\mathrm{d}x \tag{7a}$$

$$\overline{\frac{\mathrm{d}^2(\delta w)}{\mathrm{d}x^2}} = \frac{1}{l_k}\int_{\Omega_k}\frac{\mathrm{d}^2(\delta w)}{\mathrm{d}x^2}\mathrm{d}x = \frac{1}{l_k}\int_{\Gamma_k}\left(n(x)\frac{\mathrm{d}(\delta w)}{\mathrm{d}x}\right)\mathrm{d}x \tag{7b}$$

where l_k is the length of the domain Ω_k, $n(x)$ is the unit outward normal to domain Ω_k, and Γ_k is the boundary of the domain Ω_k.

Using smoothed gradient in Eq. (7), we now seek for a weak form solution of the deflection field w that satisfies the following smoothed Galerkin weak form.

$$\sum_{k=1}^{N_{\text{node}}} \frac{1}{l_k} \left[\int_{\Gamma_k} \left(n(x) \frac{\mathrm{d}(\delta w)}{\mathrm{d}x} \right) \mathrm{d}x \right] \left[EI \int_{\Gamma_k} \left(n(x) \frac{\mathrm{d}w}{\mathrm{d}x} \right) \mathrm{d}x \right] = \int_{\Omega} \delta w f \mathrm{d}x + \tilde{Q} \delta w \big|_{\Gamma_Q} - \tilde{M} \frac{\mathrm{d}(\delta w)}{\mathrm{d}x} \bigg|_{\Gamma_M} \tag{8}$$

2.3 Discretized system equations

We use a linear approximation of the deflection w in each element, same as that in standard FEM, expressed as:

$$w(x) = \left\{ N_1 \quad N_2 \right\} \mathbf{d}_e \tag{9}$$

in which

$$\mathbf{d}_e = \left\{ w_1 \quad w_2 \right\}^T \tag{10}$$

$$\begin{aligned} N_1(x) &= 1 - (x - x_{e1})/l^e \\ N_2(x) &= (x - x_{e1})/l^e \end{aligned} \tag{11}$$

where w_1, w_2 denote the nodal deflections, and $l^e = x_{e2} - x_{e1}$ is the length of the element e.

Substituting Eq. (9) into Eq. (7), smoothed gradients associated with interior nodes can be given as

$$\frac{1}{l_k} \int_{\Gamma_k} \left(n(x) \frac{\mathrm{d}w}{\mathrm{d}x} \right) \mathrm{d}x = \frac{1}{l_k} \begin{Bmatrix} -N_{1,x}^{k-1} \\ N_{1,x}^{k} - N_{2,x}^{k-1} \\ N_{2,x}^{k} \end{Bmatrix}^T \begin{Bmatrix} w_{k-1} \\ w_k \\ w_{k+1} \end{Bmatrix} = \bar{\mathbf{B}}_k \mathbf{d}_k \tag{12}$$

As only deflection constraints can be imposed at the nodes, the rotation constraints will be imposed when the smoothed gradients associated with boundary nodes are being formed. We can impose the constraint $\nabla w = 0$ as follows:

(a) For boundary node at the left end, the smoothed gradient is given by

$$\frac{1}{l_1} \int_{\Gamma_1} \left(n(x) \frac{\mathrm{d}w}{\mathrm{d}x} \right) \mathrm{d}x = \frac{1}{l_1} \begin{Bmatrix} N_{1,x} \\ N_{2,x} \end{Bmatrix}^T \begin{Bmatrix} w_1 \\ w_2 \end{Bmatrix} = \bar{\mathbf{B}}_1 \mathbf{d}_1 \tag{13a}$$

(b) For boundary node at the right end, the smoothed gradient is given by

$$\frac{1}{l_{N_{node}}}\int_{\Gamma_{N_{node}}}\left(n\left(x\right)\frac{\mathrm{d}w}{\mathrm{d}x}\right)\mathrm{d}x=\frac{1}{l_{N_{node}}}\begin{Bmatrix}-N_{1,x}\\-N_{2,x}\end{Bmatrix}^{T}\begin{Bmatrix}w_{N_{node}-1}\\w_{N_{node}}\end{Bmatrix}=\bar{\mathbf{B}}_{N_{node}}\mathbf{d}_{N_{node}} \tag{13b}$$

Substituting Eqs. (9), (12) and (13) into Eq. (8), a set of discredited algebraic system equations can be obtained in the following matrix form

$$\bar{\mathbf{K}}\mathbf{d}-\mathbf{f}=0 \tag{14}$$

where $\mathbf{d}=\{w_1,w_2,\cdots,w_{N_{node}}\}^T$ is the vector of the nodal deflection at all the nodes, $\mathbf{f}$ is the force vector defined as

$$\mathbf{f}=\int_{\Omega}\mathbf{N}^T\left(\mathbf{x}\right)f\mathrm{d}x+\int_{\Gamma_Q}\mathbf{N}^T\left(\mathbf{x}\right)\tilde{Q}\mathrm{d}\Gamma_Q-\int_{\Gamma_M}\mathbf{N}_{,x}^T\left(\mathbf{x}\right)\tilde{M}\mathrm{d}\Gamma_M \tag{15}$$

and $\bar{\mathbf{K}}$ is the (global) smoothed stiffness matrix of present method, it is assembled in the form of

$$\bar{\mathbf{K}}=\sum_{k=1}^{N_{node}}\bar{\mathbf{K}}_{(k)} \tag{16}$$

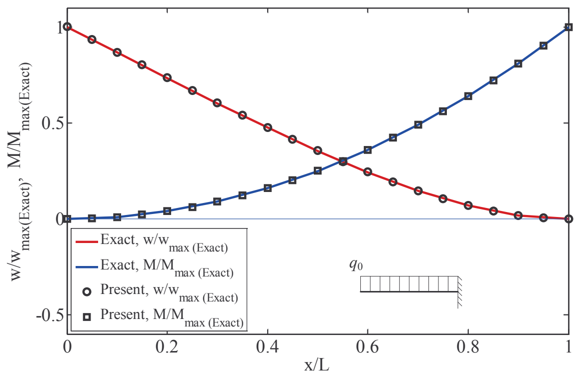
where the summation means an assembly process same as the practice in the FEM, and $\bar{\mathbf{K}}_{(k)}$ is the stiffness matrix associated with Ω_k that is computed using

$$\bar{\mathbf{K}}_{(k)}=EI\left(\bar{\mathbf{B}}_k\right)^T\bar{\mathbf{B}}_k l_k \tag{17}$$

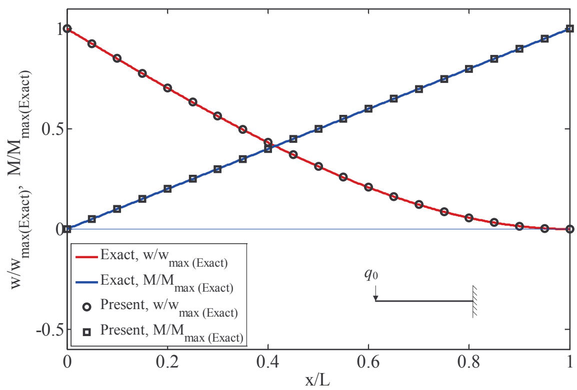
2.4 Numerical examples

A thin beam of length L=1.0 subjected to different boundary conditions is considered in this subsection. The parameters are taken as EI=1.0 and q_0=1.0. Fig.2 shows the comparison between the deflection and moment results for cantilever beam under these three kinds of loads calculated analytically and using present rotation free beam element. The plots show excellent agreements between the exact and numerical results for cantilever beams under uniformly distributed load, concentrated load and linear distributed load.

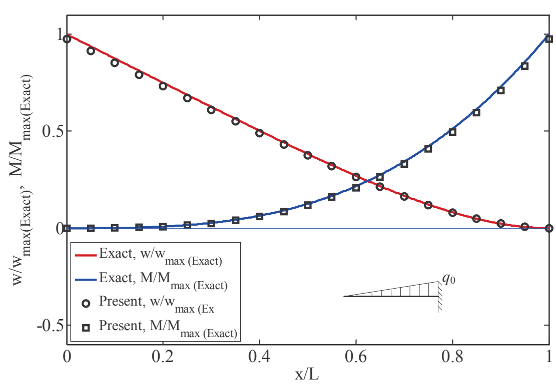
To verify the effectiveness of the present rotation free beam element for different boundary conditions, pinned-pinned thin beams and fixed-fixed beams with uniformly distributed load and concentrated load are also studied here. Fig.3 shows the comparisons between the deflection and moment results calculated analytically and using present method for pinned-pinned thin beam. Fig.4 shows the comparisons between the deflection and moment results calculated analytically and using present method for fixed-fixed thin beam. For all cases, excellent agreements results between the exact and numerical results are observed.



(a)



(b)



(c)

Figure 2: Deflections and moments of a cantilever thin beam under different loads: (a) Uniformly distributed load; (b) Concentrated load; (c) Linear distributed load.

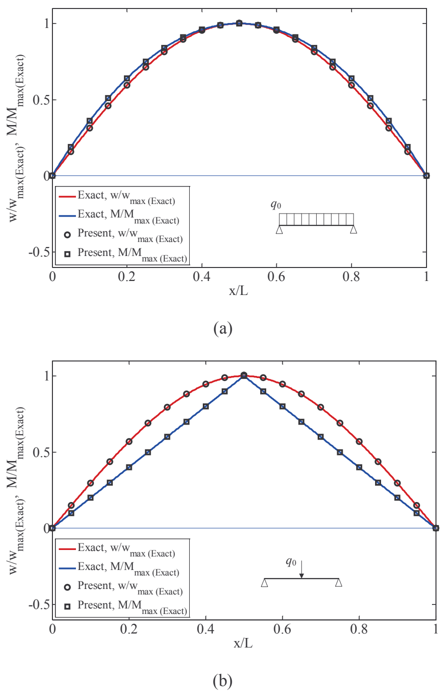


Figure 3: Deflections and moments of a pinned-pinned thin beam under different loads: (a) Uniformly distributed load; (b) Concentrated load.

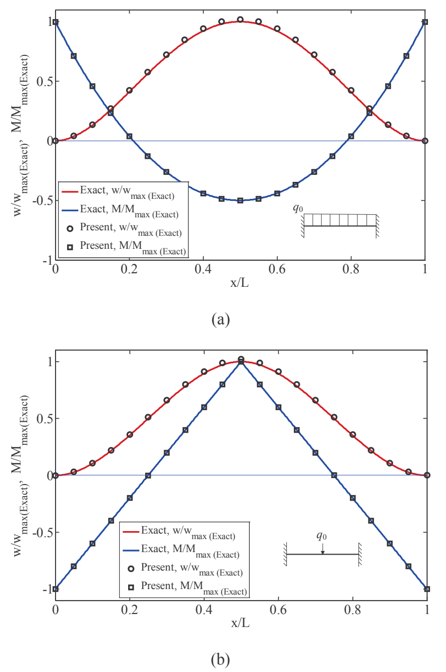


Figure 4: Deflections and moments of a fixed-fixed thin beam under different loads: (a) Uniformly distributed load; (b) Concentrated load.

3 Free vibration analysis

3.1 Smoothed Galerkin weak form

The governing equation for free vibration of the thin beam is given by

$$EI\frac{\mathrm{d}^4 w(x,t)}{\mathrm{d}x^4} - \rho A_0 \frac{\mathrm{d}^2 w(x,t)}{\mathrm{d}t^2} = 0 \quad \text{in domain } \Omega \tag{18}$$

where $w(x,t)$ is the deflection of the beam, $\rho$is the mass density, and A_0 is the cross section area.

The boundary conditions are usually the same form of Eqs. (2) and (3). The discretized dynamic equilibrium equation is obtained using the smoothed Galerkin weak form

$$\sum_{k=1}^{N_{\text{node}}} \frac{1}{l_k} \left[\int_{\Gamma_k} \left(n(x) \frac{\mathrm{d}(\delta w)}{\mathrm{d}x} \right) \mathrm{d}x \right] \left[EI \int_{\Gamma_k} \left(n(x) \frac{\mathrm{d}w}{\mathrm{d}x} \right) \mathrm{d}x \right] = \int_{\Omega} \rho A_0 \delta w \frac{\mathrm{d}^2 w(x,t)}{\mathrm{d}t^2} \mathrm{d}x \quad (19)$$

Substituting Eqs. (9), (12) and (13) into Eq. (19) yields

$$\bar{\mathbf{K}}\mathbf{d} - \mathbf{M}\ddot{\mathbf{d}} = 0 \quad (20)$$

where $\bar{\mathbf{K}}$ has the same expressions as it given in Eq. (16), and mass matrix $\mathbf{M}$ is given by

$$\mathbf{M} = \operatorname{diag}\{m_1 \ldots m_k \ldots m_{\text{node}}\} \quad (21)$$

where m_k is the mass of the smoothing domain corresponding to node k and given by

$$m_k = \rho A_0 l_k \quad (22)$$

In the free vibration analysis, a general solution to Eq. (20) can be written as

$$\mathbf{d} = \mathbf{Z}_p \mathrm{e}^{i\omega_p t} \quad (23)$$

where ω is the frequency. Substituting Eq. (23) into Eq. (20) yields the eigen equation

$$\left(\bar{\mathbf{K}} - \omega_p^2 \mathbf{M}\right) \mathbf{Z}_p = 0 \quad (24)$$

where ω_p is the natural frequency associated with the pth mode and $\mathbf{Z}_p$ is the modal vector.

3.2 Numerical examples

For free vibration analysis, thin beams with different boundary conditions are considered here. The geometrical parameters and material parameters are same as those given in subsection 2.4 and 81 uniformly distributed nodes are used. The nondimensional free vibration constant is defined as $\beta_i = \sqrt{\omega_i \sqrt{\rho/EI}}$.The first three free vibration modes are shown in Fig.5. Table 1 shows the comparison between the first eight-mode calculated analytically and using present method. It can be observed that the results obtained by the present rotation free beam element are in very good agreement with analytical solutions.

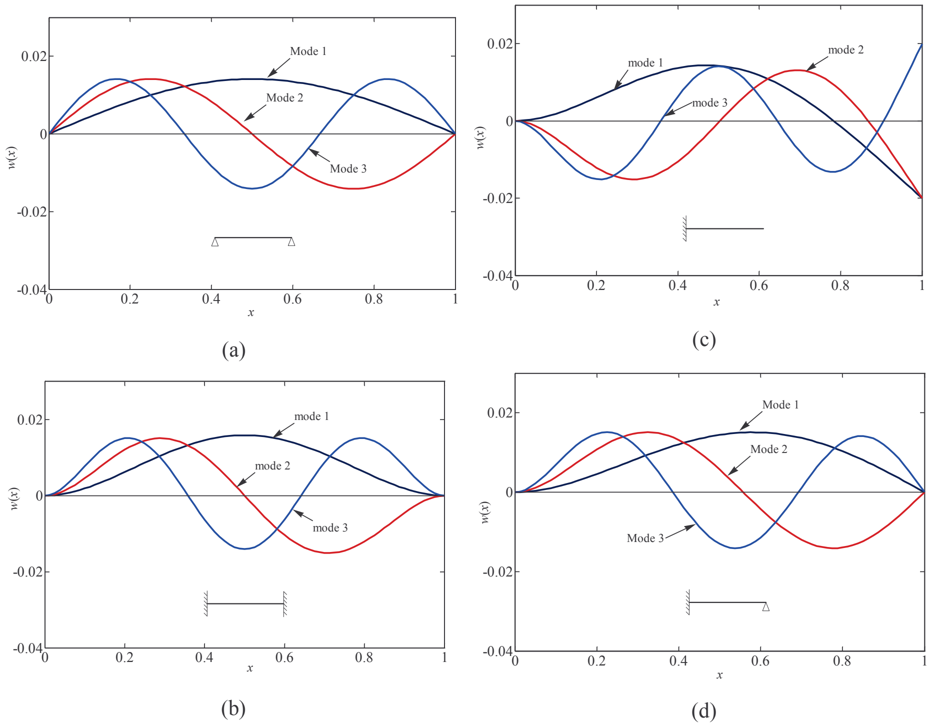


Figure 5: Free vibration modes of thin beams with different boundary conditions: (a) pinned-pinned; (b) fixed-fixed; (c) fixed-free; (d) fixed-pinned.

Table 1: Comparison of thin beam vibration constants β_i under various boundary conditions.

Modes	Pinned-Pinned		Fixed-Fixed		Fixed-Free		Pinned-Fixed	
	Analytical	Present	Analytical	Present	Analytical	Present	Analytical	Present
1	3.14159	3.14139	4.73004	4.72852	1.87510	1.87498	3.92699	3.92591
2	6.28318	6.28157	7.85398	7.84764	4.69406	4.69252	7.06858	7.06531
3	9.42477	9.41933	10.99557	10.98226	7.85398	7.84921	10.21018	10.20122
4	12.56636	12.55346	14.13717	14.11103	10.99557	10.98219	13.35177	13.33282
5	15.70795	15.68274	17.27876	17.23366	14.13717	14.11103	16.49336	16.45890
6	18.84954	18.80598	20.42035	20.34892	17.27876	17.23366	19.63495	19.57828
7	21.99113	21.92197	23.56194	23.45562	20.42035	20.34892	22.77655	22.68974
8	25.13272	25.02951	26.70354	26.55258	23.56194	23.45562	25.91814	25.79211

4 Conclusion

In this paper, a gradient field smoothed formulation is proposed to deal with a fourth-order differential equation of Bernoulli-Euler beam problems. Through the smoothing operation, the C^1 continuity requirement for fourth-order boundary value and initial value problems can be relaxed. The problem domain can be first discretized into a set of linear elements, and a linear interpolation is used for approximating the displacement field within each element. The discretized system equations are obtained using the smoothed Galerkin weak form, and the numerical integration is applied based on the smoothing domains in which gradients of the field variables are smoothed. Because of only displacements as independent variable at per node, the rotation constraints are imposed when the smoothed gradients associated with boundary nodes are being formed. Numerical examples of static and free vibration analysis for thin beams under various loads and boundary conditions are analyzed to demonstrate the effectiveness and stability of present method. It is found that the present method is very easy to implement and accurate for static and free vibration analysis of thin beams.

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