 CMES, vol.38, no.1, pp.25-38, 2008

A New Mathematical Modeling of Maxwell Equations: Complex Linear Operator and Complex Field

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Abstract: In this paper a complex matrix operator and a complex field are used to express the Maxwell equations, of which the complex field embraces all field variables and the matrix operator embraces the time and space differential operators. By left applying the operator on the complex field one can get all the *four* Maxwell equations, which are usually expressed by the vector form. The new formulation matches the Lorenz gauge condition, and its mathematical advantage is that it can incorporate the Maxwell equations into a *single* equation. The introduction of four-potential is possible only under the Lorenz gauge. In terms of the γ-ring, we found that the Maxwell equations bear certain similarity with the Dirac equation. However, we also point out their differences.

Keywords: Maxwell equations, Lorenz gauge condition, Wave equations, Jordan algebra, Complex operator, Complex field

1 Introduction

The modern mathematical formulation of Maxwell equations is due to Oliver Heaviside and Willard Gibbs, who formulated Maxwell's equations using the vector calculus:

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial x^0}, \tag{1}$$

$$\nabla \cdot \mathbf{E} = 4\pi\rho, \ \ \nabla \times \mathbf{B} = \frac{\partial \mathbf{E}}{\partial x^0} + \frac{4\pi}{c}\mathbf{J}. \tag{2}$$

In above, $x^0 := ct$, $\nabla\cdot$ is the divergence of, $\nabla\times$ is the curl of, $\mathbf{E}$ is the electric field, $\mathbf{B}$ is the magnetic field, ρ is the electric charge density, $\mathbf{J}$ is the electric vector current density, and c is the speed of light in vacuum. The vector form produces a symmetric mathematical representation that reinforced the perception of physical symmetries between electric and magnetic fields. In fact, it turns out to be an

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important clue as to the mathematical structure of Maxwell's equations [Kato and Singleton (2002)].

The electric and magnetic fields can be represented by a scalar potential ϕ and a vector potential $\mathbf{A}$ through the following relations:

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial x^0} - \nabla\phi, \tag{3}$$

$$\mathbf{B} = \nabla \times \mathbf{A}. \tag{4}$$

In order to obtain the uncoupled equations:

$$\frac{\partial^2 \phi}{\partial (x^0)^2} - \nabla^2 \phi = 4\pi\rho, \tag{5}$$

$$\frac{\partial^2 \mathbf{A}}{\partial (x^0)^2} - \nabla^2 \mathbf{A} = \frac{4\pi}{c}\mathbf{J}, \tag{6}$$

one needs to impose the Lorenz gauge condition [Jackson and Okun (2001); Jackson (2002)]:

$$\frac{\partial \phi}{\partial x^0} + \nabla \cdot \mathbf{A} = 0. \tag{7}$$

Eqs. (5) and (6) are wave equations for the scalar potential ϕ and the vector potential $\mathbf{A}$ in the Lorenz family of gauges.

There are several representations of the Maxwell equations, which emphasize the algebraic aspect of the formulations: the complex quaternion form [Majernik (1999)], the bivector form [Hestenes (1966)], the biparavector form [Misner, Thorne and Wheeler (1973); Jackson (1998); Baylis (1998)], the biquaternion form [Gsponer and Hurni (2001); Gsponer (2002)], the differential form [Castillo, Koning, Rieben and White (2004)], and the real Jordan algebra and Lie algebra [Liu (2004)], etc.

Maxwell equations is a prototype of many classical field theories. The different representations of the Maxwell equations are mathematically isomorphic. However, each specific form may bring out a further understanding of the Maxwell equations. In the computational electromagnetism [Reitich and Tamma (2004); Young, Chen and Wong (2005)] directly solving the Maxwell equations by the vector form is still the main stream. In this paper we are going to develop a new mathematical modeling of the Maxwell equations. The Maxwell equations are indispensible in the behavior study of material under electric and magnetic fields [Liu and Ku (2005); Kakimoto and Liu (2006); Ma and Walker (2006); Seiller (2007); Johnson and Owen (2007)]. However, the application of the present formulation of the Maxwell equations to the computation will not be discussed here.

This paper formulates the Maxwell equations in terms of a complex Lie operator and a complex field, which is organized as follows. In Section 2 we introduce a real Jordan algebra and a complex Jordan algebra. In Section 3 we transform the complex Jordan algebra to a Lie algebra of 4×4 complex matrix and use this algebraic system to construct an $\{\mathbf{m}\cdot\gamma\}$ ring as a geometric algebra, where $\mathbf{m}$ is a three-dimensional real vector and γ is a 4×4 matrix realization of the famous Pauli matrices. In Section 4 we use the complex Lie algebra system to express the Maxwell equations into a single equation, derive easily the wave equations, and compare the γ-matrices representation of the Maxwell equations with the Dirac equation. In Section 5 we draw some conclusions.

2 Complex Jordan algebra

Liu (2000, 2002) has considered an algebraic system of four real numbers with the following product rule:

$$\mathbf{x}\mathbf{y} = (x^0+\mathbf{x}^s)(y^0+\mathbf{y}^s) := x^0y^0+\mathbf{x}^s\cdot\mathbf{y}^s+x^0\mathbf{y}^s+y^0\mathbf{x}^s+\mathbf{x}^s\times\mathbf{y}^s. \tag{8}$$

Here, $\mathbf{x}=x^0+\mathbf{x}^s$, $\mathbf{y}=y^0+\mathbf{y}^s\in\mathbb{M}:=\mathbb{R}\oplus\mathbb{R}^3$. The algebra with the above product rule is non-commutative as well as non-associative, but it still preserves the Jacobi identity [Schafer (1995)] and other properties:

$$[\mathbf{x},\mathbf{y}] = \frac{1}{2}(\mathbf{x}\mathbf{y}-\mathbf{y}\mathbf{x}) = \mathbf{x}^s\times\mathbf{y}^s\neq\mathbf{0}, \tag{9}$$

$$[\mathbf{x},\mathbf{y},\mathbf{z}] = \frac{1}{2}\{(\mathbf{x}\mathbf{y})\mathbf{z}-\mathbf{x}(\mathbf{y}\mathbf{z})\} = \mathbf{x}^s\cdot\mathbf{y}^s\mathbf{z}^s-\mathbf{z}^s\cdot\mathbf{y}^s\mathbf{x}^s\neq\mathbf{0}, \tag{10}$$

$$[\mathbf{x}^2,\mathbf{y},\mathbf{x}] = \mathbf{0}. \tag{11}$$

In terms of

$$\hat{\mathbf{x}}^s := \begin{bmatrix} 0 & -x^3 & x^2 \\ x^3 & 0 & -x^1 \\ -x^2 & x^1 & 0 \end{bmatrix}, \tag{12}$$

we can express Eq. (8) as

$$\mathbf{x}\mathbf{y} = x^0y^0+\mathbf{x}^s\cdot\mathbf{y}^s+x^0\mathbf{y}^s+y^0\mathbf{x}^s+\hat{\mathbf{x}}^s\mathbf{y}^s. \tag{13}$$

It has been noted by Liu (2002) that the above algebra does not have a similar set of bases as that of the real quaternions. However, if we allow the imaginary number i to enter this algebraic system, then we have a set of bases $\{1,\sigma_1,\sigma_2,\sigma_3\}$ with the following relations:

$$\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = 1, \tag{14}$$

$$\sigma_1\sigma_2 = -\sigma_2\sigma_1 = i\sigma_3, \;\; \sigma_2\sigma_3 = -\sigma_3\sigma_2 = i\sigma_1, \;\; \sigma_3\sigma_1 = -\sigma_1\sigma_3 = i\sigma_2, \tag{15}$$

$$\sigma_1\sigma_2\sigma_3 = i\sigma_0. \tag{16}$$

A famous algebra of this sort is the Pauli spin matrices:

$$\sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \;\; \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \;\; \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \;\; \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{17}$$

where σ_0 denotes the unit element 1.

The four-vector $\mathbf{x}$ is of the form $x^0 + x^1\sigma_1 + x^2\sigma_2 + x^3\sigma_3$, where $x^0, \ldots, x^3$ are real numbers. When two four-vectors are multiplied together using the normal algebraic multiplication rule with the relations (14)-(16), the product is given as follows:

$$\mathbf{xy} = x^0y^0 + \mathbf{x}^s \cdot \mathbf{y}^s + x^0\mathbf{y}^s + y^0\mathbf{x}^s + i\mathbf{x}^s \times \mathbf{y}^s, \tag{18}$$

where as before $\mathbf{x}^s = x^1\sigma_1 + x^2\sigma_2 + x^3\sigma_3$ and $\mathbf{y}^s = y^1\sigma_1 + y^2\sigma_2 + y^3\sigma_3$ denote respectively the spatial parts of $\mathbf{x}$ and $\mathbf{y}$. This algebra has been named the complex four-vector algebra. More frequently, it is called the Pauli algebra, the three-dimensional Clifford algebra or geometric algebra by Hestenes (2003). However, in order to stress its algebraic behavior we name it the complex Jordan algebra. The above right-hand side includes three different objects: $x^0y^0 + \mathbf{x}^s \cdot \mathbf{y}^s$ is a scalar, $x^0\mathbf{y}^s + y^0\mathbf{x}^s$ is a vector, and $i\mathbf{x}^s \times \mathbf{y}^s$ is a bi-vector. It deserves to note that even the product rule in Eq. (18) bears certain similarity to the product rule in Eq. (13), $i\mathbf{x}^s \times \mathbf{y}^s$ is a complex bi-vector in Eq. (18), but $\mathbf{x}^s \times \mathbf{y}^s$ is a real vector in Eq. (13).

3 Complex Lie algebra

Upon employing the following isomorphisms:

$$\mathbf{y} \mapsto \mathbf{\Psi}_y := \begin{bmatrix} \mathbf{y}^s \\ y^0 \end{bmatrix}, \tag{19}$$

$$\mathbf{x} \mapsto \mathbf{M}_x := \begin{bmatrix} x^0\mathbf{I}_3 + i\hat{\mathbf{x}}^s & \mathbf{x}^s \\ (\mathbf{x}^s)^{\mathrm{T}} & x^0 \end{bmatrix}, \tag{20}$$

then the product rule in Eq. (18) can be neatly expressed as

$$\mathbf{M}_x\mathbf{\Psi}_y = \begin{bmatrix} x^0\mathbf{I}_3 + i\hat{\mathbf{x}}^s & \mathbf{x}^s \\ (\mathbf{x}^s)^{\mathrm{T}} & x^0 \end{bmatrix} \begin{bmatrix} \mathbf{y}^s \\ y^0 \end{bmatrix} = \begin{bmatrix} x^0\mathbf{y}^s + y^0\mathbf{x}^s + i\mathbf{x}^s \times \mathbf{y}^s \\ x^0y^0 + \mathbf{x}^s \cdot \mathbf{y}^s \end{bmatrix}. \tag{21}$$

It can be proved that for all $\mathbf{x} \in \mathbb{R}^4$ the corresponding $\mathbf{M}_x \in \mathbb{C}^{4\times 4}$ forms a complex Lie algebra.

In terms of the following γ matrices:

$$\gamma_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \gamma_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \tag{22}$$

$$\gamma_2 = \begin{bmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 1 \\ -i & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad \gamma_3 = \begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \tag{23}$$

where γ_0 denotes the unit element, $\mathbf{M}_x$ can be expressed by

$$\mathbf{M}_x = x^0\gamma_0 + x^1\gamma_1 + x^2\gamma_2 + x^3\gamma_3 = \mathbf{x}\cdot\gamma. \tag{24}$$

The γ matrices are Hermitian, and are the 4×4 complex matrices realization of the 2×2 Pauli matrices, satisfying the following relations as that for the Pauli matrices:

$$\gamma_1^2 = \gamma_2^2 = \gamma_3^2 = \gamma_0, \tag{25}$$

$$\gamma_1\gamma_2 = -\gamma_2\gamma_1 = i\gamma_3, \quad \gamma_2\gamma_3 = -\gamma_3\gamma_2 = i\gamma_1, \quad \gamma_3\gamma_1 = -\gamma_1\gamma_3 = i\gamma_2, \tag{26}$$

$$\gamma_1\gamma_2\gamma_3 = i\gamma_0. \tag{27}$$

Compare them with Eqs. (14)-(16) for the Pauli matrices $\sigma_0, \sigma_1, \sigma_2, \sigma_3$.

For the three-dimensional real vectors $\mathbf{m}$ and $\mathbf{F}$ we can construct a four-vector that is invariant under space reversal:

$$\mathbf{\Psi}_{mF} = \begin{bmatrix} i\hat{\mathbf{m}} & \mathbf{m} \\ \mathbf{m}^{\mathrm{T}} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{F} \\ 0 \end{bmatrix} = \begin{bmatrix} i\mathbf{m}\times\mathbf{F} \\ \mathbf{m}\cdot\mathbf{F} \end{bmatrix}, \tag{28}$$

where $\hat{\mathbf{m}}$ is defined by Eq. (12). This operation combines the usual inner product and cross product of three-dimensional vectors into a single formula. The i before $\mathbf{m}\times\mathbf{F}$ is used to stress that $\mathbf{m}\times\mathbf{F}$ is an axial vector (bi-vector) as that done in Eq. (18), which together with $\mathbf{m}\cdot\mathbf{F}$ are both invariant under space reversal, i.e., invariant under $\mathbf{m} \mapsto -\mathbf{m}$ and $\mathbf{F} \mapsto -\mathbf{F}$.

From the following identities:

$$\mathbf{m}\mathbf{m}^{\mathrm{T}} - \hat{\mathbf{m}}^2 = \mathbf{m}\cdot\mathbf{m}\mathbf{I}_3, \tag{29}$$

$$\hat{\mathbf{m}}\mathbf{m} = \mathbf{0}, \quad \mathbf{m}^{\mathrm{T}}\hat{\mathbf{m}} = \mathbf{0}^{\mathrm{T}}, \tag{30}$$

we can prove that

$$\boldsymbol{\Psi}_{mF}^{\dagger}\boldsymbol{\Psi}_{mF} = \mathbf{m}\cdot\mathbf{mF}\cdot\mathbf{F}, \tag{31}$$

where † denotes the complex conjugate and transpose of the complex four-vector. For every unit vector $\mathbf{m}$ with $\|\mathbf{m}\| = 1$, the four-vector $\boldsymbol{\Psi}_{mF}$ has a magnitude of $\|\mathbf{F}\|$.

The given vector $\mathbf{m}$ together with $\boldsymbol{\Psi}_{mF}$ provide a coordinate-free description of $\mathbf{F}$. Since the temporal components of both $\mathbf{m}$ and $\mathbf{F}$ are zeros, Eq. (28), by Eq. (24), can also be written as

$$\boldsymbol{\Psi}_{mF} = \mathbf{M}_m\boldsymbol{\Psi}_F = \mathbf{m}\cdot\gamma\boldsymbol{\Psi}_F, \tag{32}$$

where $\boldsymbol{\Psi}_F$ itself as defined by Eq. (19) can be expressed by

$$\boldsymbol{\Psi}_F = (\mathbf{F}\cdot\gamma)\boldsymbol{\Psi}_0 = \begin{bmatrix} i\hat{\mathbf{F}} & \mathbf{F} \\ \mathbf{F}^{\mathrm{T}} & 0 \end{bmatrix}\begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{F} \\ 0 \end{bmatrix} \tag{33}$$

with $\boldsymbol{\Psi}_0 = (0,0,0,1)^{\mathrm{T}}$ a unit four-vector. Inserting the above equation for $\boldsymbol{\Psi}_F$ into Eq. (32) we get

$$\boldsymbol{\Psi}_{mF} = (\mathbf{m}\cdot\gamma)(\mathbf{F}\cdot\gamma)\boldsymbol{\Psi}_0 = \begin{bmatrix} i\mathbf{m}\times\mathbf{F} \\ \mathbf{m}\cdot\mathbf{F} \end{bmatrix}. \tag{34}$$

From Eqs. (25)-(27) it follows that

$$(\mathbf{m}\cdot\gamma)(\mathbf{F}\cdot\gamma) = \boldsymbol{\Psi}_{mF}\cdot\gamma = \mathbf{m}\cdot\mathbf{F}\gamma_0 + i(\mathbf{m}\times\mathbf{F})\cdot\gamma. \tag{35}$$

The set of three-dimensional real vectors $\mathbb{R}^3 := \{\mathbf{m} = (m_1,m_2,m_3)^{\mathrm{T}}, m_i \in \mathbb{R}, i = 1,2,3\}$ with the representation $\{\mathbf{m}\cdot\gamma\}$ forms a ring under the operations of addition and the above multiplication rule in Eq. (35). That is,

$$\mathbf{m}_1\cdot\gamma + \mathbf{m}_2\cdot\gamma = \mathbf{m}_2\cdot\gamma + \mathbf{m}_1\cdot\gamma, \tag{36}$$

$$\mathbf{m}_1\cdot\gamma + (\mathbf{m}_2\cdot\gamma + \mathbf{m}_3\cdot\gamma) = (\mathbf{m}_1\cdot\gamma + \mathbf{m}_2\cdot\gamma) + \mathbf{m}_3\cdot\gamma, \tag{37}$$

$$\mathbf{0}\cdot\gamma + \mathbf{m}\cdot\gamma = \mathbf{m}\cdot\gamma + \mathbf{0}\cdot\gamma = \mathbf{m}\cdot\gamma, \tag{38}$$

$$\mathbf{m}\cdot\gamma + (-\mathbf{m}\cdot\gamma) = \mathbf{0}\cdot\gamma, \tag{39}$$

$$\mathbf{m}_1\cdot\gamma(\mathbf{m}_2\cdot\gamma\mathbf{m}_3\cdot\gamma) = (\mathbf{m}_1\cdot\gamma\mathbf{m}_2\cdot\gamma)\mathbf{m}_3\cdot\gamma, \tag{40}$$

$$\mathbf{m}_1\cdot\gamma(\mathbf{m}_2\cdot\gamma + \mathbf{m}_3\cdot\gamma) = \mathbf{m}_1\cdot\gamma\mathbf{m}_2\cdot\gamma + \mathbf{m}_1\cdot\gamma\mathbf{m}_3\cdot\gamma, \tag{41}$$

$$(\mathbf{m}_1\cdot\gamma + \mathbf{m}_2\cdot\gamma)\mathbf{m}_3\cdot\gamma = \mathbf{m}_1\cdot\gamma\mathbf{m}_3\cdot\gamma + \mathbf{m}_2\cdot\gamma\mathbf{m}_3\cdot\gamma. \tag{42}$$

This ring structure is first proved here for the γ matrices.

4 A new formulation

4.1 The Maxwell Equations

Let us consider the following complex matrix linear differential operator:

$$\mathbb{L}_c = \{\mathbb{L}\} + i[\mathbb{L}] = \begin{bmatrix} \mathbf{I}_3 \frac{\partial}{\partial x^0} + i\hat{\nabla} & \nabla \\ \nabla^{\mathrm{T}} & \frac{\partial}{\partial x^0} \end{bmatrix} \tag{43}$$

with

$$\{\mathbb{L}\} = \begin{bmatrix} \mathbf{I}_3 \frac{\partial}{\partial x^0} & \nabla \\ \nabla^{\mathrm{T}} & \frac{\partial}{\partial x^0} \end{bmatrix}, \tag{44}$$

$$[\mathbb{L}] = \begin{bmatrix} \hat{\nabla} & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix} \tag{45}$$

being symmetric and skew-symmetric operators. Upon comparing Eqs. (43) and (20), it is obvious that $\mathbb{L}_c$ is an operatorization of $\mathbf{M}_x$.

In order to derive the Maxwell equations, let us write

$$-\mathbb{F}^\star = -\mathbb{E} + i\mathbb{B} = \mathbb{L}_c\mathbb{A} = \begin{bmatrix} \nabla\phi + \frac{\partial \mathbf{A}}{\partial x^0} + i\nabla\times\mathbf{A} \\ \frac{\partial\phi}{\partial x^0} + \nabla\cdot\mathbf{A} \end{bmatrix}, \tag{46}$$

where $\mathbb{F}^\star$ denotes the complex conjugate of the following complex electromagnetic field:

$$\mathbb{F} = \mathbb{E} + i\mathbb{B} = \begin{bmatrix} \mathbf{E} \\ E^0 \end{bmatrix} + i\begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix}, \tag{47}$$

and

$$\mathbb{A} := \begin{bmatrix} \mathbf{A} \\ \phi \end{bmatrix} \tag{48}$$

is a four-potential.

By equating the real and imaginary parts in Eq. (46) we obtain Eqs. (3) and (4), as well as

$$E^0 = -\frac{\partial\phi}{\partial x^0} - \nabla\cdot\mathbf{A}. \tag{49}$$

Then, under the Lorenz gauge condition (7), i.e., $E^0 = 0$,

$$\mathbb{L}_c\mathbb{F} = \mathbb{L}_c(\mathbb{E} + i\mathbb{B}) = 4\pi\bar{\mathbb{J}} \tag{50}$$

represents the Maxwell Eqs. (1) and (2), where

$$\bar{\mathbb{J}} = \begin{bmatrix} \frac{-\mathbf{J}}{c} \\ \rho \end{bmatrix} \tag{51}$$

is the conjugate of the four-current density $\mathbb{J} = (\mathbf{J}/c, \rho)^{\mathrm{T}}$.

Eq. (50) is indeed a single representation of the Maxwell equations. This achievement requires us to consider a complex linear differential operator and a complex field in the formulation. It may be convenient for us to find the field solutions with

$$\begin{aligned} \mathbb{F} &= 4\pi\mathbb{L}_c^{-1}\bar{\mathbb{J}} \\ &= 4\pi\left((\{\mathbb{L}\} + [\mathbb{L}]\{\mathbb{L}\}^{-1}[\mathbb{L}])^{-1} - i\{\mathbb{L}\}^{-1}[\mathbb{L}](\{\mathbb{L}\} + [\mathbb{L}]\{\mathbb{L}\}^{-1}[\mathbb{L}])^{-1}\right)\bar{\mathbb{J}}. \end{aligned} \tag{52}$$

Hence, by Eq. (47) we have

$$\mathbb{E} = 4\pi(\{\mathbb{L}\} + [\mathbb{L}]\{\mathbb{L}\}^{-1}[\mathbb{L}])^{-1}\bar{\mathbb{J}}, \tag{53}$$

$$\mathbb{B} = -4\pi\{\mathbb{L}\}^{-1}[\mathbb{L}](\{\mathbb{L}\} + [\mathbb{L}]\{\mathbb{L}\}^{-1}[\mathbb{L}])^{-1}\bar{\mathbb{J}}. \tag{54}$$

How to obtain the inverse operators $\{\mathbb{L}\}^{-1}$ and $(\{\mathbb{L}\} + [\mathbb{L}]\{\mathbb{L}\}^{-1}[\mathbb{L}])^{-1}$ is a task in the near future. A further studing on the above formulation may provide us a useful method to solve the Maxwell equations.

4.2 Wave Equations

There are two ways to derive the wave equation. For self-content we first prove in the Appendix that the two operators $\{\mathbb{L}\}$ and $[\mathbb{L}]$ are commutative as made by Liu (2004), which leads to $\mathbb{L}_c$ and $\mathbb{L}_c^\star$ commutative and $\mathbb{L}_c\mathbb{L}_c^\star = \mathbb{L}_c^\star\mathbb{L}_c$ a real operator.

Substituting Eq. (43) for $\mathbb{L}_c$ and its complex conjugate for $\mathbb{L}_c^\star$ into $\mathbb{L}_c\mathbb{L}_c^\star$ we obtain

$$\begin{aligned} \mathbb{L}_c\mathbb{L}_c^\star &= (\{\mathbb{L}\} + i[\mathbb{L}])(\{\mathbb{L}\} - i[\mathbb{L}]) \\ &= \{\mathbb{L}\}^2 + [\mathbb{L}]^2 + i([\mathbb{L}]\{\mathbb{L}\} - \{\mathbb{L}\}[\mathbb{L}]) = \{\mathbb{L}\}^2 + [\mathbb{L}]^2. \end{aligned} \tag{55}$$

The imaginary part is zero due to Eq. (A.11) derived in the Appendix. By the same token we have

$$\mathbb{L}_c^\star\mathbb{L}_c = (\{\mathbb{L}\} - i[\mathbb{L}])(\{\mathbb{L}\} + i[\mathbb{L}]) = \{\mathbb{L}\}^2 + [\mathbb{L}]^2. \tag{56}$$

Therefore, we have proved that $\mathbb{L}_c$ and $\mathbb{L}_c^\star$ are commutative and that $\mathbb{L}_c\mathbb{L}_c^\star = \mathbb{L}_c^\star\mathbb{L}_c$ is a real operator.

Next, we applying the operator $\mathbb{L}_c^\star$ to Eq. (46) and taking the complex conjugate of Eq. (50) to obtain

$$\mathbb{L}_c^\star \mathbb{L}_c \mathbb{A} = -\mathbb{L}_c^\star \mathbb{F}^\star = -4\pi \bar{\mathbb{J}}, \tag{57}$$

where $\bar{\mathbb{J}}^\star = \bar{\mathbb{J}}$ since $\bar{\mathbb{J}}$ is a real four-vector. Inserting Eq. (56) into the above equation we thus have a wave equation for $\mathbb{A}$:

$$(\{\mathbb{L}\}^2 + [\mathbb{L}]^2)\mathbb{A} = -4\pi\bar{\mathbb{J}}. \tag{58}$$

Finally, when applying the operator $\mathbb{L}_c^\star$ on Eq. (50), reminding that $\mathbb{L}_c^\star \mathbb{L}_c$ is a real operator, and equating the real and imaginary parts on both the sides, we obtain the wave equations for $\mathbb{E}$ and $\mathbb{B}$:

$$(\{\mathbb{L}\}^2 + [\mathbb{L}]^2)\mathbb{E} = 4\pi\{\mathbb{L}\}\bar{\mathbb{J}}, \tag{59}$$

$$(\{\mathbb{L}\}^2 + [\mathbb{L}]^2)\mathbb{B} = -4\pi[\mathbb{L}]\bar{\mathbb{J}}. \tag{60}$$

When comparing the derivations revealed in Liu (2004), it is obvious that the current approaches are more straightforward. Taking the inverses of the above two equations we get formal solutions of $\mathbb{E}$ and $\mathbb{B}$:

$$\mathbb{E} = 4\pi(\{\mathbb{L}\}^2 + [\mathbb{L}]^2)^{-1}\{\mathbb{L}\}\bar{\mathbb{J}}, \tag{61}$$

$$\mathbb{B} = -4\pi(\{\mathbb{L}\}^2 + [\mathbb{L}]^2)^{-1}[\mathbb{L}]\bar{\mathbb{J}}. \tag{62}$$

Another approach to the wave equation is given below. The nabla operator ∇ acting on the usual three-dimensional field which depending on $\mathbf{x}^s$ can be extended to include a time-like part to form a four-dimensional gradient operator $\blacktriangledown$, which is acting on the four-dimensional field with a dependency on $\mathbf{x}$:

$$\blacktriangledown := \begin{bmatrix} \nabla \\ \frac{\partial}{\partial x^0} \end{bmatrix}. \tag{63}$$

Comparing with Eq. (19), it can be seen that $\blacktriangledown$ is an operatorization of $\mathbf{\Psi}_x$. According to the notation used in Eq. (24) and the identification specified in Eq. (19) we can recast Eqs. (46) and (50) into the following forms:

$$-(\gamma \cdot \blacktriangledown)\mathbb{A} = \mathbf{\Psi}_{\mathbb{F}}^\star, \tag{64}$$

$$(\gamma \cdot \blacktriangledown)\mathbf{\Psi}_{\mathbb{F}} = 4\pi\bar{\mathbb{J}}. \tag{65}$$

Taking the complex conjugate of Eq. (64) and substituting it for $\mathbf{\Psi}_F$ into Eq. (65), we obtain

$$(\gamma \cdot \blacktriangledown)(\gamma^\star \cdot \blacktriangledown)\mathbb{A} = -4\pi\bar{\mathbb{J}}. \tag{66}$$

This equation is a compact form of the wave equation for $\mathbb{A}$.

4.3 Comparing with the Dirac equation

Eq. (65) is superficially similar to the Dirac equation on its left-hand side:

$$(\gamma \cdot \blacktriangledown)\mathbf{\Psi} = \frac{mc}{\hbar}\mathbf{\Psi}, \tag{67}$$

where the Dirac matrices are given by

$$\gamma^0 = \begin{bmatrix} i\sigma_0 & \mathbf{0} \\ \mathbf{0} & -i\sigma_0 \end{bmatrix} = \begin{bmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{bmatrix}, \tag{68}$$

$$\gamma^1 = \begin{bmatrix} \mathbf{0} & i\sigma_1 \\ -i\sigma_1 & \mathbf{0} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix}, \tag{69}$$

$$\gamma^2 = \begin{bmatrix} \mathbf{0} & i\sigma_2 \\ -i\sigma_2 & \mathbf{0} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \tag{70}$$

$$\gamma^3 = \begin{bmatrix} \mathbf{0} & i\sigma_3 \\ -i\sigma_3 & \mathbf{0} \end{bmatrix} = \begin{bmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{bmatrix}. \tag{71}$$

However, for the Maxwell equations we have $(\gamma_0)^2 = 1$ and also $\gamma_0\gamma_n = \gamma_n\gamma_0, n = 1,2,3$, such that the γ in Eq. (65) as given in Eqs. (22) and (23) fails to satisfy the conditions $(\gamma^0)^2 = -1$ and $\gamma^0\gamma^n = -\gamma^n\gamma^0, n = 1,2,3$, which are essential for the Dirac Eq. (67). From Eqs. (68)-(71) we observe that $(\gamma^0)^\dagger = -\gamma^0$ and $(\gamma^n)^\dagger = \gamma^n, n = 1,2,3$. The above properties are also different from $(\gamma_n)^\dagger = \gamma_n, n = 0,1,2,3$, for the γ-matrices of the Maxwell equations.

5 Conclusions

We have transformed the complex Jordan algebra to a complex Lie algebra, and in terms of it we can derive the Maxwell equations and the wave equations through a symmetric linear differential operator $\{\mathbb{L}\}$ and a skew-symmetric linear differential operator $[\mathbb{L}]$. Then, in terms of a complex electromagnetic field we were able to put the four Maxwell equations into a single one. It is superficially similar to the Dirac equation. However, we point out the differences.

Acknowledgement: The financial support provided by the National Science Council under the Grant NSC 93-2212-E-019-006 is gratefully acknowledged.

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Appendix

Define the operators

$$\mathbb{L} = \{\mathbb{L}\} + [\mathbb{L}] = \begin{bmatrix} \mathbf{I}_3 \frac{\partial}{\partial x^0} + \hat{\nabla} & \nabla \\ \nabla^{\mathrm{T}} & \frac{\partial}{\partial x^0} \end{bmatrix}, \tag{A.1}$$

$$\mathbb{L}^{\mathrm{T}} = \{\mathbb{L}\} - [\mathbb{L}] = \begin{bmatrix} \mathbf{I}_3 \frac{\partial}{\partial x^0} - \hat{\nabla} & \nabla \\ \nabla^{\mathrm{T}} & \frac{\partial}{\partial x^0} \end{bmatrix}. \tag{A.2}$$

Applying them to any differentiable four potential, say the $\mathbb{A}$ in Eq. (48), we obtain

$$\mathbb{L}\mathbb{A} = \begin{bmatrix} \nabla\phi + \frac{\partial \mathbf{A}}{\partial x^0} + \nabla\times\mathbf{A} \\ \frac{\partial\phi}{\partial x^0} + \nabla\cdot\mathbf{A} \end{bmatrix}, \tag{A.3}$$

$$\mathbb{L}^{\mathrm{T}}\mathbb{A} = \begin{bmatrix} \nabla\phi + \frac{\partial \mathbf{A}}{\partial x^0} - \nabla\times\mathbf{A} \\ \frac{\partial\phi}{\partial x^0} + \nabla\cdot\mathbf{A} \end{bmatrix}. \tag{A.4}$$

Left multiplying Eq. (A.3) by $\mathbb{L}^{\mathrm{T}}$ and applying the operational rule in Eq. (A.4), we obtain

$$\mathbb{L}^{\mathrm{T}}\mathbb{L}\mathbb{A} = \begin{bmatrix} \nabla\left(\frac{\partial\phi}{\partial x^0} + \nabla\cdot\mathbf{A}\right) + \frac{\partial}{\partial x^0}\left(\nabla\phi + \frac{\partial \mathbf{A}}{\partial x^0} + \nabla\times\mathbf{A}\right) - \nabla\times\left(\nabla\phi + \frac{\partial \mathbf{A}}{\partial x^0} + \nabla\times\mathbf{A}\right) \\ \frac{\partial}{\partial x^0}\left(\frac{\partial\phi}{\partial x^0} + \nabla\cdot\mathbf{A}\right) + \nabla\cdot\left(\nabla\phi + \frac{\partial \mathbf{A}}{\partial x^0} + \nabla\times\mathbf{A}\right) \end{bmatrix}. \tag{A.5}$$

Similarly, left multiplying Eq. (A.4) by $\mathbb{L}$ and applying the operational rule in Eq. (A.3), we obtain

$$\mathbb{L}\mathbb{L}^{\mathrm{T}}\mathbb{A} = \begin{bmatrix} \nabla\left(\frac{\partial\phi}{\partial x^0} + \nabla\cdot\mathbf{A}\right) + \frac{\partial}{\partial x^0}\left(\nabla\phi + \frac{\partial \mathbf{A}}{\partial x^0} - \nabla\times\mathbf{A}\right) + \nabla\times\left(\nabla\phi + \frac{\partial \mathbf{A}}{\partial x^0} - \nabla\times\mathbf{A}\right) \\ \frac{\partial}{\partial x^0}\left(\frac{\partial\phi}{\partial x^0} + \nabla\cdot\mathbf{A}\right) + \nabla\cdot\left(\nabla\phi + \frac{\partial \mathbf{A}}{\partial x^0} - \nabla\times\mathbf{A}\right) \end{bmatrix}. \tag{A.6}$$

Due to

$$\nabla\cdot(\nabla\times\mathbf{A}) = 0, \tag{A.7}$$

$$\nabla\times(\nabla\phi) = \mathbf{0}, \tag{A.8}$$

we can prove that

$$\mathbb{L}^{\mathrm{T}}\mathbb{L}\mathbb{A} = \mathbb{L}\mathbb{L}^{\mathrm{T}}\mathbb{A}. \tag{A.9}$$

Substituting Eqs. (A.1) and (A.2) for $\mathbb{L}$ and $\mathbb{L}^{\mathrm{T}}$ into the above identity, leads to

$$\begin{aligned}(\mathbb{L}^{\mathrm{T}}\mathbb{L} - \mathbb{L}\mathbb{L}^{\mathrm{T}})\mathbb{A} =& (\{\mathbb{L}\} - [\mathbb{L}])(\{\mathbb{L}\} + [\mathbb{L}])\mathbb{A} - (\{\mathbb{L}\} + [\mathbb{L}])(\{\mathbb{L}\} - [\mathbb{L}])\mathbb{A} \\ =& \{\mathbb{L}\}\{\mathbb{L}\}\mathbb{A} + \{\mathbb{L}\}[\mathbb{L}]\mathbb{A} - [\mathbb{L}]\{\mathbb{L}\}\mathbb{A} - [\mathbb{L}][\mathbb{L}]\mathbb{A} \\ & - \{\mathbb{L}\}\{\mathbb{L}\}\mathbb{A} + \{\mathbb{L}\}[\mathbb{L}]\mathbb{A} - [\mathbb{L}]\{\mathbb{L}\}\mathbb{A} + [\mathbb{L}][\mathbb{L}]\mathbb{A} \\ =& 2\{\mathbb{L}\}[\mathbb{L}]\mathbb{A} - 2[\mathbb{L}]\{\mathbb{L}\}\mathbb{A} \\ =& \mathbf{0}. \end{aligned} \tag{A.10}$$

Accordingly, we can prove that $\{\mathbb{L}\}$ and $[\mathbb{L}]$ are commutative:

$$\{\mathbb{L}\}[\mathbb{L}] = [\mathbb{L}]\{\mathbb{L}\}, \tag{A.11}$$

since $\mathbb{A}$ is an arbitrary four-potential.