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A Study of Boundary Conditions in the Meshless Local Petrov-Galerkin (MLPG) Method for Electromagnetic Field Computations

Meiling Zhao[1] **and Yufeng Nie**[2]

Abstract: Meshless local Petrov-Galerkin (MLPG) method is successfully applied for electromagnetic field computations. The moving least square technique is used to interpolate the trial and test functions. More attention is paid to imposing the essential boundary conditions of electromagnetic equations. A new coupled meshless local Petrov-Galerkin and finite element (MLPG-FE) method is presented to enforce the essential boundary conditions. Unlike the conventional coupled technique, this approach can ensure the smooth blending of the potential variables as well as their derivatives in the transition region between the meshless and finite element domains. Then the boundary singular weight method is proposed to enforce the boundary conditions for electromagnetic field equations accurately. Practical examples in engineering, including the computations of the electric-field intensity of the cross section of long straight metal slot, the end region of a power transformer and axisymmetric problem in the electromagnetic field, are solved by the presented approaches. All numerical verification and all kinds of comparison analysis show that the MLPG method is a promising alternative numerical approach for electromagnetic field computations, and the proposed techniques can be good candidates for imposing essential boundary conditions.

Keyword: Meshless local Petrov-Galerkin (MLPG) method, electromagnetic field computation, essential boundary, coupled method.

1 Introduction

In the past decade meshless methods have emerged as very promising numerical approaches for computation mechanics. Meshless methods can eliminate the

[1] Department of Mathematics, Beijing University of Aeronautics & Astronautics, Beijing, 100083, P. R. China

[2] Department of Applied Mathematics, Northwestern Polytechnic University, Xi'an, Shaanxi, 710072, P. R. China

time-consuming human labor process of constructing meshes or generating adaptive meshes in solving many problems with moving boundaries, high gradients and so on, and alleviate or deal with the known drawbacks of the finite element (FE) method, such as locking, element distortion and other problems related with the finite element method. Among many meshless methods, a meshless local Petrov-Galerkin (MLPG) method, which was presented in Atluri and Zhu (1998), does not need any mesh, either for the interpolation of the solution variables or for the integration of the weak forms. The MLPG method can use various interpolation schemes, such as moving least square (MLS) [Lancaster and Salkauskas (1981)], the partition of unity (PU) [Babuska ans Melenk (1997)] or Shepard functions and etc., and can be based on a local weak formulation to get the symmetric or unsymmetric local weak forms, which has been developed as a general framework for solving partial differential equations in Atluri and Shen (2002a, b).

In recent years, the MLPG method has been widely applied for various problems, including those in elasto-statics [Li, Shen, Han and Atluri (2003); Han and Atluri (2003ab, 2004a)], fracture mechanics [Kim and Atluri, (2000)], elasto-dynamics [Han and Atluri (2004b)], nonlinear problems [Han, Rajendran, Atluri (2005)], fluid mechanics [Lin and Atluri (2001); M. Haji Mohammadi (2008)] and heat conduction problems [Sladek, Sladek, Atluri (2004a), Wu, Shen, Tao (2007)] and other fileds [Han, et al.. (2006); Atluri, et al.. (2006a, b); Liu, et al.. (2006)]. Though the MLPG method has achieved remarkable successes in many fields, it is seldom used to electromagnetic computations. In this work, the MLPG method is successfully applied for practical problems of electromagnetic field. The moving least square approximation is used to construct the shape functions. The special attention is paid to the enforcement of the essential boundary conditions of electromagnetic equations.

While using the meshless approximation without the Kronecker-Delta properties, such as MLS, Shepard function and PU, the imposition of the essential boundary conditions (EBC) is not as straightforward as for the finite element method. In many literatures, a Lagrange multiplier technique has widely used to impose the essential boundary. However, it produces a stiffness matrix without banded and positive definite properties. The penalty parameter technique by Zhu and Atluri (1998) does not need other additional unknown variables, but needs an appropriate choice of the penalty parameter. Many MLPG methods usually use this approach to obtain the weak form and enforce the boundary. A modified collocation method is also a very important technique to enforce the essential boundary conditions in the MLPG method [Zhu and Atluri (1998)]. In this approach, second derivatives of the shape functions are usually needed in constructing the global stiffness matrix for the interior nodes. The local boundary integral equation method, as a special

MLPG, can relatively easily enforce the essential boundary conditions, although it involves a singular integral [Zhu, Zhang and Atluri (1998a, b)]. A transformation method presented by Atluri, Kim and Cho (1999) investigated the relation between the actual nodal values of the interpolant and the fictitious nodal ones, and used the linear transformation to make the shape functions satisfy the Kronecker-δ properties and impose the EBC at the cost of involving the inverse of a matrix. Because of its accuracy enforcement of the essential boundary, the transformation method is taken into consideration for the comparison analysis in this paper. In our work, two techniques, as good alternatives to impose the essential boundary conditions, are proposed in the meshless local Petrov-Galerkin method and successfully applied for electromagnetic field computations. Firstly, a new coupled MLPG-FE method is presented to impose the EBC. In this approach, the analysis domain is divided into two regions where the FE method and the MLPG method are used separately. It defines a transition domain between the two regions. In the transition part, a new ramp function, which is different from the conventional function in Belystchko (1996), is chosen to combine the shape functions of the two methods. Using this technique, the continuity conditions of potential variables and their derivatives are satisfied, while in the conventional method the derivatives undergo a jump across the interface. Then a boundary singular weight function method (BSW) is proposed to impose the EBC in the MLPG method. Using both methods, the essential boundary conditions can be accurately enforced. Finally, all above techniques are successfully applied for practical electromagnetic field computations, and the results validate the efficiency of the proposed approaches. The new coupled MLPG-FE method gives full play of the advantages of both the MLPG and FE method, and successfully imposes the EBC and avoids the discontinuity of the derivatives of the potential variables in the traditional technique. By the comparison investigation, it can be seen that the singular weight function technique can save considerable computational time.

2 MLPG method

2.1 Moving least square approximation (MLS)

The MLS method is usually used to interpolate random data with appropriate accuracy, and the property of MLS has been widely discussed in literatures [Atluri and Zhu (1998), Jin, Li and Aluru (2001)]. Consider a function $u(x)$ in Ω. The MLS approximation $u^h(x)$ is defined by

$$u^h(x) = \sum_{j=1}^{m} p_j(x)a_j(x) = \mathbf{p}^{\mathrm{T}}(x)\mathbf{a}(x), \tag{1}$$

where m is the number of basis functions. Minimizing some weighted discrete L^2 norm by least square theorem, we can obtain $\mathbf{a}(x)$. Finally we have

$$u^h(x) = \sum_{i=1}^{n} \Phi_i(x) u^*(x_i), \tag{2}$$

where the MLS shape function is

$$\Phi_i(x) = \sum_{j=1}^{m} p_j(x)(A^{-1}(x)B(x))_{ji}. \tag{3}$$

In Equation (3), $A(x)$ and $B(x)$ are defined by

$$A(x) = \sum_{i=1}^{n} w_i(x) p(x_i) p^{\mathrm{T}}(x_i), \tag{4}$$

$$B(x) = [w_1(x)p(x_1), w_2(x)p(x_2), \ldots, w_n(x)p(x_n)], \tag{5}$$

where $w_i(x)$ is the supported compacted weighted function. In general, $\Phi_i(x_j) \neq \delta_{ij}$.

2.2 Local Petrov-Galerkin integral equation

Consider the boundary value problem in the static field

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = \overline{u} & \text{on } \Gamma_u \\ \frac{\partial u}{\partial n} = q = \overline{q} & \text{on } \Gamma_q \end{cases}, \tag{6}$$

where $\Gamma = \Gamma_u \cup \Gamma_q$. A local weak formulation can be written as

$$\int_{\Omega_s} (\Delta u + f) v d\Omega = 0. \tag{7}$$

Choose $v(x, x_I)$ as test function in every sub-domain, and we can obtain the linear system

$$\mathbf{K} \bullet \mathbf{u}^* = \mathbf{f}, \tag{8}$$

where

$$K_{IJ} = \int_{\Omega_s} (\Phi_{J,x}(x) v_{,x}(x, x_I) + \Phi_{J,y}(x) v_{,y}(x, x_I)) d\Omega - \int_{\Gamma_{su}} \frac{\partial \Phi_J(x)}{\partial n} v(x, x_I) ds, \tag{9}$$

$$f_I = \int_{\Gamma_{sq}} \overline{q} v(x, x_I) ds + \int_{\Omega_s} f v(x, x_I) d\Omega. \tag{10}$$

In Eq. (9) and Eq. (10), $I = 1, 2, \ldots, N$, $J = 1, 2, \ldots, M$. N denotes the total number of nodes of Ω, and M is the number of test function centered at x_I, which do not vanish at x_J.

3 Coupling of MLPG and FE

As shown in Figure 1,$\Omega = \Omega_{FE} \cup \Omega_{MLPG} \cup \Omega_T$, and Ω_T is the transition region between Ω_{FE} and Ω_{MLPG}, where the FE method and the MLPG method are used separately. In order to make the modified shape function transit smoothly from one region to another, we redefine the shape function as

$$\tilde{\Phi}_i(x) = \begin{cases} \Phi_i(x) & \text{in } \Omega_{MLPG} \\ \Phi_i(x) + R(x)(N_i(x) - \Phi_i(x)) & \text{in } \Omega_T \\ N_i(x) & \text{in } \Omega_{FE}. \end{cases} \tag{11}$$

Its corresponding derivative is

$$\tilde{\Phi}_{i,j}(x) = \begin{cases} \Phi_{i,j}(x) & \text{in } \Omega_{MLPG} \\ \Phi_{i,j}(x) + R(x)(N_{i,j}(x) - \Phi_{i,j}(x)) + R_{,j}(x)(N_i(x) - \Phi_i(x)) & \text{in } \Omega_T \\ N_{i,j}(x) & \text{in } \Omega_{FE}. \end{cases} \tag{12}$$

Instead of traditional ramp functions, a new $R(x)$ is proposed in this paper as following

$$R(x) = \left(\sum_i N_i(x) \right)^2, \quad x_i \in \Gamma_{TF}. \tag{13}$$

In Eq. (13), $R(x)$ satisfies

$$R(x) = \begin{cases} 1 & x \in \Gamma_{TF} \\ 0 & x \in \Gamma_{TM} \end{cases} \tag{14}$$

and

$$R_{,j}(x) = 0 \quad x \in \Gamma_{TM}, \tag{15}$$

Eq. (14) and Eq. (15) ensure the smooth blending of the shape function and its derivatives between the two sub-domains. Moreover, the modified shape function has the property of Kronecker-δ everywhere in the essential boundary. Figure 2 shows the comparison of the new coupled method and the traditional method in 1D. It can be seen from Figure 2(a) that, although the traditional modified shape function is continuous at the interface, but it has a discontinuity of the derivative at the interface, which is displayed by the tangent slope of the imaginary line, while the new coupled MLPG-FE shape function, which is shown in Figure 2(b), ensures the continuity of the shape function and its derivative simultaneously.

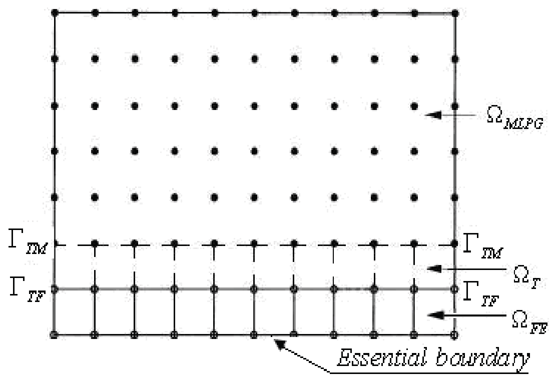

Figure 1: Schematics of the MLPG-FE method

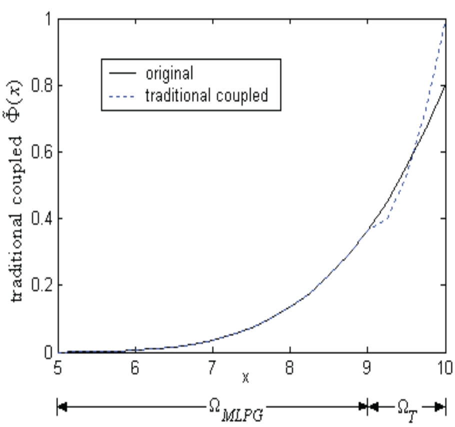

Figure 2a: Comparison of the original MLPG and the traditional coupled MLPG-FE shape functions in 1D

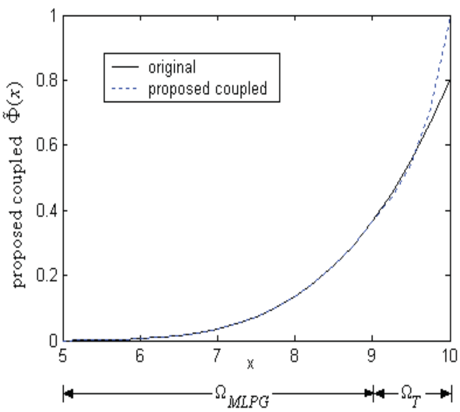

Figure 2b: Comparison of the original MLPG and the new coupled MLPG-FE shape functions in 1D

4 Boundary singular weight function method

The original paper of MLS method by Lancaster and Salkauskas (1981) as suggested that by introducing a singularity into the weight function, and the approximation leads to interpolation. This concept was first introduced by Kaljevic and Saigal (1997) to the element free Galerkin method (EFGM) function. In their approach, singular weight functions are employed all discrete nodes, and the Kronecker-δ properties are recovered in the MLS shape functions. In this paper, the MLPG shape functions are constructed with singularities introduced only to the constrained boundary nodes. As such, this method does not generate interpolation functions at the interior nodes, but it is sufficient to obtain nodal values at the restrained boundary nodes for direct imposition of the EBC.

A singularity is introduced to the weight functions with a designated node I located at on the essential boundary

$$\tilde{w}(x-\tilde{x}_i, y-\tilde{y}_i) = \frac{w(x-\tilde{x}_i, y-\tilde{y}_i)}{f(x-\tilde{x}_i, y-\tilde{y}_i)}, \tag{16}$$

where $f(0,0)=0$, and the superposed $\sim$ on the nodal coordinate denotes a node with singularity imposed in the associated shape function. The function f is chosen to have the following form

$$f(x-\tilde{x}_i, y-\tilde{y}_i) = \left[\left(\frac{x-\tilde{x}_i}{a_x}\right)^2 + \left(\frac{y-\tilde{y}_i}{a_y}\right)^2\right]^p, \quad p>0 \tag{17}$$

where p reflects the order of singularity, and a_x, a_y

are the parameters to adjusting the size of influence domain.

Using Eq. (16), we get the shape function associated with the weight $\tilde{w}(x-\tilde{x}_i, y-\tilde{y}_i)$ as following

$$\tilde{\Phi}_i(x) = p^T(x)\tilde{\mathbf{A}}^{-1}(x)p(\tilde{x}_i)\tilde{w}(x-\tilde{x}_i), \tag{18}$$

where

$$\tilde{\mathbf{A}}(x) = \sum_{j\notin\Gamma_u} p(x_j)p^T(x_j)w(x-x_j) + \sum_{k\in\Gamma_u, k\neq I} p(\tilde{x}_k)p^T(\tilde{x}_k)w(x-\tilde{x}_k) + p(\tilde{x}_i)p^T(\tilde{x}_i)\tilde{w}(x-\tilde{x}_i). \tag{19}$$

Other shape functions are

$$\Phi_j(x) = p^T(x)\tilde{\mathbf{A}}^{-1}(x)p(x_j)w(x-x_j) \quad j\notin\Gamma_u, \tag{20}$$

$$\tilde{\Phi}_k(x) = p^T(x)\tilde{\mathbf{A}}^{-1}(x)p(\tilde{x}_k)\tilde{w}(x-\tilde{x}_k), k\in\Gamma_u, \quad k\neq i. \tag{21}$$

The singular weight shape functions $\tilde{\Phi}_i(x)$ have the following property

$$\tilde{\Phi}_i(x\to\tilde{x}_i) = 1. \tag{22}$$

Other shape functions have the following property

$$\Phi_j(x\to\tilde{x}_i) = 0, \tag{23}$$

$$\tilde{\Phi}_k(x \to \tilde{x}_i) = 0, k \in \Gamma_u, \quad k \neq i. \tag{24}$$

Recall the approximation of the potential variable

$$u^h(\tilde{x}_i) = \sum_{j=1, j\neq i}^{n} \Phi_j(\tilde{x}_i)u_j + \tilde{\Phi}_i(\tilde{x}_i)u_i = u_i. \tag{25}$$

Compared with the approach by Kaljevic, the proposed boundary singular weight method provides exact nodal values at the constraint boundary nodes, and partly saves computational cost.

5 Numerical examples

5.1 Numerical validation and comparison analysis

The grounding metal slot with the square domain in the dimension $1m \times 1m$, in which the upper wall insulated with the earth has the electric potential $\varphi = 10\sin(\pi x)$, and the side wall and the bottom wall both have the electric potential of zero, is shown in Figure3. This electrostatic model is used for verification of the accuracy and convergence of the presented methods. We use regularly distributed $441(21 \times 21)$, $121(11 \times 11)$, $36(6 \times 6)$ nodes for this model. The appropriate radius of the influence domain is chosen according to Nie and Atluri (2006). 3×3 Gaussian quadrature is used to integrate the energy in each small partition over the intersection of Ω_{te}^{I} and Ω_{tr}^{J}. The proposed MLPG-FE method is applied to compute the model, in which the bilinear Lagrange rectangular elements are used to enforce the essential boundary conditions.

For the convergence study, we separately use MLPG, MLPG-FE and the FE with the same order interpolation to solve this problem, and the following relative error is defined as:

$$R_e = \sqrt{\sum_{i=1}^{N} (\overline{\varphi}_i - \varphi_i)^2 \Big/ \sum_{i=1}^{N} \varphi_i^2}, \tag{26}$$

where $\overline{\varphi}$ and φ are the numerical solution and the analytical solution respectively, and N is the number of nodes set in the studied domain. As can be seen, Figure 4 shows the proposed MLPG-FE method has better convergence than the FE method. Moreover, from the Figure 5 we can see that the electric-field intensity, which is the derivative of electric potential variable, is continuously distributed in the analysis field.

To study the computational effort required for the proposed methods, the problem is computed by the MLPG with penalty, the transformation method (TM), the proposed MLPG-FE and the boundary singular weight function method (BSW) separately, and the comparison of normalized error and CPU time using these methods are listed in Table 1. In the table, the error and CPU time are normalized by those in the case of the penalty method. The results indicate that the BSW has saved computational effort in a degree.

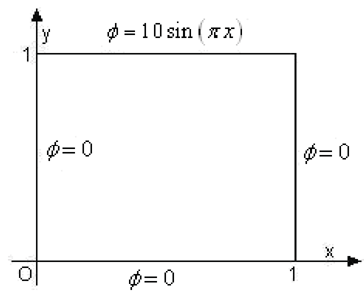

Figure 3: The cross-section of long straight metal slot

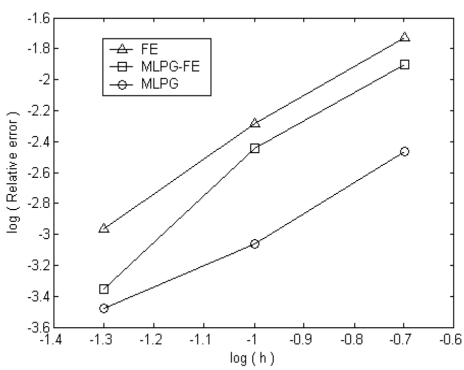

Figure 4: The convergence curve of the electric potential

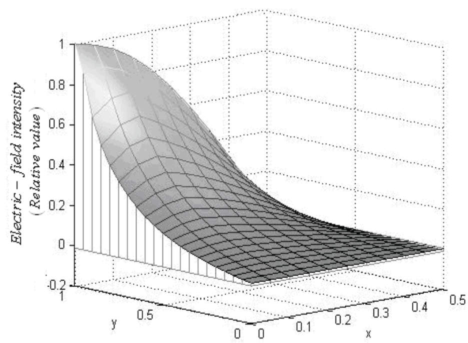

Figure 5: The distribution of the electric-field intensity in the analysis field (the half field)

Table 1: Comparison of normalized error and CPU time

	Penalty	MLPG-FE	TM	BSW
Normalized Error	1.000	0.984	0.975	1.107
Normalized CPU Costs	1.000	1.002	1.105	0.783

5.2 The end region of a power transformer

To validate the proposed techniques further, they are used to compute the end fields of a power transformer in Yang and Ni (2003), which is illustrated in Figure 6.

First the new coupled MLPG-FE method is applied to solve this problem. In the meshless region, the linear basis is used. At the same time, Γ_1 and Γ_3 are the essential boundaries which are enforced by bilinear Lagrange rectangular elements. The appropriate radius of the influence domain is chosen according to Nie and Atluri (2006). Figure 7 gives the arrangement of nodes and elements needed in the studied region. The reliability of the proposed method is implicated in Figure 8, which shows the accurate comparison of the numerical results for the FE method and the proposed method respectively, and verifies the reliability of the new coupled MLPG-FE method.

Then we use the FE method with biquadratic Lagrange rectangular elements to solve the model, and take the solution as the reference value, which is usually thought closer to the real value than the solutions by the FE method with bilinear Lagrange rectangular elements and the MLPG-FE method. The electric potential distributions along Line *AB* are given in Figure 9, which shows that the proposed MLPG-FE method possesses a high accuracy.

Then the BSW in MLPG is used to solve the model. From the comparison between the BSW in MLPG and the FE method in Figure 10, we can see that the results of the transformation method and the FE method are quite close to each other.

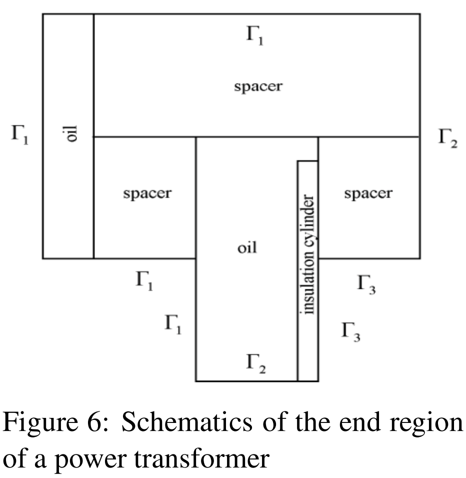

Figure 6: Schematics of the end region of a power transformer

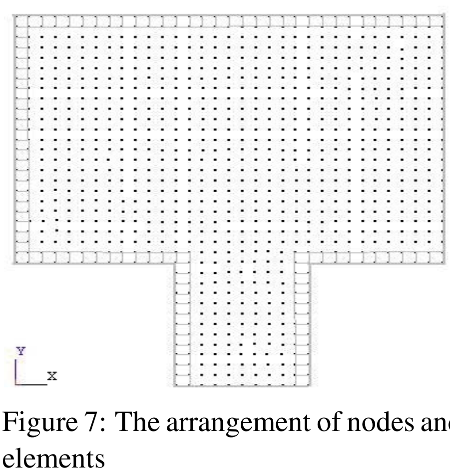

Figure 7: The arrangement of nodes and elements

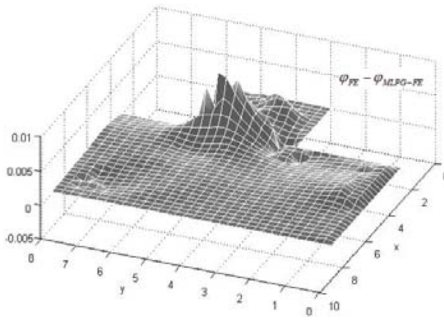

Figure 8: Comparison of results with the proposed MLPG-FE method and the FE method

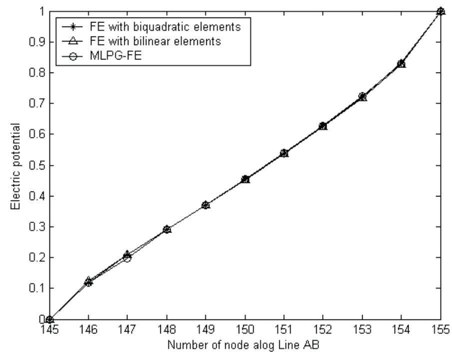

Figure 9: Comparison of electric potential along Line *AB*

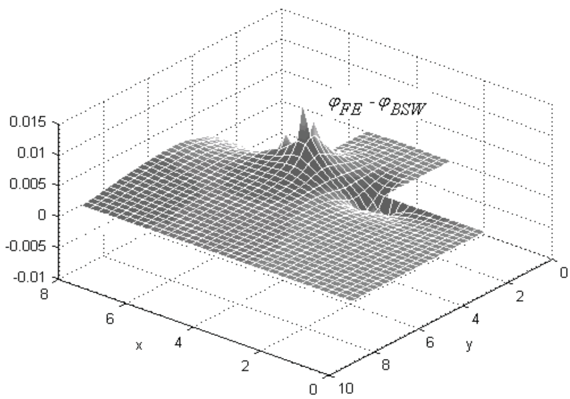

Figure 10: Comparison of numerical results of the proposed BSW in MLPG and the FE method

5.3 Axisymmetric case

Another class of problems that can be analyzed using the proposed technique is that of axisymmetric problems. Axially symmetric geometries, which are known as bodies of revolution, not only exist but are very common. Here we consider the geometry of Figure 11 in Jianming Jin (2002), where two coaxial waveguides having different inner radius are joined. This geometry is rotationally symmetric with respect to the $z-$axis, so in the $\rho z-$plane the potential satisfies

$$-\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\varepsilon_r\rho\frac{\partial\varphi}{\partial\rho}\right)-\frac{\partial}{\partial z}\left(\varepsilon_r\frac{\partial\varphi}{\partial z}\right)=\frac{\rho_c}{\varepsilon_0}, \tag{27}$$

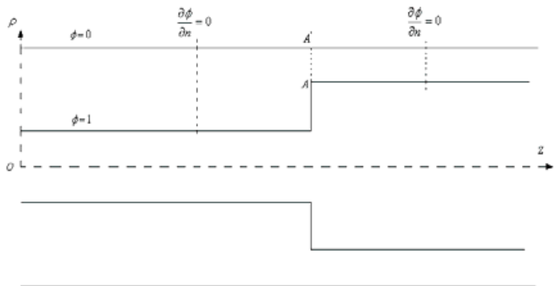

Figure 11: Cross section in the $\rho z-$ plane of the join between two coaxial waveguides

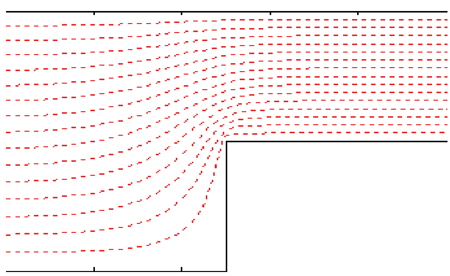

Figure 12: Equipotential lines by the FE method near the join between two coaxial waveguides

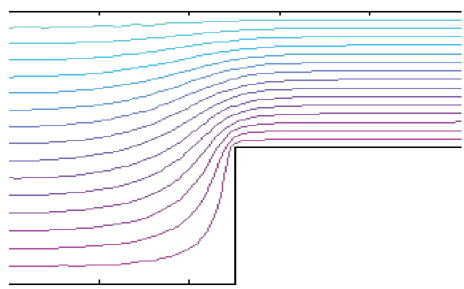

Figure 13: Equipotential lines by BSW near the join between two coaxial waveguides

where ρ_c is charge density, ρ and z are cylindrical coordinate variables, ε_r is relative dielectric constant of medium, and ε_0 is vacuum dielectric constant. Since the perturbation is confined near the join, the potential at some distance away from the join should be the same as in the unperturbed case. Therefore, the potential far enough away from the join is independent of z, or in other words, it satisfies the condition

$$\frac{\partial \varphi}{\partial z} = 0. \tag{28}$$

This can be used as the boundary condition to terminate the solution domain.

We use the FE method, the MLPG with penalty formulation and the MLPG with BSW to solve the waveguides model separately. The equipotential contours are plotted in Figure 12 and Figure 13, which show the proposed BSW in MLPG gives good results as well as the accuracy enforcement of essential boundary. The electric

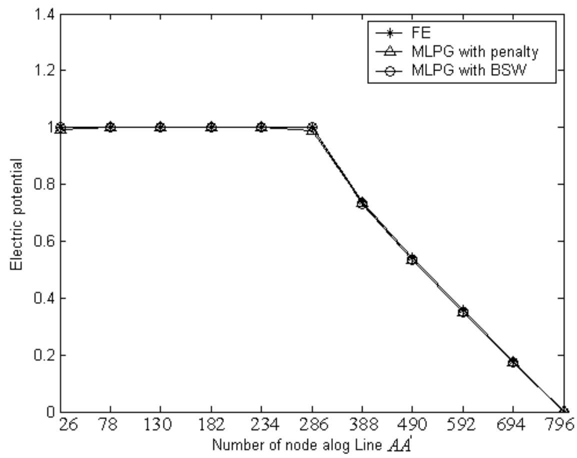

Figure 14: Comparison of electric potential along Line AA'

potential distributions along Line AA' can be seen in Figure 14. Comparison of the numerical results shows that BSW method has better approximation in the essential boundary than the MLPG with penalty parameter.

6 Conclusions

In this paper, the MLPG method has been successfully used to solve the electromagnetic problems. In addition, a new coupled MLPG-FE method and a boundary singular weight function method in the MLPG are developed. Comparison research with existing approaches has been analyzed by computing electromagnetic examples. Both the proposed approaches can directly enforce the essential boundary conditions. The new coupled MLPG-FE method provides smooth transition of the potential variable and its derivatives between the MLPG and FE domains. It makes numerical solution more reasonable and objective. The singular weight function method has saved computational time to a great extent. All the numerical results show that the MLPG method is promising in solving electromagnetic problems, especially such as models where part of the domain changes in its geometrical shape, shape optimizations of electromagnetic equipments, coupled field problems and so on, and the proposed techniques are effective and efficiency as good alternatives to enforce the essential boundary conditions.

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