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Steady-state Response of the Wave Propagation in a Magneto-Electro-Elastic Square Column

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Abstract: The steady-state response of the wave propagation in a magneto-electro-elastic square column (MEESC) was studied. Some new characteristics were discovered: the guided waves are classified in the forms of the Quasi-P, Quasi-SV and Quasi-SH waves and ordered by the standing wave number, and the three type guided waves are corresponding to the extension, twist and shear modes of the body vibration; the induced electric and magnetic fields can be aroused by the propagating stress wave. We proposed a self-adjoint method, by which the guided-wave restriction condition was derived. After finding the corresponding orthogonal sets, the analytic dispersion equation was obtained. In the end, an example was presented. The dispersive spectrum, the group velocity curve and the steady-state response curve of MEESC were plotted.

Keyword: magneto-electro-elastic square column, guided-wave restriction condition, dispersive equation, group velocity equation, steady-state response.

1 Introduction

Magneto-electro-elastic dielectrics are widely used in the aerospace structures, the equipments of the energy and chemical industries, and they are the main part of the electron elements and the microelectronics mechanical systems, and they play an important role in the material science progressively. This kind of medium has piezoelectric, piezomagnetic, elastic, dielectric and electromagnetic properties, and there are no free electrons and electric flux in it. The electric and magnetic influences can only transmit in the forms of the induced electric and magnetic fields. But these media are very sensitive to the outside physical fields, and they have many response patterns. In general, there are coupling stress waves propagation and coupling mechanical and electromagnetic signs transmission. These physical

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phenomena will affect the operation reliability and the control precision of these equipments and systems mentioned above.

Because these media have coupled piezoelectric, piezomagnetic and anisotropic elastic constitutive relations, etc., the characteristics of the stress wave propagation are very complicated. If the effects of the boundary conditions are considered, the coupling waves will reflect and refract at the boundaries, thus forming complicated interferences and causing dispersive phenomena. Furthermore, the dynamic problem will become more difficult and complicated in magneto-electro-elastic structures.

At present, the Bessel-Fourier expansion method [Karl 1991, Paul 1997] and the Stroh method [Ting 2002] are widely used to study the dynamic problems of the magneto-electro-elastic and piezoelectric structures, but these two methods have some limitations. The Bessel-Fourier expansion method can be only applied in cylindrical structures and it requires that the physical property of the cross section is isotropic. The Stroh method is a complex method. Theoretically, it can solve all kinds of magneto-electro-elastic structures, but this method is only applicable to two space arguments' problems. The dispersive equations derived from these two methods contain transcendental functions, so it is very difficult to plot the dispersive spectrum.

Although the magneto-electro-elastic guided-wave problems are very complicated, and there is not a complete dispersive spectrum for any 3D magneto-electro-elastic structure, there are some achievements, which can be used for reference.

Paul and Venkatesan [Paul and Venkatesan 1987, 1989] applied the Bessel-Fourier expansion method to studying the dispersive equation of a piezoelectric cylinder with a slightly changing cross section. It is shown that the dispersive equation changes with the slightly changing cross section. But it is very difficult to plot the dispersive curves, so the difference between two slightly changing cross sections cannot be compared.

Chen et al. [1998] and Ding & Xu [2002] studied the vibration of composite plates. In order to get the iteration matrix they selected two boundary conditions (elastic simply supported and rigid sliding supported) in the iterative process. It is very easy to find two group orthogonal sets for these two boundary conditions, but it is very difficult to obtain the orthogonal sets for other boundary conditions to realize the iterative process. These imply that there are internal relations between the boundary condition and the orthogonal sets. We have applied these two boundary conditions in studying the wave propagation in a piezoelectric cylinder [Wei and Su 2005].

Jun and Mauro [2003] researched the elastic wave reflection on biological tissue. Although in their model they used the averaged material properties for thin layer, we can see that changes in micro-level physical and geometrical parameters affect the reflectivity of the thin layer. Verbis, Tsinopoulos and Polyzos [2002] compared the elastic wave propagation in fiber reinforced composite materials between non-uniform distribution of fibers and the uniform distribution of fibers. We can also find that the slight change of physical properties can arise the large change of wave propagation.

We have also applied an approximate model in solving the guided-wave problems of the piezoelectric and magneto-electro-elastic cylinders [Wei and Su 2005, 2006], obtained the lower order modes and found that the electric and magnetic physical properties have different influences on the stress wave propagation in the piezoelectric and magneto-electro-elastic dielectric media.

In this paper the self-adjoint method is presented in studying the steady-state response of the wave propagation in MEESC. The physical property of the magneto-electro-elastic dielectric medium, the space structure and boundary condition of the guided-wave system are combined together in this method, and a guided-wave restriction condition is derived to describe the inner relations of the guided-wave system in mathematics. After finding the orthogonal sets, which satisfy the guided-wave restriction condition, and mapping them from the time domain to the frequency domain, the analytic dispersive equation, the group velocity equation and the steady-state response are obtained completely.

2 Basic equation

We consider the magneto-electro-elastic dynamic equations in a rectangular coordinate system. The relation between deformation and displacement is

$$\mathbf{s}(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + \mathbf{u}\nabla). \tag{1}$$

Relation between electric field and electric potential is

$$\mathbf{E}(\phi) = -\nabla\phi. \tag{2}$$

Relation between magnetic field and magnetic potential is

$$\mathbf{H}(\psi) = -\nabla\psi. \tag{3}$$

where $\mathbf{s} = [s_{ij}]$, $\mathbf{u} = [u_i]$, $\mathbf{E} = [E_i]$, $\mathbf{H} = [H_i]$, ϕ and ψ are the strain tensor, displacement vector, electric field vector, magnetic field vector, electric potential and magnetic potential, respectively.

The magneto-electro-elastic constitutive relations are

$$\boldsymbol{\sigma} = \mathbf{c}\cdot\cdot\mathbf{s}(\mathbf{u}) - \mathbf{E}(\phi)\cdot\mathbf{f} - \mathbf{H}(\psi)\cdot\mathbf{d}, \tag{4}$$

$$\mathbf{D}=\mathbf{f}^T\cdot\cdot\mathbf{s}(\mathbf{u})+\boldsymbol{\varepsilon}\cdot\mathbf{E}(\phi)+\mathbf{h}\cdot\mathbf{H}(\psi), \tag{5}$$

$$\mathbf{B}=\mathbf{d}^T\cdot\cdot\mathbf{s}(\mathbf{u})+\mathbf{h}\cdot\mathbf{E}(\phi)+\boldsymbol{\mu}\cdot\mathbf{H}(\psi), \tag{6}$$

where $\boldsymbol{\sigma}=[\sigma_{ij}]$, $\mathbf{D}=[D_i]$ and $\mathbf{B}=[B_i]$ are the stress tensor, electric displacement vector and magnetic induction vector, respectively. $\mathbf{c}=[c_{ijkl}]$, $\mathbf{f}=[f_{ijk}]$, $\mathbf{d}=[d_{ijk}]$, $\boldsymbol{\varepsilon}=[\varepsilon_{ij}]$, $\mathbf{h}=[h_{ij}]$ and $\boldsymbol{\mu}=[\mu_{ij}]$ are the constants of the elasticity tensor, piezoelectric tensor, piezomagnetic tensor, dielectric tensor, electromagnetic tensor and permeability tensor, respectively. The symmetric and transpose relations of the above tensors are $c_{ijkl}=c_{jikl}=c_{ijlk}=c_{klij}$, $\Box_{ijk}=\Box_{ikj}$, $\Box_{ijk}^T=\Box_{jki}$ and $\Box_{ij}=\Box_{ji}$.
The governing equations in the magneto-electro-elastic dielectric are

$$\nabla\cdot\boldsymbol{\sigma}=\rho\ddot{\mathbf{u}}, \tag{7}$$

$$\nabla\cdot\mathbf{D}=0, \tag{8}$$

$$\nabla\cdot\mathbf{B}=0, \tag{9}$$

where ρ is the mass density. The superimposed dot indicates the differentiation with respect to the time parameter t. Here body force is neglected for simplicity.
Substituting eqs. (1-6) into eqs. (7-9), the governing equation in the form of the displacements, electric and magnetic potentials is

$$\mathbf{L}[\mathbf{U}(\mathbf{x},t)]=0, \tag{10}$$

where $\mathbf{U}(\mathbf{x},t)=\mathbf{U}(x_1,x_2,x_3,t)=\begin{bmatrix}\mathbf{u} & \phi & \psi\end{bmatrix}^T$. The differential operator is

$$\mathbf{L}=\begin{bmatrix} c_{i1k1}\partial_{ik}-\rho\partial_{tt} & c_{i1k2}\partial_{ik} & c_{i1k3}\partial_{ik} & f_{i1k}\partial_{ik} & d_{i1k}\partial_{ik} \\ c_{i2k1}\partial_{ik} & c_{i2k2}\partial_{ik}-\rho\partial_{tt} & c_{i2k3}\partial_{ik} & f_{i2k}\partial_{ik} & d_{i2k}\partial_{ik} \\ c_{i3k1}\partial_{ik} & c_{i3k2}\partial_{ik} & c_{i3k3}\partial_{ik}-\rho\partial_{tt} & f_{i3k}\partial_{ik} & d_{i3k}\partial_{ik} \\ f_{ij1}\partial_{ij} & f_{ij2}\partial_{ij} & f_{ij3}\partial_{ij} & -\varepsilon_{ij}\partial_{ij} & -h_{ij}\partial_{ij} \\ d_{ij1}\partial_{ij} & d_{ij2}\partial_{ij} & d_{ij3}\partial_{ij} & -h_{ij}\partial_{ij} & -\mu_{ij}\partial_{ij} \end{bmatrix}, \tag{11}$$

where $\partial_{ij}=\frac{\partial^2}{\partial x_i\partial x_j}$, $(i,j=1,2,3)$, $\partial_{tt}=\frac{\partial^2}{\partial t^2}$. In derivation of the overall formulas, we take Einstein notation; where we do not take, a special note will be given.

3 Self-adjoint method

If the steady-state response of the structure is considered, namely, $\mathbf{U}(\mathbf{x},t)$ is the harmonic wave in formula (10), the coefficient matrix of $\mathbf{L}[\mathbf{U}(\mathbf{x},t)]=0$ will be an Hermite's matrix. The characteristic of the Hermite's matrix is that all eigenvalues are real, and the eigenvectors are orthogonal one another for different eigenvalues. Eringen and Suhubi [1975] got that the squares of the wave frequencies are positive real numbers in studying the differential operator **L** of isotropic material, and the eigenvectors construct a complete orthogonal set. Gurtin [1972] proposed a method to construct the orthogonal sets for the isotropic guided-wave systems, and he applied this method to getting some group orthogonal sets in one- and two-dimensional endless guided-wave systems. In this paper we still apply the characteristic of the Hermite's matrix and construct the complete orthogonal sets to solve the magneto-electro-elastic guided-wave problem. The differential operator **L** is a self-adjoint operator, so it satisfies the following integral.

$$\int_{\Omega} \{\mathbf{V}^T(\mathbf{x},t)\cdot\mathbf{L}[\mathbf{U}(\mathbf{x},t)] - \mathbf{U}^T(\mathbf{x},t)\cdot\mathbf{L}[\mathbf{V}(\mathbf{x},t)]\}dV = 0, \tag{12}$$

where $\mathbf{U}(\mathbf{x},t) = \begin{bmatrix}\mathbf{u} & \phi_1 & \psi_1\end{bmatrix}^T$ and $\mathbf{V}(\mathbf{x},t) = \begin{bmatrix}\mathbf{v} & \phi_2 & \psi_2\end{bmatrix}^T$ are two arbitrary orthogonal modes of structure Ω.

Substituting the two orthogonal modes into the differential operator (11) and calculating the following formula yield

$$\begin{aligned}
&(\mathbf{x},t)\cdot\mathbf{L}[\mathbf{V}(\mathbf{x},t)]\\
&= \nabla\cdot\{\mathbf{v}\cdot[\mathbf{c}\cdot\cdot\mathbf{s}(\mathbf{u})+\nabla\phi_1\cdot\mathbf{f}+\nabla\psi_1\cdot\mathbf{d}]\}+\nabla\cdot\{\phi_2[\mathbf{f}^T\cdot\cdot\mathbf{s}(\mathbf{u})-\nabla\phi_1\cdot\boldsymbol{\varepsilon}-\nabla\psi_1\cdot\mathbf{h}]\}\\
&+\nabla\cdot\{\psi_2[\mathbf{d}^T\cdot\cdot\mathbf{s}(\mathbf{u})-\nabla\phi_1\cdot\mathbf{h}-\nabla\psi_1\cdot\boldsymbol{\mu}]\}-\nabla\cdot\{\mathbf{u}\cdot[\mathbf{c}\cdot\cdot\mathbf{s}(\mathbf{v})+\nabla\phi_2\cdot\mathbf{f}+\nabla\psi_2\cdot\mathbf{d}]\}\\
&-\nabla\cdot\{\phi_1[\mathbf{f}^T\cdot\cdot\mathbf{s}(\mathbf{v})-\nabla\phi_2\cdot\boldsymbol{\varepsilon}-\nabla\psi_2\cdot\mathbf{h}]\}-\nabla\cdot\{\psi_1[\mathbf{d}^T\cdot\cdot\mathbf{s}(\mathbf{v})-\nabla\phi_2\cdot\mathbf{h}-\nabla\psi_2\cdot\boldsymbol{\mu}]\}\\
&-\rho\mathbf{v}\cdot\ddot{\mathbf{u}}+\rho\mathbf{u}\cdot\ddot{\mathbf{v}}
\end{aligned} \tag{13}$$

If we consider the harmonic wave propagation in this guided-wave system, the displacements, electric and magnetic potentials will be written as

$$\mathbf{U}(\mathbf{x},t) = \mathbf{B}\cdot\bar{\mathbf{U}}(\mathbf{x})e^{-i\omega t}, \tag{14}$$

where ω is the circular frequency. $i=\sqrt{-1}$. $\mathbf{B}\cdot\bar{\mathbf{U}}(\mathbf{x})$ is the mode. $\mathbf{B}$ is the wave amplitude matrix. $\bar{\mathbf{U}}(\mathbf{x}) = (U_1,U_2,U_3,U_4,U_5)^T$ is the orthogonal sets.

Substituting formula (13) into integral (12), and considering the harmonic wave form (14), we obtain

$$\begin{aligned}
&\int_{\Omega}\{\mathbf{V}(\mathbf{x},t)\cdot\mathbf{L}[\mathbf{U}(\mathbf{x},t)]-\mathbf{U}^T(\mathbf{x},t)\cdot\mathbf{L}[\mathbf{V}(\mathbf{x},t)]\}dV\\
&=\int_{\Omega}\{\bar{\mathbf{V}}(\mathbf{x})\cdot\mathbf{T}[\bar{\mathbf{U}}(\mathbf{x})]\cdot n-\bar{\mathbf{U}}(\mathbf{x})\cdot\mathbf{T}[\bar{\mathbf{V}}(\mathbf{x})]\cdot n\}dV
\end{aligned} \tag{15}$$

where $\mathbf{n}$ is the external normal vector of the structure surface $\partial\Omega$,

$$\mathbf{T}[\bar{\mathbf{U}}(\mathbf{x})] = \begin{bmatrix} \mathbf{c}\cdot\mathbf{s}(\mathbf{u}) + \nabla\phi_1\cdot\mathbf{f} + \nabla\psi_1\cdot\mathbf{d} \\ \mathbf{f}^T\cdot\mathbf{s}(\mathbf{u}) - \nabla\phi_1\cdot\boldsymbol{\varepsilon} - \nabla\psi_1\cdot\mathbf{h} \\ \mathbf{d}^T\cdot\mathbf{s}(\mathbf{u}) - \nabla\phi_1\cdot\mathbf{h} - \nabla\psi_1\cdot\boldsymbol{\mu} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\sigma}[\bar{\mathbf{U}}(\mathbf{x})] \\ \mathbf{D}[\bar{\mathbf{U}}(\mathbf{x})] \\ \mathbf{B}[\bar{\mathbf{U}}(\mathbf{x})] \end{bmatrix},$$

$$\mathbf{T}[\bar{\mathbf{V}}(\mathbf{x})] = \begin{bmatrix} \mathbf{c}\cdot\mathbf{s}(\mathbf{v}) + \nabla\phi_1\cdot\mathbf{f} + \nabla\psi_1\cdot\mathbf{d} \\ \mathbf{f}^T\cdot\mathbf{s}(\mathbf{v}) - \nabla\phi_1\cdot\boldsymbol{\varepsilon} - \nabla\psi_1\cdot\mathbf{h} \\ \mathbf{d}^T\cdot\mathbf{s}(\mathbf{v}) - \nabla\phi_1\cdot\mathbf{h} - \nabla\psi_1\cdot\boldsymbol{\mu} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\sigma}[\bar{\mathbf{V}}(\mathbf{x})] \\ \mathbf{D}[\bar{\mathbf{V}}(\mathbf{x})] \\ \mathbf{B}[\bar{\mathbf{V}}(\mathbf{x})] \end{bmatrix}. \qquad (16)$$

On the structure surface, in the form of the harmonic wave, integral (15) equals

$$\bar{\mathbf{V}}^T(\mathbf{x})\cdot\mathbf{T}[\bar{\mathbf{U}}(\mathbf{x})]\cdot\mathbf{n} - \bar{\mathbf{U}}^T(\mathbf{x})\cdot\mathbf{T}[\bar{\mathbf{V}}(\mathbf{x})]\cdot\mathbf{n} = 0, \text{ on } \partial\Omega \qquad (17)$$

and formula (17) can be written in the following determinant form:

$$\det\begin{pmatrix} \mathbf{U}_m & \mathbf{T}_m(\bar{\mathbf{U}}) \\ \mathbf{V}_m & \mathbf{T}_m(\bar{\mathbf{V}}) \end{pmatrix} = 0, \text{ on } \partial\Omega \qquad (18)$$

where blocking matrix $\mathbf{U}_m = diag(U_1, U_2, U_3, U_4, U_5)$, $\mathbf{V}_m = diag(V_1, V_2, V_3, V_4, V_5)$ and $\mathbf{T}_m = diag(T_{1j}n_j, T_{2j}n_j, T_{3j}n_j, T_{4j}n_j, T_{5j}n_j)$ are all diagonal matrixes. Furthermore we obtain

$$a_iU_i + b_iT_{ij}(\bar{\mathbf{U}})n_j = 0, \qquad (19)$$

$$a_iV_i + b_iT_{ij}(\bar{\mathbf{V}})n_j = 0, \text{ i from 1 to 5 (do not do summation over i on } \partial\Omega)$$

where a_i and b_i $(i = 1,2,3,4,5)$ are dimensional constants, and will not equal zero at the same time.

Because the orthogonal sets $\bar{\mathbf{U}}(\mathbf{x})$ and $\bar{\mathbf{V}}(\mathbf{x})$have the same form, only one of the two formulas (19) needs to be taken,

$$a_iU_i + b_iT_{ij}(\bar{\mathbf{U}})n_j = 0, \text{ i from 1 to 5 (do not do summation over i on } \partial\Omega) \qquad (20)$$

Now, formula (20) contains a lot of information. First, it holds on the structure surface, involving every boundary condition; second, it satisfies arbitrary structures, because of the external normal vector; third, it fully reflects the constitutive relation of the magneto-electro-elastic dielectric medium; fourth, it gives the restriction condition between the displacements, electric and magnetic potentials and the stress, electric displacement and magnetic induction. So we name formula (20) the guided-wave restriction condition in the magneto-electro-elastic guided-wave system. We can also get that there exists a group of different orthogonal set matching every boundary condition. This proves that different boundary conditions adopt different orthogonal sets in refs. [6∼8].

When a_i and b_i ($i = 1,2,3,4,5$) only equal 1 or 0, there are 32(=2^5) group of boundary conditions in the same normal direction. But most of them only have meaning in mathematics, and will not appear in a real situation. We name these boundary conditions self-adjoint boundary conditions to make a distinction with the other boundary conditions. Formula (20) implicates the constitutive relation. If the boundary condition only has the displacements, electric and magnetic potentials, namely, $a_i = 1$ and $b_i = 0$ ($i = 1,2,3,4,5$), which means that the constraint of the constitutive relation is removed, we need to add the constitutive relation to constrain the solution space in this case.

So we know that the self-adjoint method describes the inner relations among the physical property of the magneto-electro-elastic dielectric, the space structure and the boundary condition of the guided-wave system. Then we need only find the corresponding orthogonal sets to thoroughly solve the guided-wave problem. In detail, for a real guided-wave system, if the corresponding orthogonal sets are found, by which the displacements, electric and magnetic potentials that obey the operator equation (10) are derived, then the differential operator (11) is applied to mapping wave propagation from the time domain to the frequency domain. The relation between the wave number and the frequency is decided, namely, the dispersive equation is obtained.

4 Wave propagation in MEESC

Square column is an easily manufactured simple structure with a widely applying range. It is very easy to install the detecting devices on its boundary. If the solution of the wave propagation and the mechanical and electromagnetic signs transmission (energy transportation) characteristic is obtained, it will help us in applying the magneto-electro-elastic dielectric medium and structure.

The oxy plane is the cross section of the infinite square column; the x- and y-axes are normal to the boundary. The square column is $2L_1$ high and $2L_2$ wide. Figure 1 is the map of the exhibition of the MEESC.

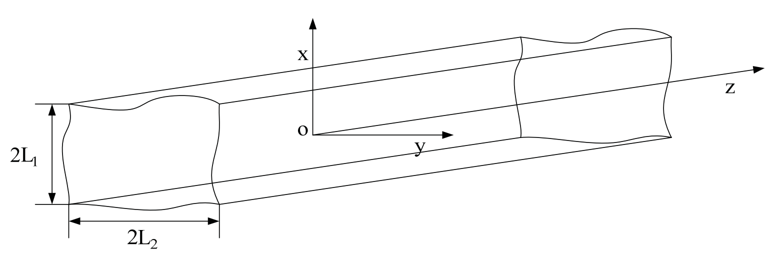

Figure 1: The map of the exhibition of MEESC.

4.1 Wave propagation governing equation in MEESC

The magneto-electro-elastic constitutive relation considered in this paper, is

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \\ D_x \\ D_y \\ D_z \\ B_x \\ B_y \\ B_z \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & & & & & & -f_{31} & & & -d_{31} \\ c_{12} & c_{11} & c_{13} & & & & & & -f_{31} & & & -d_{31} \\ c_{13} & c_{13} & c_{33} & & & & & & -f_{33} & & & -d_{33} \\ & & & c_{44} & & & & -f_{15} & & & -d_{15} & \\ & & & & c_{44} & & -f_{15} & & & -d_{15} & & \\ & & & & & c_{55} & & & & & & \\ & & & & f_{15} & & \varepsilon_{11} & & & h_{11} & & \\ & & & f_{15} & & & & \varepsilon_{11} & & & h_{11} & \\ f_{31} & f_{31} & f_{33} & & & & & & \varepsilon_{33} & & & h_{33} \\ & & & & d_{15} & & h_{11} & & & \mu_{11} & & \\ & & & d_{15} & & & & h_{11} & & & \mu_{11} & \\ d_{31} & d_{31} & d_{33} & & & & & & h_{33} & & & \mu_{33} \end{bmatrix} \begin{bmatrix} s_{xx} \\ s_{yy} \\ s_{zz} \\ 2s_{yz} \\ 2s_{zx} \\ 2s_{xy} \\ E_x \\ E_y \\ E_z \\ H_x \\ H_y \\ H_z \end{bmatrix}. \quad (21)$$

Substituting constitutive relation (21) into differential operator (11) yields

$$
\mathbf{L} = \begin{bmatrix}
c_{11}\frac{\partial^2}{\partial x^2}+c_{55}\frac{\partial^2}{\partial y^2}+c_{44}\frac{\partial^2}{\partial z^2}-\rho\frac{\partial^2}{\partial t^2} & (c_{12}+c_{55})\frac{\partial^2}{\partial x\partial y} & (c_{13}+c_{44})\frac{\partial^2}{\partial x\partial z} & (f_{15}+f_{31})\frac{\partial^2}{\partial x\partial z} & (d_{15}+d_{31})\frac{\partial^2}{\partial x\partial z} \\
(c_{12}+c_{55})\frac{\partial^2}{\partial x\partial y} & c_{55}\frac{\partial^2}{\partial x^2}+c_{11}\frac{\partial^2}{\partial y^2}+c_{44}\frac{\partial^2}{\partial z^2}-\rho\frac{\partial^2}{\partial t^2} & (c_{13}+c_{44})\frac{\partial^2}{\partial y\partial z} & (f_{15}+f_{31})\frac{\partial^2}{\partial y\partial z} & (d_{15}+d_{31})\frac{\partial^2}{\partial y\partial z} \\
(c_{13}+c_{44})\frac{\partial^2}{\partial x\partial z} & (c_{13}+c_{44})\frac{\partial^2}{\partial y\partial z} & c_{44}(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2})+c_{33}\frac{\partial^2}{\partial z^2}-\rho\frac{\partial^2}{\partial t^2} & f_{15}(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2})+f_{33}\frac{\partial^2}{\partial z^2} & d_{15}(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2})+d_{33}\frac{\partial^2}{\partial z^2} \\
(f_15+f_{31})\frac{\partial^2}{\partial x\partial z} & (f_15+f_{31})\frac{\partial^2}{\partial y\partial z} & f_{15}(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2})+f_{33}\frac{\partial^2}{\partial z^2} & -\varepsilon_{11}(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2})-\varepsilon_{33}\frac{\partial^2}{\partial z^2} & -h_{11}(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2})-h_{33}\frac{\partial^2}{\partial z^2} \\
(d_{15}+d_{31})\frac{\partial^2}{\partial x\partial z} & (d_{15}+d_{31})\frac{\partial^2}{\partial y\partial z} & d_{15}(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2})+d_{33}\frac{\partial^2}{\partial z^2} & -h_{11}(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2})-h_{33}\frac{\partial^2}{\partial z^2} & -\mu_{11}(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2})-\mu_{33}\frac{\partial^2}{\partial z^2}
\end{bmatrix} \qquad (22)
$$

So

$$\mathbf{L}[\mathbf{U}(\mathbf{x},t)] = \mathbf{0} \qquad (23)$$

is the wave propagation governing equation about the constitutive relation (21), where the displacements, electric and magnetic potentials are

$$\mathbf{U}(\mathbf{x},t) = \mathbf{U}(x,y,z,t) = (\mathbf{u},\phi,\psi)^T = (u,v,w,\phi,\psi)^T. \qquad (24)$$

On the square column boundaries, the guided-wave restriction condition can be simplified as

$$
\begin{array}{l}
a_1 u \pm b_1\sigma_{xx} = 0 \\
a_2 v \pm b_2\sigma_{yx} = 0 \\
a_3 w \pm b_3\sigma_{zx} = 0 \\
a_4\phi \pm b_4 D_x = 0 \\
a_5\psi \pm b_5 B_x = 0
\end{array}
\ , \text{ while } x = \pm L_1, \qquad (25a)
$$

$$\begin{array}{l} a_1 u \pm b_1 \sigma_{xy} = 0 \\ a_2 v \pm b_2 \sigma_{yy} = 0 \\ a_3 w \pm b_3 \sigma_{zy} = 0 \\ a_4 \phi \pm b_4 D_y = 0 \\ a_5 \psi \pm b_5 D_y = 0 \end{array}, \text{ while } y = \pm L_2. \tag{25b}$$

Now, after giving the boundary condition, the guided-wave system will be decided. If the orthogonal sets satisfying the restriction condition (25) can be found, the dynamic problem will be thoroughly solved.

4.2 Dispersive equation

Here, we select a group self-adjoint boundary condition B1

$$u = 0, \sigma_{xy} = 0, \sigma_{xz} = 0, D_x = 0, B_x = 0, x = \pm L_1,$$

$$v = 0, \sigma_{yx} = 0, \sigma_{yz} = 0, D_y = 0, B_y = 0, y = \pm L_2. \tag{26}$$

It means that the square column surface cannot move in x- and y-axes directions, but it can slide along z-axis direction, and the electric and magnetic fields are in an open circuit state at the boundary.

Because the square column is limited in x- and y-axes directions, we apply the standing wave superposition method to forming the wave propagation in x- and y-axes directions, and thus we assume the displacements, electric and magnetic potentials in the following forms.

$$\begin{aligned} \mathbf{U}(\mathbf{x},t) &= (\mathbf{u}, \phi, \psi)^T = (u, v, w, \phi, \psi)^T = \sum_{m,n} \mathbf{U}_{mn}(\mathbf{x},t) \\ &= \sum_{m,n} \mathbf{U}_{mn}(x,y) e^{i(kz-\omega t)} = \sum_{m,n} \mathbf{B}_{mn} \bar{\mathbf{U}}_{mn}(x,y) e^{i(kz-\omega t)} \\ &= \left[\sum_{m,n} u_{mn}(\mathbf{x},t) \quad \sum_{m,n} v_{mn}(\mathbf{x},t) \quad \sum_{m,n} w_{mn}(\mathbf{x},t) \quad \sum_{m,n} \phi_{mn}(\mathbf{x},t) \quad \sum_{m,n} \psi_{mn}(\mathbf{x},t) \right]^T \\ &= \left[\sum_{m,n} u_{mn}(x,y) \quad \sum_{m,n} v_{mn}(x,y) \quad \sum_{m,n} w_{mn}(x,y) \quad \sum_{m,n} \phi_{mn}(x,y) \quad \sum_{m,n} \psi_{mn}(x,y) \right]^T e^{i(kz-\omega t)} \\ &= \left[\sum_{m,n} A_{mn} \bar{u}_{mn} \quad \sum_{m,n} B_{mn} \bar{v}_{mn} \quad \sum_{m,n} C_{mn} \bar{w}_{mn} \quad \sum_{m,n} D_{mn} \bar{\phi}_{mn} \quad \sum_{m,n} E_{mn} \bar{\psi}_{mn} \right]^T e^{i(kz-\omega t)} \end{aligned} \tag{27}$$

where $\mathbf{B}_{mn}\bar{\mathbf{U}}_{mn}(x,y)$ is the mode, $\mathbf{B}_{mn}$ is the wave amplitude matrix, $\bar{\mathbf{U}}_{mn}(x,y)$ is the orthogonal sets, A_{mn}, B_{mn}, C_{mn}, D_{mn} and $E_{mn}$are the coefficients of the displacements, electric and magnetic potentials, respectively. $\mathbf{b} = (A_{mn}, B_{mn}, C_{mn}, D_{mn}, E_{mn})^T$ is the wave amplitude vector and $\mathbf{B}_{mn} = diag(A_{mn}, B_{mn}, C_{mn}, D_{mn}, E_{mn})$. m and n are the standing wave numbers; $m\pi/L_1$ and $n\pi/L_2$ will be nonnegative integers. k is the propagating wave number along z-axis direction. In our model, the wave, which keeps the standing wave form, propagates along the propagating wave direction. For the same standing wave form, the propagating waves are independent for different propagating wave numbers.

From the above assumption, we find the orthogonal sets satisfying the self-adjoint boundary condition B1, which are

$$\bar{u}_{mn}=\left\{\begin{bmatrix}\sin\frac{m\pi x}{2L_1}\\ \cos\frac{m\pi x}{2L_1}\end{bmatrix}\right\}\left\{\begin{bmatrix}\cos\frac{n\pi y}{2L_2}\\ \sin\frac{n\pi y}{2L_2}\end{bmatrix}\right\},\quad \bar{v}_{mn}=\left\{\begin{bmatrix}\cos\frac{m\pi x}{2L_1}\\ \sin\frac{m\pi x}{2L_1}\end{bmatrix}\right\}\left\{\begin{bmatrix}\sin\frac{n\pi y}{2L_2}\\ \cos\frac{n\pi y}{2L_2}\end{bmatrix}\right\},$$

$$\bar{w}_{mn}=\left\{\begin{bmatrix}\cos\frac{m\pi x}{2L_1}\\ \sin\frac{m\pi x}{2L_1}\end{bmatrix}\right\}\left\{\begin{bmatrix}\cos\frac{n\pi y}{2L_2}\\ \sin\frac{n\pi y}{2L_2}\end{bmatrix}\right\},\quad \bar{\phi}_{mn}=\left\{\begin{bmatrix}\cos\frac{m\pi x}{2L_1}\\ \sin\frac{m\pi x}{2L_1}\end{bmatrix}\right\}\left\{\begin{bmatrix}\cos\frac{n\pi y}{2L_2}\\ \sin\frac{n\pi y}{2L_2}\end{bmatrix}\right\},$$

$$\bar{\psi}_{mn}=\left\{\begin{bmatrix}\cos\frac{m\pi x}{2L_1}\\ \sin\frac{m\pi x}{2L_1}\end{bmatrix}\right\}\left\{\begin{bmatrix}\cos\frac{n\pi y}{2L_2}\\ \sin\frac{n\pi y}{2L_2}\end{bmatrix}\right\}, \tag{28}$$

when m is even, the first lines are adopted; when m is odd, the second lines are adopted. n has the same regular. When $m \neq a$ and $n \neq b$, we obtain

$$\int\limits_{S_{oxy}} \bar{\mathbf{U}}_{mn}^{T}(x,y)\cdot\bar{\mathbf{U}}_{ab}(x,y)dS=0. \tag{29}$$

where S_{oxy} is the middle plate of the square column.

Whether the different modes are orthogonal one another is decided by the standing wave number. So we get the complete orthogonal sets.

Substituting (27) and (28) into eq. (23), for fixed standing wave numbers m and n, we obtain

$$\mathrm{L}_1(\omega,k,m,n)\cdot\mathbf{U}_{mn}(x,y)=\mathbf{0}, \tag{30}$$

where the matrix $\mathbf{L}_1(\omega,k;m,n)$ is an Hermite's matrix. Its elements are in Appendix.

The differential operator (22) is applied to mapping the wave propagation from the time domain to the frequency domain, so matrix $\mathbf{L}_1(\omega,k;m,n)$ gives the relation between the wave number and the frequency of the guided-wave system. When the wave arrives, the displacements, electric and magnetic fields will not equal zero simultaneously. So the dispersive equation of the square column about the boundary condition B1 is

$$F(\omega,k;m,n)=\det[\mathbf{L}_1(\omega,k;m,n)]=0. \tag{31}$$

The elements of the matrix $\mathbf{L}_1(\omega,k;m,n)$ are all polynomials. $F(\omega,k;m,n)=0$ is a cubic polynomial equation about the circular frequency ω^2, so the ω^2 can be solved and analytically expressed by the parameters m, n and k, in other words, we get the analytical dispersive equation. There are three ω for every group wave number m, n and k. We denote the three wave number-frequency groups as $(\omega_i,k;m,n), i=1,2,3$.

There is the relation $\omega = sc$ among the frequency, the wave number $s=\sqrt{m^2+n^2+k^2}$ and the phase velocity c, so

$$F(sc,k;m,n)=0 \tag{32}$$

is the phase velocity equation.

4.3 Group velocity equation

The x- and y-axes directions are the standing wave directions. When the wave number m and n are given, the energy exchange in the standing wave plane is zero in one period time. The group velocity can be defined as $c_g=\frac{d\omega}{dk}$. The relation between the group velocity and the wave number-frequency group is that the propagating wave keeps the standing wave form and propagates along the z-axis direction with the speed of this group velocity. Differentiating eq. (31) produces $dF=\frac{\partial F}{\partial k}dk+\frac{\partial F}{\partial \omega}d\omega=0$ and $\frac{\partial F}{\partial k}+\frac{\partial F}{\partial \omega}c_g=0$, then eliminating ω with eq. (31), the group velocity equation is

$$G(c_g,k;m,n)=0. \tag{33}$$

4.4 Steady-state response

The relation between the wave number-frequency groups $(\omega_i,k;m,n), i=1,2,3$ and the matrix $\mathbf{L}_1(\omega,k;m,n)$ will be discussed again. First the matrix $\mathbf{L}_1(\omega,k;m,n)$ is blocked, and then it becomes

$$\mathbf{L}_1(\omega,k;m,n)=\begin{bmatrix}\mathbf{A}-\rho\omega^2\mathbf{I} & \mathbf{R}\\ \bar{\mathbf{R}}^T & \mathbf{S}\end{bmatrix}, \tag{34}$$

where $\mathbf{A}-\rho\omega^2\mathbf{I}=\begin{bmatrix}L_{111} & L_{112} & L_{113}\\ L_{121} & L_{122} & L_{123}\\ L_{131} & L_{132} & L_{133}\end{bmatrix}$, $\mathbf{R}=\begin{bmatrix}L_{114} & L_{115}\\ L_{124} & L_{125}\\ L_{134} & L_{135}\end{bmatrix}$, $\mathbf{S}=\begin{bmatrix}L_{144} & L_{145}\\ L_{154} & L_{155}\end{bmatrix}$.

Because $\det[\mathbf{L}_1(\omega,k;m,n)]=0$, the equation

$$\mathbf{L}_1(\omega,k;m,n)\mathbf{b}=\mathbf{0} \tag{35}$$

has nonzero solution. The vector is denoted as $\mathbf{b} = (b_1, b_2, b_3, b_4, b_5)^T = (\mathbf{b}_1^*, b_4, b_5)^T = (\mathbf{b}_1^*, \mathbf{b}_2^*)^T$. Eq. (35) changes to

$$\begin{bmatrix} \mathbf{A} - \rho\omega^2\mathbf{I} & \mathbf{R} \\ \bar{\mathbf{R}}^T & \mathbf{S} \end{bmatrix} \begin{bmatrix} (\mathbf{b}_1^*)^T \\ (\mathbf{b}_2^*)^T \end{bmatrix} = \mathbf{0}, \tag{36}$$

so

$$(\mathbf{A} - \rho\omega^2\mathbf{I})(\mathbf{b}_1^*)^T + \mathbf{R}(\mathbf{b}_2^*)^T = \mathbf{0}, \tag{37a}$$

$$\bar{\mathbf{R}}^T(\mathbf{b}_1^*)^T + \mathbf{S}(\mathbf{b}_2^*)^T = \mathbf{0}, \tag{37b}$$

then we obtain

$$(\mathbf{A} - \mathbf{R}\mathbf{S}^{-1}\bar{\mathbf{R}}^T - \rho\omega^2\mathbf{I})(\mathbf{b}_1^*)^T = \mathbf{0}, \tag{38a}$$

$$(\mathbf{b}_2^*)^T = -\mathbf{S}^{-1}\bar{\mathbf{R}}^T(\mathbf{b}_1^*)^T. \tag{38b}$$

In formula (38a), the matrix $\mathbf{A} - \mathbf{R}\mathbf{S}^{-1}\bar{\mathbf{R}}^T$ is an Hermite's matrix, and $\rho\omega^2$ is the eigenvalue, so the eigenvector must exist, denoted by $\mathbf{b}_1^*(\omega_1), \mathbf{b}_1^*(\omega_2), \mathbf{b}_1^*(\omega_3)$. And we have

$$[\mathbf{b}_1^*(\omega_i)]^T \cdot \mathbf{b}_1^*(\omega_j) = 0, i \neq j. \tag{39}$$

In formula (39), the stress wave amplitude vectors, which are formed by the three wave amplitudes about the displacement, are orthogonal at the same group wave number with the different frequencies. This wave amplitude vector decides the orthogonal modes when the three group wave numbers are the same. If the matrix $\mathbf{R}\mathbf{S}^{-1}\bar{\mathbf{R}}^T = \mathbf{0}$, this modal will degenerate to the elastic guided-wave system. The influences of the induced electromagnetic field come from matrix $\mathbf{R}\mathbf{S}^{-1}\bar{\mathbf{R}}^T$. In formula (38b), the wave amplitude vectors $\mathbf{b}_2^*(\omega_1), \mathbf{b}_2^*(\omega_2), \mathbf{b}_2^*(\omega_3)$ are derived, so the wave amplitude vectors $\mathbf{b}(k, \omega_i; m, n), i = 1, 2, 3$ are obtained. But we need point out that $[\mathbf{b}(\omega_i)]^T \cdot \mathbf{b}(\omega_j) \neq 0, i \neq j$ and $[\mathbf{b}_2^*(\omega_i)]^T \cdot \mathbf{b}_2^*(\omega_j) \neq 0, i \neq j$ at the same group wave number. This phenomenon comes from the assumption of this modal; in detail, it comes from formulas (8) and (9). But it surpasses the discussion range in this paper, so we will not debate it any more. But in formula (38b), it is hard to get vector $\mathbf{b}_1^*(\omega)$ by vector$\mathbf{b}_2^*(\omega)$. If only an electric or magnetic signal is known, we will obtain the wave numbers m, n and k, and know the wave type, so a coefficient of vector $\mathbf{b}_1^*(\omega)$ can be derived. This character will be used in studying the transient-state response.

When the wave amplitude vector is obtained, for arbitrary standing wave numbers m and n, the steady-state response is derived

$$\mathbf{B}_{mn}(\omega)\cdot\bar{\mathbf{U}}_{mn}(x,y) = \begin{bmatrix} \mathbf{B}_{1mn}(\omega) & & \\ & b_{4mn}(\omega) & \\ & & b_{5mn}(\omega) \end{bmatrix} \begin{bmatrix} \bar{\mathbf{U}}_{1mn}(x,y) \\ \bar{\phi}_{mn}(x,y) \\ \bar{\psi}_{mn}(x,y) \end{bmatrix}$$

(do not do summation over m and n), (40)

where

$$\mathbf{B}_{1mn}(\omega) = diag[b_{1mn}(\omega), b_{2mn}(\omega), b_{3mn}(\omega)]$$

and

$$b_{1mn}^2(\omega) + b_{2mn}^2(\omega) + b_{3mn}^2(\omega) = 1.$$

5 Dispersive spectrum, group velocity curve and steady-state response curve

In this part, the spectrum, the group velocity curve and the steady-state response curve will be plotted. In computation, the following parameters are taken: $c_{11} = 166Gp_a$, $c_{12} = 77Gp_a$, $c_{13} = 78Gp_a$, $c_{33} = 162Gp_a$, $c_{44} = 43Gp_a$, $c_{55} = (c_{11} - c_{22})/2$, $f_{15} = 11.6C/m^2$, $f_{31} = -4.4C/m^2$, $f_{33} = 18.6C/m^2$, $\varepsilon_{11} = 11.2nF/m$, $\varepsilon_{33} = 12.6nF/m$, $d_{15} = 550N/Am$, $d_{31} = 580.3N/Am$, $d_{33} = 699.7N/Am$, $h_{11} = 5.0\times 10^{-12}Ns/VC$, $h_{33} = 3.0\times 10^{-12}Ns/VC$, $\mu_{11} = 5.0\times 10^{-5}Ns^2/C^2$, $\mu_{33} = 1.0\times 10^{-5}Ns^2/C^2$ and $\rho = 7500kg/m^3$. Before plotting the figures, we introduce the non-dimensional frequency $\Omega = \omega/c_S$ and group velocity$C_g = c_g/c_S$, where $c_S = \sqrt{c_{44}/\rho}$.

Figure 2 shows the dispersive spectrum of MEESC. The positive and negative parts of the abscissas k represent the real and imaginary numbers. For every standing wave number group (m, n), there are three curves, and they are corresponding to the Quasi-P, Quasi-SV and Quasi-SH waves, and they do not cross one another. In this figure, the dispersive curves of standing wave numbers m=1 and n from 0 to 12 are plotted. The same type curves are ordered by the standing wave number, and they do not cross one another. The slopes of line S and line P are corresponding to the phase velocities c_S and

$$c_P = \sqrt{[c_{33} + (d_{33}^2\varepsilon_{33} + f_{33}^2\mu_{33} - 2f_{33}d_{33}h_{33})/(\varepsilon_{33}\mu_{33} - h_{33}^2)]/c_{44}},$$

which are the phase velocities in infinite body in z-axis direction under constitutive relation (21).

It is how we named the Quasi-P waves. In this figure, the whole property of the dispersive spectrum is displayed.

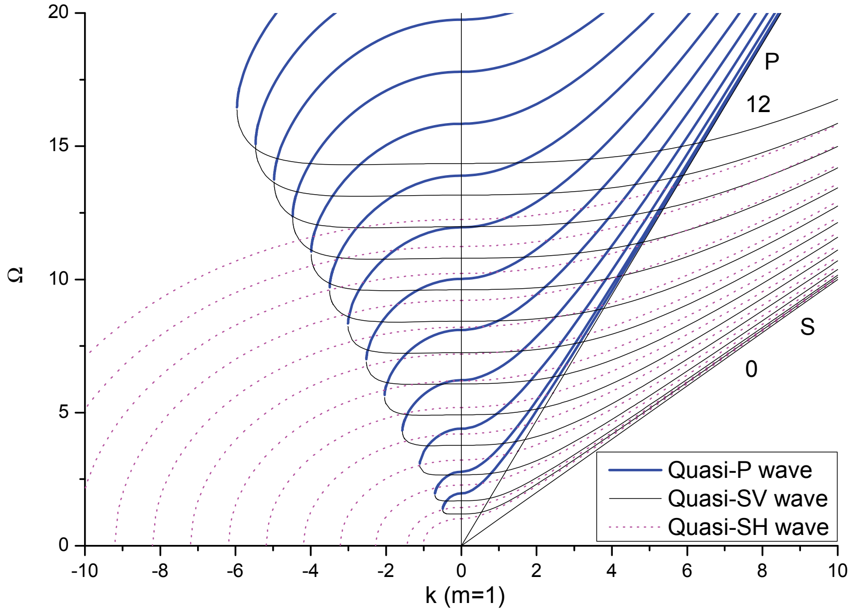

Figure 2: Dispersive spectrum of MEESC.

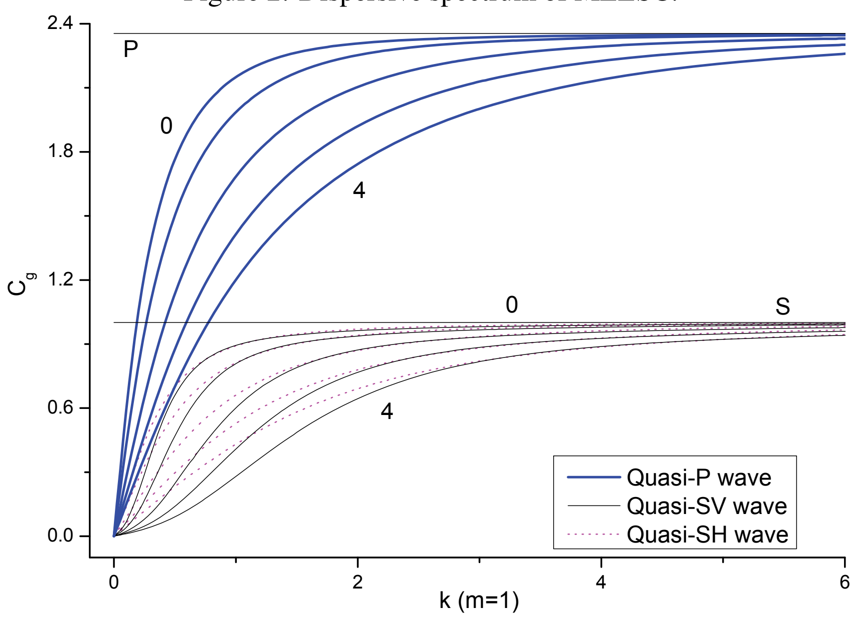

Figure 3: Group velocity curves of MEESC.

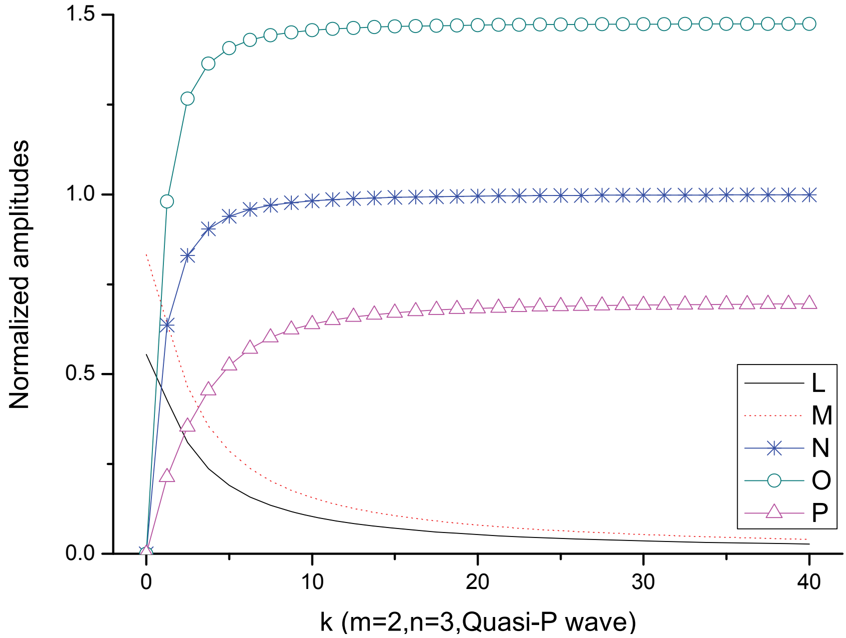

Figure 4: Steady-state response curves of MEESC.

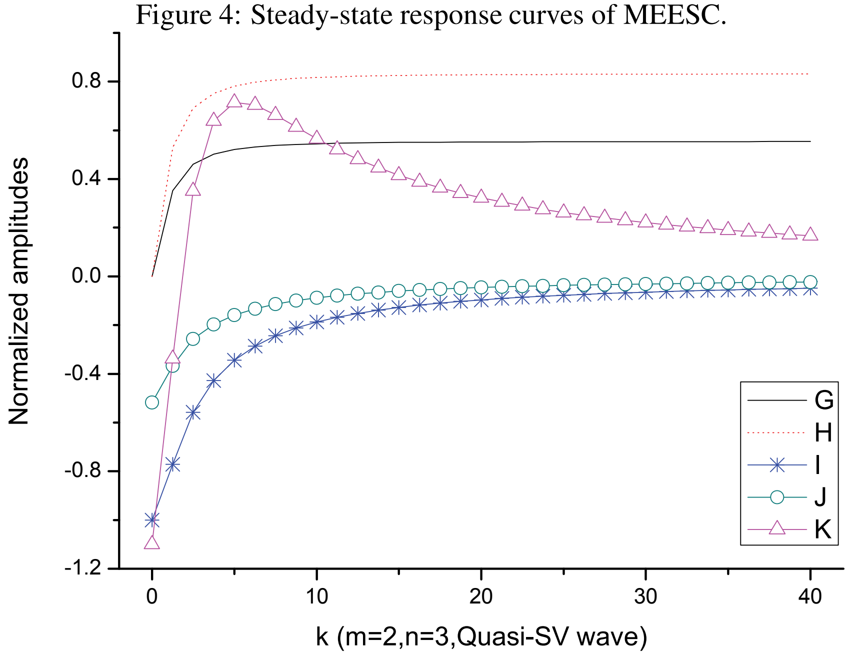

Figure 5: Steady-state response curves of MEESC.

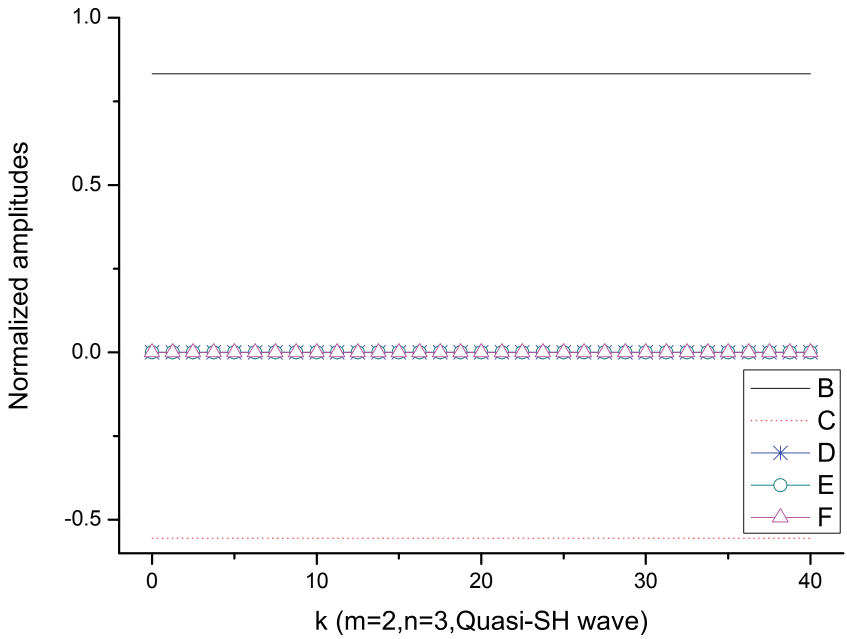

Figure 6: Steady-state response curves of MEESC.

In figure 3, the group velocity curves are displayed, which are classified by the wave type and ordered by the standing wave number. Line P and line S are corresponding to the group velocities c_P and c_S, which are the group velocities in infinite body in z-axis direction under constitutive relation (21).

In figure 2 and 3, the curves asymptotically tend to the line P and line S when the propagating wave number k increases to infinite. It means that when the wave number is larger, the wave length is shorter, the square column can be treated as infinite body, so the phase velocities and group velocities in the square column will tend toc_P and c_S.

Figure 4, 5 and 6 show the steady-state response curves corresponding to the Quasi-P, Quasi-SV and Quasi-SH waves of m=2 and n=3, where the normalized amplitudes of the displacement are plotted, and in order to display the amplitudes of the electric and magnetic potentials, so the amplitudes of the electric and magnetic potentials are reduced by 10^9 and 10^8 times in figure 4; the amplitudes of the electric and magnetic potentials are reduced by $2{*}10^9$ and 10^7 times in figure 5.

The three steady-state response modes are corresponding to the extensional, thickness-twist and thickness-shear modes of the square column. It is how we named the Quasi-SH waves. In figure 4, the curves of$w$, ϕ and ψhave the similar curve, it is means that the induced electric and magnetic fields are deeply aroused by the Quasi-P stress wave. In figure 5, the curves of$w$, ϕ and ψtend to zero, it is means that the induced electric and magnetic fields are slightly aroused by the Quasi-SV stress wave in high frequency band. But in low frequency band the amplitude of ψ has a peak value, this phenomenon need pay attention. Because the Quasi-SH wave can't be affected by the induced electric and magnetic fields, or the induced electric and magnetic fields can't be aroused by the Quasi-SH stress wave, and the amplitude of w tends to zero, so the three curves of$w$, ϕ and ψ are superposition in one line in figure 6.

6 Conclusions

The stress wave propagations affected by the induced electric and magnetic fields in the MEESC were studied and some new characteristics about the guided waves in it were discovered. They are

1. The guided stress waves are classified in the forms of the Quasi-P, Quasi-SV and Quasi-SH waves corresponding to the extensional, thickness-twist and thickness-shear modes of the square column, and are ordered by the standing wave number.

2. The propagating waves in the channel formed by the Quasi-P, Quasi-SV and Quasi-SH waves affected by the induced electric and magnetic fields are obviously different.

3. The Quasi-P waves are thoroughly affected by the induced electric and magnetic fields.

4. The Quasi-SV waves are affected by the induced electric and magnetic fields only in the lower frequency band.

5. The Quasi-SH wave is hardly affected by the induced electric and magnetic fields.

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Appendix

The elements of $\mathbf{L}_{1ij}(\omega,k,m,n)$:

$$\mathbf{L}_{111} = c_{11}\frac{m^2\pi^2}{4\mathbf{L}_1^2} + c_{55}\frac{n^2\pi^2}{4\mathbf{L}_2^2} + c_{44}k^2 - \rho\omega^2, \quad \mathbf{L}_{112} = (-1)^{m+n}(c_{12}+c_{55})\frac{mn\pi^2}{4\mathbf{L}_1\mathbf{L}_2},$$

$$\mathbf{L}_{113} = i(-1)^m (c_{13}+c_{44})\frac{\pi mk}{2\mathbf{L}_1}, \quad \mathbf{L}_{114} = i(-1)^m (f_{15}+f_{31})\frac{\pi mk}{2\mathbf{L}_1},$$

$$\mathbf{L}_{115} = i(-1)^m (d_{15}+d_{31})\frac{\pi mk}{2\mathbf{L}_1}, \quad \mathbf{L}_{121} = (-1)^{m+n} (c_{12}+c_{55})\frac{mn\pi^2}{4\mathbf{L}_1\mathbf{L}_2},$$

$$\mathbf{L}_{122} = c_{55}\frac{m^2\pi^2}{4\mathbf{L}_1^2} + c_{11}\frac{n^2\pi^2}{4\mathbf{L}_2^2} + c_{44}k^2 - \rho\omega^2, \quad \mathbf{L}_{123} = i(-1)^n (c_{13}+c_{44})\frac{\pi nk}{2\mathbf{L}_2},$$

$$\mathbf{L}_{124} = i(-1)^n (f_{15}+f_{31})\frac{\pi nk}{2\mathbf{L}_2}, \quad \mathbf{L}_{125} = i(-1)^n (d_{15}+d_{31})\frac{\pi nk}{2\mathbf{L}_2},$$

$$\mathbf{L}_{131} = i(-1)^{m+1} (c_{13}+c_{44})\frac{\pi mk}{2\mathbf{L}_1}, \quad \mathbf{L}_{132} = i(-1)^{n+1} (c_{13}+c_{44})\frac{\pi nk}{2\mathbf{L}_2},$$

$$\mathbf{L}_{133} = c_{44}\frac{m^2\pi^2}{4\mathbf{L}_1^2} + c_{44}\frac{n^2\pi^2}{4\mathbf{L}_2^2} + c_{33}k^2 - \rho\omega^2, \quad \mathbf{L}_{134} = f_{15}\frac{m^2\pi^2}{4\mathbf{L}_1^2} + f_{15}\frac{n^2\pi^2}{4\mathbf{L}_2^2} + f_{33}k^2,$$

$$\mathbf{L}_{135} = d_{15}\frac{m^2\pi^2}{4\mathbf{L}_1^2} + d_{15}\frac{n^2\pi^2}{4\mathbf{L}_2^2} + d_{33}k^2, \quad \mathbf{L}_{141} = i(-1)^{m+1} (f_{15}+f_{31})\frac{\pi mk}{2\mathbf{L}_1},$$

$$\mathbf{L}_{142} = i(-1)^{n+1} (f_{15}+f_{31})\frac{\pi nk}{2\mathbf{L}_2}, \quad \mathbf{L}_{143} = f_{15}\frac{m^2\pi^2}{4\mathbf{L}_1^2} + f_{15}\frac{n^2\pi^2}{4\mathbf{L}_2^2} + f_{33}k^2,$$

$$\mathbf{L}_{144} = -\varepsilon_{11}\frac{m^2\pi^2}{4\mathbf{L}_1^2} - \varepsilon_{11}\frac{n^2\pi^2}{4\mathbf{L}_2^2} - \varepsilon_{33}k^2, \quad \mathbf{L}_{145} = -h_{11}\frac{m^2\pi^2}{4\mathbf{L}_1^2} - h_{11}\frac{n^2\pi^2}{4\mathbf{L}_2^2} - h_{33}k^2,$$

$$\mathbf{L}_{151} = i(-1)^{m+1} (d_{15}+d_{31})\frac{\pi mk}{2\mathbf{L}_1}, \quad \mathbf{L}_{152} = i(-1)^{n+1} (d_{15}+d_{31})\frac{\pi nk}{2\mathbf{L}_2},$$

$$\mathbf{L}_{153} = d_{15}\frac{m^2\pi^2}{4\mathbf{L}_1^2} + d_{15}\frac{n^2\pi^2}{4\mathbf{L}_2^2} + d_{33}k^2, \quad \mathbf{L}_{154} = -h_{11}\frac{m^2\pi^2}{4\mathbf{L}_1^2} - h_{11}\frac{n^2\pi^2}{4\mathbf{L}_2^2} - h_{33}k^2,$$

$$\mathbf{L}_{155} = -\mu_{11}\frac{m^2\pi^2}{4\mathbf{L}_1^2} - \mu_{11}\frac{n^2\pi^2}{4\mathbf{L}_2^2} - \mu_{33}k^2.$$

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