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Integral Method for Contact Problem of Bonded Plane Material with Arbitrary Cracks

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Abstract: A problem for bonded plane material with a set of curvilinear cracks, which is under the action of a rigid punch with the foundation of convex shape, has been considered in this paper. Kolosov-Muskhelishvili complex potentials are constructed as integral representations with the Cauchy kernels with respect to derivatives of displacement discontinuities along the crack contours and pressure under the punch. The contact of crack faces is considered. The considered problem has been transformed to a system of complex Cauchy type singular integral equations of first and second kind. The presented approach allows to consider various configurations of cracks and the punch foundation. Numerical results for the bonded plane material with a vertical crack or a horizontal crack under the frictionless flat punch are obtained.

Keyword: Cracks, contact, rigid punch, singular integral equation, numerical solution.

1 Introduction

The problem of contact interaction (contact problem) as one of the most popular means of transmitting external forces is important for both fundament developments on solid mechanics and various branches of modern engineering. Typical contact problem such as the contact between running pulley and rollay, gear outer ring and bearing box in bearing systems, gears and connecting elements, frictional contact between wheel and rail in the rolling process, the contact interaction between different rock formations and the coupling contact between structure and soil ground in the rock-soil engineering, are well known [Christensen, Klarbring, and Pang (1998); Liu, Liu, L., and Mahadevan (2007); Keppas, Giannopoulos, and Anifantis (2008); Zhang, He, and Li (2008)].

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During the contact of the units of machines and structures, dynamic cracks [Maiti and Geubelle (2004); Nishioka (2005); Gao, Liu, and Liu, Y. (2006); Guo and Nairn (2006); Liu, Han, Rajendran, and Atluri (2006); Rabczuk and Areias (2006); Rabczuk and Belytschko (2006, 2007); Song, Areias, and Belytschko (2006); Le, Mai-Duy, and Tran-Cong (2008); Xu, Dong, and Zhang (2008); Zhang and Chen (2008)] and static cracks [Shah, Tan, and Wang (2006); Rabczuk and Zi (2007); Zi, Rabczuk, and Wall (2007); Ma, Levy, and Perl (2008)] often arise and propagate in the contact zones. These cracks cause the loss of the units serviceability. Thus, it is important for engineering practice to study the fracture of elastic [Miers and Telles (2006)], gradient elasticity [Karlis, Tsinopoulos, Polyzos, and Beskos (2008)], elastic-plastic [Hagihara, Tsunori, Ikeda, and Miyazaki (2007); Long, Liu, and Li (2008)], elasto visco-plastic [Fujimoto and Nishioka (2006)], piezoelectric solids [Ozaki, Hashiguchi, Okayasu, and Chen (2007); Sanz, Solis, and Dominguez (2007); Sladek, Sladek, V. and Zhang (2007)], nonhomogeneous material [Sladek, Sladek, V. and Krivacek (2005); Liu, Long, and Li (2008); Wen, Aliabadi, and Liu (2008)], shell structure [Gato and Shie (2008)] and nonlinear elastic-plastic material [Nishioka, Kobayashi, and Fujimoto (2007)].

To approximate the contact interaction between rigid punches and a vertical crack, the method generally used is the dual integral equation method. For example, [Tonoyan and Melkumian (1973) and Tonoyan and Minasian (1976)] have used this method to study the contact problem of indentation of two rigid punches into an elastic half-plane with a vertical semi-infinite crack and a infinite crack, respectively. In their study, the initial problem was reduced to Fredholm integral equations of the second kind finally. On the other hand, some researchers use the conformal mapping method to investigate rigid punches and an inclined crack. For example, Qian and Hasebe (1997) have used this method to study contact problem of indentation of circular rigid punch into a semi-infinite plane with an oblique edge crack subjected to concentrated forces or point dislocations. The above work neglects the contact of crack faces.

When a crack is loaded, there have a contact interval (or intervals) between the crack faces. Ioakimidis (1982) investigated the classical problem of a finite single straight crack inside an infinite isotropic elastic plane under a polynomial normal loading distribution along both crack faces. He derived a quantifier-free formula of general validity concerning the conditions for contact/lack of contact [Ioakimidis (1997)] of the crack faces along the whole crack in terms of the loading parameters, taking into consideration that the crack opening displacement should satisfy the physically posed inequality constraint that it must be a continuously positive quantity (in the popular case of no contact). Contact problems along circular-arc-shaped cracks were also investigated [Toya (1974); Chao and Laws (1992); Chao and Rau

(1995)], where the phenomenon of contact along the crack under appropriate tensile loadings at infinity only becomes completely clear. Panasyuk, Datsyshyn, and Marchenko (2000) investigated contact problem for an elastic half-plane with edge or a set of curvilinear internal cracks under action of a rigid punch. In their study, it is assumed the crack faces are under conditions of either stick or smooth contact on contact parts. The problems of the crack edges contact interaction in 2-D and 3-D elastodynamics, with consideration for the contact interaction of the crack faces have been investigated in Guz and Zozulya (2002); Guz, Menshykov, and Zozulya (2003); Guz, Zozulya, and Menshikov (2004); Menshykov and Guz, I (2006, 2007); Guz, Menshykov, Zozulya, and Guz, I (2007); Guz and Zozulya (2007). The aforementioned studies were limited to the contact problems of a single isotropic infinite plane(space) or a single isotropic half-plane(half-space).

For contact problem of composite laminated plates, one of the most widely used laws is that proposed by Yang and Sun (1982). In their work, it is assumed that the contact pressure and contact area could be obtained from the usual formulas for isotropic materials, but with the isotropic modulus of elasticity replaced by the orthotropic modulus in the loading direction. Cairns and Lagace (1987) studied the thick composite laminates subjected to lateral loading by the stress function proposed by Leknitskii (1963). Wu and Yen (1994) and Chao and Tu (1999) used Pagano's solution as a Green's function approximating the static indentation of a cross-ply laminate to the contact force exerted by a rigid sphere. Recently, Chen, Xiong, and Shen (2008) proposed a modified Hertz contact law for the contact problem of a laminated plate indented by a rigid sphere. They analyzed the effects of the thickness, in-plane dimensions and boundary conditions of the plate. To the authors' knowledge contact problem for the composite bonded plane material with rectilinear arbitrary cracks under rigid punch action has not been studied yet.

In this paper, the singular integral equation method [Panasyuk, Savruk, and Datsyshyn (1976, 1977); Savruk (1981); Panasyuk, Datsyshyn, and Marchenko (1995, 2000)] is used to solve the contact problem for the bonded plane material with rectilinear arbitrary cracks under rigid punch action with the foundation of convex shape. Kolosov-Muskhelishvili complex potentials are constructed as integral representations with the Cauchy kernels with respect to derivatives of displacement discontinuities along the crack contours and pressure under the punch. As a result, the considered problem is reduced to a system of singular integral equations of first and second kind. An algorithm which is based on using the method of mechanical quadratures and an iterative procedure has been applied to find solution of these equations. The presented approach allows to consider various configurations of cracks and the punch foundation and also general conditions of interaction between the crack faces and between the punch and the bonded plane material. Besides, this approach allows to find effective numerical solution of obtained singular integral equation. Numerical results for bonded plane material with a vertical crack or a horizontal crack under the frictionless flat punch are obtained.

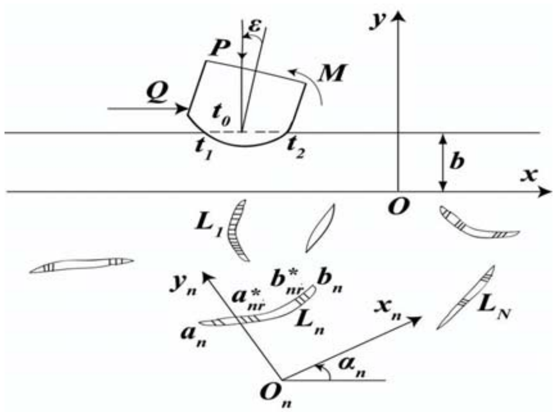

Figure 1: General scheme of the problem

2 Problem statement

Consider a problem of indention of a punch with convex foundation into an elastic strip bonded to a half-plane with cracks along the entire real axis Ox (Figure 1). The elastic constants of the strip and half-plane are κ_1, μ_1 and κ_2, μ_2 respectively. The contact area is γ_0. The positive direction of γ_0 is from t_1 to t_2. There are N cracks of arbitrary shapes in the half-plane, and these cracks are smooth and non-intersecting segment $L_k (k = 1, \cdots, N)$, the positive direction of L_k is from a_k to b_k. The bonded plane material is related to the system of coordinates xOy, while the cracks related to local systems $x_n O_n y_n$, connected with basic system xOy by the relation $z = z_n e^{i\alpha_n} + z_n^0$ $(z = x + i \cdot y, z_n = x_n + i \cdot y_n)$, z_n^0 is the affix of the point O_n in the basic system of coordinates, α_n is the angle of the $O_n x_n$-axis with Ox.

There are three kinds of loadings act on the punch, (i) a vertical force P with abscissa of the application point t_0, (ii) a horizontal force Q, (iii) a moment M. The boundary of bonded material is free outside the contact area γ_0. Due to the external loading, the punch is displaced transitionally in the direction parallel to the Oy-axis with simultaneous rotation through a certain small angle ε. Between the punch and the bonded plane material, frictional forces arise by the Coulomb friction; $Q = \rho P$, where ρ is the coefficient of friction. The punch fulfills following equilibrium conditions

$$\int_{\gamma_0} p_0(\zeta)d\zeta = -(P+iQ) \tag{1}$$

$$\int_{\gamma_0} (\zeta - t_0)p_0(\zeta)d\zeta = -M \tag{2}$$

Here, $p_0(\zeta) = -(1+i\rho)p(\zeta)$, $p(\zeta)$ is the unknown pressure under the punch.

Due to the punch indentation, the contact of crack faces may take place in compression zone. Thus, contact stress propagates. Outside the contact parts the crack faces are unstressed. We will limit our investigation to two limiting cases: (i) stick contact of the crack faces, (ii) smooth contact of the crack faces.

Let $a^*_{nr}b^*_{nr}(r = 1,\cdots,R_n)$ denotes the rth no contact zone on the nth crack. Thus , the whole no contact zones on the nth crack can be denoted by

$$L^*_n = \bigcup_{r=1}^{R_n} a^*_{nr}b^*_{nr} \quad (n = 1,\cdots N,\ r = 1,\cdots,R_n) \tag{3}$$

In addition, the points a^*_{nr}, b^*_{nr} at $r = 1$ and $r = R_n$ may coincide with crack tips a_n, b_n. Obviously, for any open crack, $L^*_n = L_n$.

Generally speaking, the borders of the contact parts between the crack faces and between the punch foundation and the bonded plane material boundary are unknown beforehand and can be found from additional equations. The borders of the contact parts of the cracks can be found from following equations

$$K_{In}(t_{nr}) = 0, \quad t_{nr} = a^*_{nr}, b^*_{nr}, \quad r = 1,\cdots,R_n,\ a^*_{nr} \neq a_n,\ b^*_{nr} \neq b_n \tag{4}$$

where K_{In} denotes the Model I stress intensity factor for the nth crack. Employing the condition that the contact pressure, $p(t) = -\sigma_y(t)(p(t) \geq 0, t \in \gamma_0$ for physical sense), at the edge points t_1 and $t_2$of this area is equal to zero, i.e.

$$p(t_1) = 0, \quad p(t_2) = 0 \tag{5}$$

the unknown width of the contact area between the punch foundation and the bonded plane material boundary can be determined.

The boundary conditions of the bonded plane material are following

$$\frac{dv(t)}{dt} = f'(t) + \varepsilon, \quad \tau_{xy}(t) = -\rho\sigma_y(t), \quad t \in \gamma_0 \tag{6}$$

$$\sigma_y(t) + i\tau_{xy}(t) = 0, \quad t \notin \gamma_0 \tag{7}$$

where u and v denote displacements on the Ox- and Oy-axes within the basic coordinate system, respectively. The punch foundation contour is described by smooth function $f(t)$.

Kolosov-Muskhelishvili complex potentials of the stated problem can be represented as function sums [Panasyuk, Datsyshyn, and Marchenko (2000)]

$$\Phi_0(z) = \Phi_1(z) + \Phi_2(z) \quad \Psi_0(z) = \Psi_1(z) + \Psi_2(z) \tag{8}$$

The functions $\Phi_1(z)$ and $\Psi_1(z)$ describe the stress state of the uncracked bonded plane material with boundary loaded by the contact stress $p_0(t)$.

The functions $\Phi_2(z)$ and $\Psi_2(z)$ define the bonded plane material stress state due to displacement discontinuities along the crack contours $L_n (n = 1, \cdots, N)$.

The boundary conditions have to satisfy along the interface (x-axes) of the bonded plane material are the continuity of the normal and shear stress components, as well as continuity of displacements, which can be written as [Ioakimidis, Theocaris (1979)]

$$\Phi_i^+(x) + \overline{\Phi_i^+(x)} + x\Phi_i^{'+}(x) + \Psi_i^+(x) = \Phi_i^-(x) + \overline{\Phi_i^-(x)} + x\Phi_i^{'-}(x) + \Psi_i^-(x) \tag{9}$$

$$\frac{1}{\mu_1}\left[\Phi_i^+(x) - \kappa_1\overline{\Phi_i^+(x)} + x\Phi_i^{'+}(x) + \Psi_i^+(x)\right] = \frac{1}{\mu_2}\left[\Phi_i^-(x) - \kappa_2\overline{\Phi_i^-(x)} + x\Phi_i^{'-}(x) + \Psi_i^-(x)\right] \tag{10}$$

where i =0,1,2, x denotes the points of x-axes, μ is the shear modulus, $\kappa = 3-4\nu$ for plane strain, $\kappa = \frac{3-\nu}{1+\nu}$ for plane stress, and ν is Poisson's ratio.

These functions are represented as follows:

$$\Phi_1(z) = \begin{cases} -\frac{1-A}{2\pi i}\int_{\gamma_0}\frac{p_0(\zeta)}{\zeta-z}d\zeta, & 0 < Imz \le b \\ -\frac{1}{2\pi i}\int_{\gamma_0}\frac{p_0(\zeta)}{\zeta-z}d\zeta + \frac{A\cdot b}{\pi}\int_{\gamma_0}\frac{\overline{p_0(\zeta)}}{(\overline{\zeta}-z)^2}d\zeta, & Imz < 0 \end{cases} \tag{11}$$

$$\Psi_1(z) = \begin{cases} \frac{A_0}{2\pi i}\int_{\gamma_0}\frac{\overline{p_0(\zeta)}}{\overline{\zeta}-z}d\overline{\zeta} + \frac{1-A}{2\pi i}\int_{\gamma_0}\frac{\overline{\zeta}p_0(\zeta)}{(\zeta-z)^2}d\zeta, & 0 < Imz \le b \\ \frac{1}{2\pi i}\int_{\gamma_0}\frac{\overline{\zeta}p_0(\zeta)}{(\zeta-z)^2}d\zeta & \\ +\frac{1}{2\pi i}\int_{\gamma_0}\left[\frac{A_0}{\overline{\zeta}-z} - \frac{A}{\overline{\zeta}-z} - \frac{2\cdot A\cdot i\cdot b\cdot(\overline{\zeta}+z)}{(\overline{\zeta}-z)^3}\right]\overline{p_0(\zeta)}d\zeta, & Imz < 0 \end{cases} \tag{12}$$

$$\Phi_2(z) = \begin{cases} \sum\limits_{k=1}^{N} A_1\{L_k\}g_k'(\tau_k) & 0 < Imz \le b \\ \sum\limits_{k=1}^{N} A_2\{L_k\}g_k'(\tau_k) & Imz < 0 \end{cases} \tag{13}$$

$$\Psi_2(z) = \begin{cases} \sum\limits_{k=1}^{N} B_1\{L_k\}g_k'(\tau_k) & 0 < Imz \le b \\ \sum\limits_{k=1}^{N} B_2\{L_k\}g_k'(\tau_k) & Imz < 0 \end{cases} \tag{14}$$

$$g_k'(\tau_k) = \frac{2\mu_2}{i(1+\kappa_2)}\frac{d}{d\tau_k}[u_k^+(\tau_k) - u_k^-(\tau_k) + i(u_k^+(\tau_k) - u_k^-(\tau_k))] \tag{15}$$

where bi-material constants A, A_0, the operators $A_1\{L_k\}, A_2\{L_k\}, B_1\{L_k\}, B_2\{L_k\}$ are given in Appendix.

The functions $\Phi_{1,2}(z)$ and $\Psi_{1,2}(z)$ satisfy the boundary conditions Eqs. (9) and (10) identically. Thus, $\Phi_0(z)$ and $\Psi_0(z)$ satisfy boundary conditions Eqs. (9) and (10) identically, also.

Singular integral equations for each case between interactions of the crack faces will be constructed.

3 Stick of the crack faces in contact

The boundary conditions on the crack faces in case of stick are as follows:

$$N_n^{\pm}(t_n) + iT_n^{\pm}(t_n) = 0, \quad t_n \in L_n^*, \quad n = 1, \cdots, N \tag{16}$$

$$T_n^{\pm}(t_n) = 0, \quad v_v^+(t_n) - v_v^-(t_n) = 0, \quad t_n \in L_n \backslash L_n^*, \quad n = 1, \cdots, N \tag{17}$$

$$N_n^+(t_n) - N_n^-(t_n) = 0, \quad t_n \in L_n \backslash L_n^*, \quad n = 1, \cdots, N \tag{18}$$

Here, N_n and T_n are the normal and tangential stresses on the nth crack faces on the O_nx_n- and O_ny_n-axes in the local coordinate system $x_nO_ny_n$, v_v the normal component of displacements of the crack faces. The superscripts (+) or (-) denote the boundary values of quantities when approaching the crack contours from left or from the right. Note that the positive direction of the tangential traction T_n coincides with that of tracing the region boundary when the region stays all the time on the left [Muskhelishvili (1975)].

In accordance with the boundary condition (17), the functions (13), (14) can be rewritten as

$$\Phi_2(z) = \begin{cases} \sum\limits_{k=1}^{N} A_1\{L_k^*\}g_k'(\tau_k) & 0 < Imz \le b \\ \sum\limits_{k=1}^{N} A_2\{L_k^*\}g_k'(\tau_k) & Imz < 0 \end{cases} \tag{19}$$

$$\Psi_2(z) = \begin{cases} \sum\limits_{k=1}^{N} B_1\{L_k^*\}g_k'(\tau_k) & 0 < Imz \le b \\ \sum\limits_{k=1}^{N} B_2\{L_k^*\}g_k'(\tau_k) & Imz < 0 \end{cases} \tag{20}$$

The stress distribution along the nth crack faces takes the form [Muskhelishvili (1975)]

$$2Re\Phi_0^{\pm}(T_n') + \frac{d\overline{T_n'}}{dT_n'}[T_n'\overline{\Phi_0^{'\pm}(T_n')} + \overline{\Psi_0^{\pm}(T_n')}] = N_n^{\pm}\left(T_n'\right) + iT_n^{\pm}\left(T_n'\right), \quad T_n' \in L_n \tag{21}$$

Define

$$N_n + iT_n = \frac{1}{2}\left[N_n^+ + N_n^- + i(T_n^+ + T_n^-)\right] \tag{22}$$

Substituting complex potentials (11), (12), (19) and (20) into Eq. (21), resulting

$$N_n(t_n) + iT_n(t_n) = \frac{1}{\pi}[\Omega_n(t_n) - P_n(t_n)], \quad t_n \in L_n \tag{23}$$

where

$$T_n' = t_n e^{i\alpha_n} + z_n^0 \tag{24}$$

$$\Omega_n(t_n) = \sum_{k=1}^{N} D_n\{L_k\} g_k'(\tau_k) \tag{25}$$

$$D_n\{L_k\}\psi_k(\tau_k) = \int_{L_k}[R_{nk}(\tau_k,t_n)\psi_k(\tau_k)d\tau_k + S_{nk}(\tau_k,t_n)\overline{\psi_k(\tau_k)}d\bar{\tau}_k] \tag{26}$$

$$P_n(t_n) = \int_{\gamma_0}(-K_{31}(\zeta,t_n) + K_{32}(\zeta,t_n))p(\zeta)d\zeta \tag{27}$$

where the operator $R_{nk}(\tau_k,t_n), S_{nk}(\tau_k,t_n)$, $K_{31}(\zeta,t_n)$ and $K_{32}(\zeta,t_n)$ are given in Appendix.

Considering boundary conditions (16) in view of Eq. (23), we obtain a system of N singular integral equations of the first kind

$$\Omega_n(t_n) - P_n(t_n) = 0, \quad t_n \in L_n^*, n = 1,\cdots,N \tag{28}$$

for the functions $g_n'(t_n)$ on the open parts $a_{nr}^*, b_{nr}^*(n = 1,\cdots N, r = 1,\cdots,R_n)$ of the crack and the pressure $p(t)$. Here,

$$\Omega_n(t_n) = \sum_{k=1}^{N} D_n\{L_k^*\} g_k'(\tau_k) \tag{29}$$

With respect to t, we write derivative of complex combination of displacements on the bonded plane material edge as [Muskhelishvili (1975)]

$$2\mu_1[\frac{du(t)}{dt} + i\frac{dv(t)}{dt}] = \kappa_1\Phi_0^-(t) - \overline{\Phi_0^-(t)} - [t\overline{\Phi_0^{'-}(t)} + \overline{\Psi_0^-(t)}] \tag{30}$$

Now, substituting the complex potentials (11), (12), (19), (20) into Eq. (30), we obtain the right hand part of Eq. (30)

$$2\mu_1[\frac{du(t)}{dt}+i\frac{dv(t)}{dt}]=\frac{k_1+1}{2}(1-A)p_0(t)+\frac{1}{2\pi}W_1(t)+\frac{i}{2\pi}W_2(t) \tag{31}$$

where

$$W_1(t)=\sum_{k=1}^{N}u\{L_k^*\}g_k'(\tau_k) \tag{32}$$

$W_2(t)$, $u\{L_k\}$ are given in Appendix.

Considering the first of the boundary conditions (6), we obtain a singular integral equation of the second kind

$$ImW_1(t)+ReW_2(t)-\pi\cdot k_1\cdot(1-A)\cdot\rho\cdot p(t)=4\pi\mu_1[f'(t)+\varepsilon],t\in r_0 \tag{33}$$

To complete the system of equations (28), (31) , it is necessary to add the displacement continuity conditions at the ends of the open parts

$$\int_{a_{nr}^*b_{nr}^*}g_n'(\tau_n)d\tau_n=0,\quad n=1,\cdots,N,r=1,\cdots,R_n \tag{34}$$

and the conditions of punch equilibrium conditions (1), (2).

The singular integral equations (28), (31) and conditions (1), (2), (34) allow to find the unknown functions $g_n'(\tau_n)$ $(n=1,\cdots,N)$, $p(t)$ and ε.

On the basis of the obtained solution considering Eqs. (17) and (23), we can obtain a formulae to determine contact stresses on the contact parts of the crack faces as

$$N_n(t_n)+iT_n(t_n)=\frac{1}{\pi}[\Omega_n(t_n)-P_n(t_n)],\quad t_n\in L_N/L_N^* \tag{35}$$

Using the solution of the singular integral equations constructed, we can calculate the stress intensity factors (SIF) at the crack tips and at the points where the crack faces begin to contact by

$$K_{In}(t_{nr})-iK_{IIn}(t_{nr})=\mp\lim_{t_n\to t_{nr}}\left[\sqrt{2\pi(t_n-t_{nr})}g_n'(\tau_n)\right]\quad t_{nr}=a_{nr}^*b_{nr}^*,$$
$$n=1,\cdots,N,\ r=1,\cdots,R_n \tag{36}$$

The upper sign (-) conforms to the points a_{nr}^*, while the lower sign (+) conforms to the points b_{nr}^*.

4 Smooth of the crack faces in contact

The boundary conditions on the crack faces in case of smooth are as follows:

$$N_n^{\pm}(t_n) + iT_n^{\pm}(t_n) = 0, \quad t_n \in L_n^*, \quad n = 1, \cdots, N \tag{37}$$

$$T_n^{\pm}(t_n) = 0, \quad v_\nu^+(t_n) - v_\nu^-(t_n) = 0, \quad t_n \in L_n \backslash L_n^*, \quad n = 1, \cdots, N \tag{38}$$

$$N_n^+(t_n) - N_n^-(t_n) = 0, \quad t_n \in L_n \backslash L_n^*, \quad n = 1, \cdots, N \tag{39}$$

In this case, the unknown functions $g_n'(t_n)$ $(n = 1, \cdots, N)$ in Eq. (15) can be represented as sums of two functions [Savruk (1981)]

$$g_n'(t_n) = g_{1n}'(t_n) + g_{2n}'(t_n) \tag{40}$$

These functions are expressed by discontinuity of the normal V_ν and tangential V_s components of displacement vector to the crack contour L_n by

$$g_{1n}(t_n) = \frac{2\mu_2}{1+\kappa_2}\left[v_\nu^+(t_n) - v_\nu^-(t_n)\right]\frac{dt_n}{ds_n} \tag{41}$$

$$g_{2n}(t_n) = -\frac{2i\mu_2}{1+\kappa_2}\left[v_s^+(t_n) - v_s^-(t_n)\right]\frac{dt_n}{ds_n} \tag{42}$$

where s_n is the arc abscissa of a point t_n on the contour L_n. Considering the second group of the boundary conditions (38), the complex potentials $\Phi_2(z)$, $\Psi_2(z)$ can be rewritten as

$$\Phi_2(z) = \begin{cases} \sum\limits_{k=1}^{N} \left[A_1\{L_k^*\} g_{1k}'(\tau_k) + A_1\{L_k\} g_{2k}'(\tau_k)\right] & 0 < Imz \le b \\ \sum\limits_{k=1}^{N} \left[A_2\{L_k^*\} g_{1k}'(\tau_k) + A_2\{L_k\} g_{2k}'(\tau_k)\right] & Imz < 0 \end{cases} \tag{43}$$

$$\Psi_2(z) = \begin{cases} \sum\limits_{k=1}^{N} \left[B_1\{L_k^*\} g_{1k}'(\tau_k) + B_1\{L_k\} g_{2k}'(\tau_k)\right] & 0 < Imz \le b \\ \sum\limits_{k=1}^{N} \left[B_2\{L_k^*\} g_{1k}'(\tau_k) + B_2\{L_k\} g_{2k}'(\tau_k)\right] & Imz < 0 \end{cases} \tag{44}$$

Substituting the complex potentials (11), (12), (43), (44) into the boundary conditions (37), (38) in view of Eq. (21), we obtain a system of $2N$ singular integral equations as follows

$$Re\left[\Omega_n(t_n) - P_n(t_n)\right] = 0, \quad t_n \in L_n^*,\ n = 1, \cdots, N \tag{45}$$

$$Im\left[\Omega_n(t_n) - P_n(t_n)\right] = 0, \quad t_n \in L_n,\ n = 1, \cdots, N \tag{46}$$

Here

$$\Omega_n(t_n) = \sum_{k=1}^{N} [D_n\{L_k^*\}g'_{1k}(\tau_k) + D_n\{L_k\}g'_{2k}(\tau_k)] \tag{47}$$

The operator $D_n\{L_k\}$ is determined by Eq. (26), and the quantity $P_n(t_n)$ is determined by Eq. (27).

To complete the system of equations (45), (46), it is necessary to add equations

$$Im[g_{1n}(t_n)\frac{d\overline{t_n}}{ds_n}] = 0, \quad t_n \in L_n^*,\ n = 1, \cdots, N \tag{48}$$

$$Re[g_{2n}(t_n)\frac{d\overline{t_n}}{ds_n}] = 0, \quad t_n \in L_n,\ n = 1, \cdots, N \tag{49}$$

and the equation (33) is also necessary, which can be simplified as

$$ImW_1(t) + ReW_2(t) = 4\pi\mu_1[f'(t) + \varepsilon], \quad t \in r_0 \tag{50}$$

where

$$W_1(t) = \sum_{k=1}^{N} [u\{L_k^*\}g'_{1k}(\tau_k) + u\{L_k\}g'_{2k}(\tau_k)] \tag{51}$$

The expression of the function $W_1(t)$ can be obtained by analogy with the previous case on the basis of the first of the boundary condition (6) using Eqs. (11), (12), (43) and (44).

The displacement continuity conditions at the ends of the open parts

$$\int_{a_{nr}^* b_{nr}^*} g'_{1n}(\tau_n)d\tau_n = 0, \quad n = 1, \cdots, N,\ r = 1, \cdots, R_n \tag{52}$$

$$\int_{L_n} g'_{2n}(\tau_n)d\tau_n = 0, \quad n = 1, \cdots, N \tag{53}$$

must be satisfied.

The singular integral equations (45), (46), (50) and conditions (1), (2), (48), (49), (52), (53) allow to find the functions $g'_{1n}(\tau_n)$ along the open parts $a_{nr}^* b_{nr}^*$ ($n = 1, \cdots, N, r = 1, \cdots, R_n$) of the cracks, the functions $g'_{2n}(\tau_n)$ along the contours L_n ($n = 1, \cdots, N$) and the pressure $p(t)$.

On the solution of the obtained equations in view of Eqs. (38) and (21), we take a relation

$$N(t_n) = \frac{1}{\pi} Re[\Omega_n(t_n) - P_n(t_n)], \quad t_n \in L_n \backslash L_n^* \tag{54}$$

to determine normal stress on the contact parts of the crack faces.

The SIF can be calculated by formulae Eq. (36).

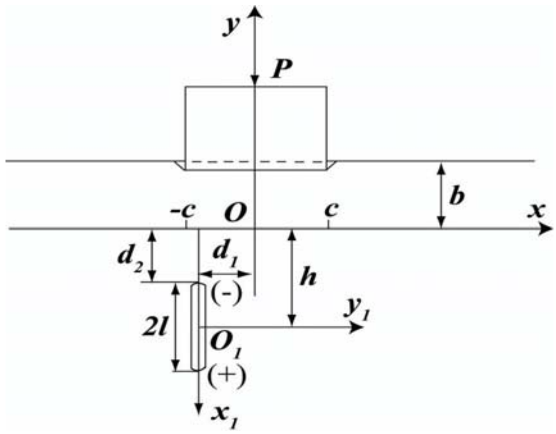

Figure 2: Bonded plane material with a vertical crack under punch action

5 Numerical results

5.1 Bonded plane material with a vertical crack under punch action

We consider a problem of indentation of a frictionless flat punch into an elastic strip bonded to a half-plane with a near-interface vertical crack of length $2l$. The bonded plane material and the crack are related to the system of coordinates xOy, $x_1O_1y_1$, respectively. A vertical force P acts on the punch, whose line coincides with the punch axis and Oy-axis. It is assumed that the punch does not rotate under the loading (i.e. $\varepsilon = 0$).

Employing the algorithm proposed by Panasyuk, Datsyshyn, and Marchenko (2000), it has been established the crack faces contact throughout its length for all the positions of the vertical crack near the punch foundation.

We investigate the case of smooth contact of the crack faces. Considering Eqs. (45)-(47), (50) and (51), we get following singular integral equations can be obtained

$$\int_{-l}^{l} Q(\tau_1,t_1)\varphi(\tau_1)d\tau_1 + \int_{-c}^{c} U(\zeta_1,t_1)p(\zeta_1)d\zeta_1 = 0, \quad |t_1| < l \tag{55}$$

$$\int_{-l}^{l} V(\tau_1,x)\varphi(\tau_1)d\tau_1 - \int_{-c}^{c} \left[\frac{k_1(1-A)}{\zeta_1 - x} - Re\frac{A_0}{\zeta_1 - x + 2bi}\right] p(\zeta_1)d\zeta_1 = 0, \quad |x| \le c \tag{56}$$

where the kernels $Q(\tau_1,t_1)$, $U(\zeta_1,t_1)$, $V(\tau_1,x)$ are given in Appendix. The unknown functions are

$$\varphi(t_1) = -ig_2(t_1) = -\frac{2\mu_2}{1+\kappa_2}\left[u^+(t_1) - u^-(t_1)\right], \quad |t_1| < l \tag{57}$$

and $p(t)(t = x + i \cdot b, |x| \leq c)$.

The additional conditions are needed to complete the system equations (55), (56)

$$\int_{-l}^{l} \varphi(\tau_1)d\tau_1 = 0 \tag{58}$$

$$\int_{-c}^{c} p(\zeta_1)d\zeta_1 = P \tag{59}$$

$$\int_{-c}^{c} \zeta_1 p(\zeta_1)d\zeta_1 = -M \tag{60}$$

The moment M which restrains the punch from rotation can be determined by the condition (60).

The singular integral equations (55), (56) can be solved by the methods given in Savruk (1981) and Panasyuk, Datsyshyn, and Marchenko (2000). To reduce Eqs. (55) and (56) to a standard form, the intervals are normalized by

$$\begin{aligned} &\tau_1 = l\xi, \quad \zeta_1 = c\xi, \quad |\xi| \leq 1, \\ &t_1 = l\eta, \quad x = c\eta, \quad |\eta| < 1 \end{aligned} \tag{61}$$

The solution of equations (55), (56) can be written as

$$\varphi(\eta) = \frac{u(\eta)}{\sqrt{1-\eta^2}}, p(\eta) = \frac{q(\eta)}{\sqrt{1-\eta^2}} \tag{62}$$

The stress intensity factors at the crack tips can be defined as

$$K_{II}^{\pm} = \pm\sqrt{\pi/l}u(\pm 1) \tag{63}$$

The pressure in the contact area can be defined as

$$p(\eta) = -\frac{T_N(\eta)}{N\sqrt{1-\eta^2}} \sum_{k=1}^{N} (-1)^k \frac{\sqrt{1-\xi_k^2}}{\eta - \xi_k} q(\xi_k), \quad |\eta| < 1 \tag{64}$$

In Eq. (63), the upper sign (+) concerns the right crack tip, while the lower sign (-) concerns the left crack tip, and the function $u(\xi)$ at the points $\xi = +1, -1$ takes the values

$$u(1) = -\frac{1}{N} \sum_{k=1}^{N} (-1)^k \cot\left(\frac{2k-1}{4N}\pi\right) u(\xi_k) \tag{65}$$

$$u(-1) = \frac{1}{N}\sum_{k=1}^{N}(-1)^{k+N}\tan(\frac{2k-1}{4N}\pi)u(\xi_k) \tag{66}$$

Here, ξ_k are the zeros of the Chebyshev polynomials of the first kind, $T_N(\xi) = \cos(N\cos^{-1}\xi)$.

The numerical calculations of the normalized Mode II *SIF*, $F_{II}^{\pm} = K_{II}^{\pm}\sqrt{c}/(P\sqrt{\pi})$ are conducted. The normalized Mode II SIF depends on relative distance d_1/c of the crack from the punch axis for different crack lengths and distances from the $x-$axes. Plane stress state is considered here. Young's moduleses of the strip and half-plane are taken as $E_1 = 140$Mpa, $E_2 = 280$Mpa respectively. Poisson's ratios are taken as $\mu_1 = 0.2$, $\mu_2 = 0.4$, respectively.

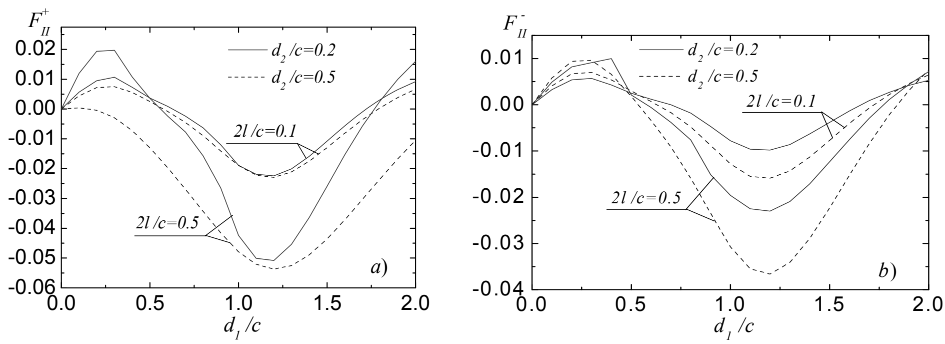

Figure 3: Normalized Model II stress intensity factors at the tips of vertical crack, $F_{II}^{\pm} = K_{II}^{\pm}\sqrt{c}/(P\sqrt{\pi})$, $b/c = 0.2$, solid lines are for$d_2/c = 0.2$, while dashed lines are for $d_2/c = 0.5$. (a) F_{II}^{+} for the right tip, (b) F_{II}^{-} for the left tip

Figure 3(a) and 3(b) indicate that the absolute value of the Mode II stress intensity reach maximum when the crack approaches the punch edge $d_1/c \approx 1.1$. It is shown the factors K_{II} equal to zero at both of the crack tips when $d_1/c = 0$ for the reason of problem symmetry. The crack is no longer a stress concentrator and has no effects on the pressure distribution under the punch when which is placed along the punch axis.

Figure 4(c) and 4(d) illustrate the variation of the normalized Mode II SIF with relative length$2l/c$ for different relative strip thickness b/c at the most dangerous crack position ($d_1/c = 1.1$). It is assumed $d_1/c = 0.2$. When increasing the relative thickness b/c of the strip, the normalized Mode II SIF increases. Both F_{II}^{+} and F_{II}^{-} in magnitude increase with the increasing of the value of $2l/c$.

To verify the validity of the procedure, we compare the numerical values of the pressure $p(t)$ under the punch of the uncracked bonded material plane with close

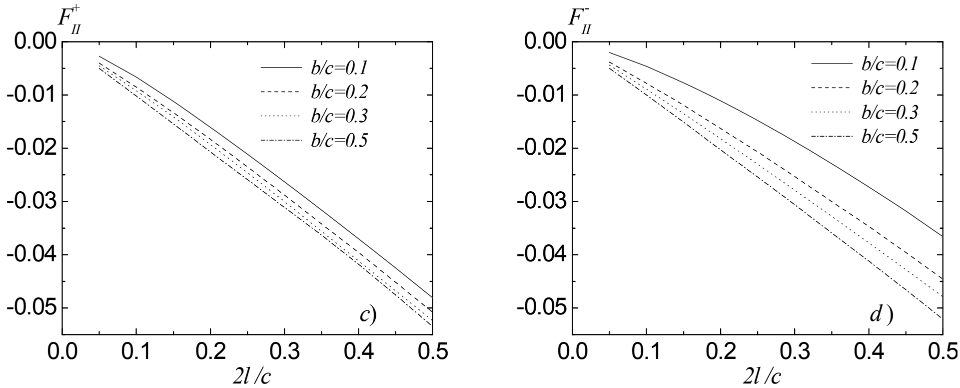

Figure 4: Influence of crack length on the normalized Mode II SIF $F^{\pm}_{II} = K^{\pm}_{II}\sqrt{c}/(P\sqrt{\pi})$ for different strip thickness b/c. (c) F^{+}_{II} for the right tip, (d) F^{-}_{II} for the left tip

solution provided by Liu (1992). From table 1, it's observed that the present results coincide with those of Liu (1992) well. In Table 1, the numerical values of the pressure $p(t)$ under the punch for the most dangerous crack positions are also given when the SIF in magnitude are maximum. It is seen that in comparison with the uncracked bonded material plane, the crack presence has an insignificant effect on the value.

5.2 Bonded plane material with a horizontal crack under punch action

We consider another case of indentation of a frictionless flat punch into the bonded plane material. There is a horizontal crack of length $2l$ in the half-plane, which is placed symmetrically about the punch axis. Employing the algorithm proposed by Panasyuk, Datsyshyn, and Marchenko (2000), it has been established that the crack faces contact throughout its length, also.

For the case of smooth contact of the crack faces, the considering problem is finally reduced to solving the singular integral equations (55), (56) with the kernels

$$Q(\tau_1,t_1) = \frac{1}{\tau_1 - t_1} - \frac{A-B}{2} Re\,\omega^{-1}(\tau_1,t_1) + A\cdot Re\left[\omega^{-1}(\tau_1,t_1) - \frac{4h^2}{\omega^3(\tau_1,t_1)}\right] + 2A\cdot h\cdot Im\frac{1}{\omega^2(\tau_1,t_1)}, \quad \omega(\tau_1,t_1) = \tau_1 - t_1 - 2\cdot i\cdot h \quad (67)$$

Table 1: Pressure $cp(x)/P$ under a frictionless flat punch which is indented to bonded plane materials with a vertical crack

	$2l/c$	d_1/c	d_2/c	b/c	$x/c=-0.9$	$x/c=-0.5$	$x/c=0$	$x/c=0.5$	$x/c=0.9$
Liu's paper	0	-	-	-	0.7303	0.3676	0.3183	0.3676	0.7303
Present paper	0	-	-	-	0.7303	0.3676	0.3183	0.3676	0.7303
	0.1	1.2	0.2	0.2	0.7303	0.3676	0.3183	0.3675	0.7302
	0.1	1.2	0.5	0.2	0.7303	0.3676	0.3183	0.3675	0.7303
	0.5	1.2	0.2	0.2	0.7308	0.3677	0.3182	0.3668	0.7302
	0.5	1.2	0.5	0.2	0.7305	0.3675	0.3180	0.3670	0.7310
	0.5	1.2	0.2	0.1	0.7309	0.3678	0.3183	0.3669	0.7297
	0.5	1.2	0.2	0.3	0.7307	0.3677	0.3181	0.3669	0.7306
	0.5	1.2	0.2	0.5	0.7305	0.3675	0.3180	0.3671	0.7310

$$U(\zeta_1,t_1) = -Re\frac{A_0 - A}{2(\zeta - \overline{T'})} - Re\frac{\zeta - T_1'}{2\left(\overline{\zeta} - \overline{T'}\right)^2} + A\cdot b\cdot Im\frac{\zeta + \overline{T_1'} - 2T_1'}{(\zeta - \overline{T'})^3}$$
$$\zeta = \zeta_1 + i\cdot b, T' = t_1 - i\cdot h \quad (68)$$

$$V(\tau_1,x) = Re\left[\frac{k_1(1-B)}{T-t} - \frac{1-A}{\overline{T}-\overline{t}}\right] + 2Im\frac{(1-A)h + b(1-B)}{\left(\overline{T}-\overline{t}\right)^2}$$
$$t = x + i\cdot b, T = \tau_1 - i\cdot h \quad (69)$$

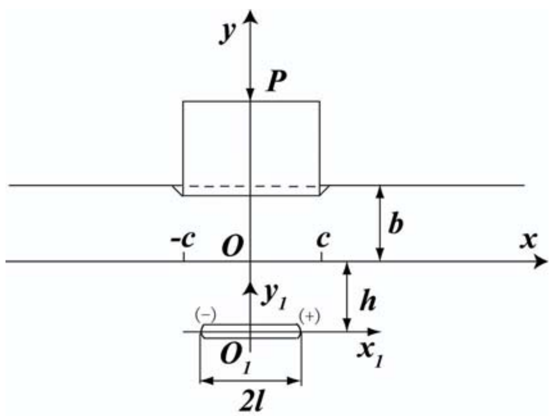

Figure 5: Bonded plane material with a horizontal crack under punch action

The influences of the value of relative crack length $2l/c$ and relative distance h/c of the crack from the x-axis on the normalized Mode II SIF, $F_{II}^{\pm} = K_{II}^{\pm}\sqrt{c}/(P\sqrt{\pi})(F_{II}^{-} = -F_{II}^{+})$ are shown in Figure 6. It can be found the values of F_{II}^{+} in magnitude decrease with the increasing of the value of h/c. The values of F_{II}^{+} in magnitude increase with the increasing of crack length.

A It's shown in Table 2, the existence of the crack has no effects on the pressure distribution under the punch.

6 Conclusions

In this paper, the singular integral equation method has been applied to study the contact problem of the bonded plane material with curvilinear cracks, which is

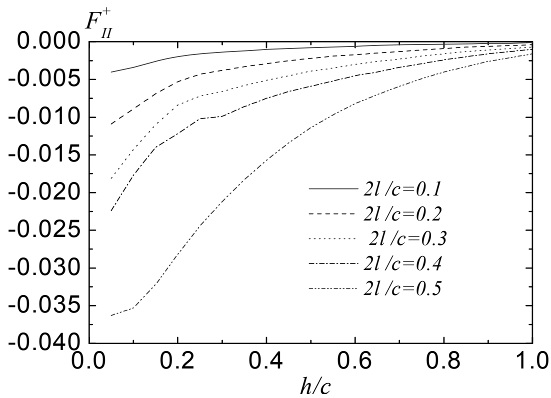

Figure 6: Influence of strip height h/c on the normalized Mode II SIF, $F_{II}^{\pm} = K_{II}^{\pm}\sqrt{c}/(P\sqrt{\pi})$ for $b/c = 0.2$ and different crack length $2l/c$

Table 2: Pressure $cp(x)/P$ under a frictionless flat punch, which is indented to bonded plane materials with a horizontal crack

$2l/c$	h/c	b/c	$x/c = -0.9$	$x/c = -0.5$	$x/c = 0$	$x/c = 0.5$	$x/c = 0.9$
0	-	-	0.7303	0.3676	0.3183	0.3676	0.7303
0.1	0.05	0.2	0.7303	0 .3676	0.3183	0.3676	0.7302
0.2	0.05	0.2	0.7303	0 .3676	0.3183	0.3676	0.7302
0.3	0.05	0.2	0.7303	0 .3676	0.3183	0.3676	0.7302
0.4	0.05	0.2	0.7303	0 .3676	0.3183	0.3676	0.7302
0.5	0.05	0.2	0.7303	0 .3676	0.3183	0.3676	0.7302

under the action of a rigid punch with the foundation of convex shape. Kolosov-Muskhelishvili complex potentials are constructed to satisfy the continuity of stresses and displacements along the interface automatically. The crack faces may contact, two limiting cases either stick or smooth contact on the contact parts is considered. The considered problem is finally reduced to solving a system of complex Cauchy type singular integral equations of the first and second kind. Numerical analysis has been made for the particular cases of indentation of a frictionless punch with flat foundation into the bonded plane material weakened by a vertical crack or a horizontal crack. It has been established that for various positions of the considered cracks, they are fully closed.

For the case of smooth contact of the crack faces, numerical results show: (i) The thickness of the strip influences the Mode II stress intensity at the tips of the vertical crack; (ii) The crack configuration has an sinificant effect on the Mode II stress intensity; (iii) The existence of the vertical crack or horizontal crack near the punch foundation has an insignificant effect on the pressure distribution under the punch.

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Appendix

Expression of the bi-material constants A, A_0 in Eq. (11) and Eq. (12)

$$A = 1 - \frac{\mu_1(\kappa_2+1)}{\mu_2+\kappa_2\mu_1} = \frac{\mu_2-\mu_1}{\mu_2+\kappa_2\mu_1}, \quad A_0 = \frac{\kappa_1\mu_2(1-A)-\mu_1(A+\kappa_2)}{\mu_2-\mu_1} \tag{70}$$

Expressions of operators $A_1\{L_k\}, A_2\{L_k\}, B_1\{L_k\}, B_2\{L_k\}$ in Eq. (13) and Eq. (14)

$$A_1\{L_k\}\psi_k(\tau_k) = \frac{1-B}{2\pi}\int_{L_k}\frac{1}{T_k-z}e^{ia_k}\psi_k(\tau_k)d\tau_k \tag{71}$$

$$A_2\{L_k\}\psi_k(\tau_k) = \frac{1}{2\pi}\int_{L_k}\left\{\left[\frac{1}{T_k-z}-\frac{A}{\overline{T}_k-z}\right]e^{i\alpha_k}\psi_k(\tau_k)d\tau_k + \frac{A\cdot 2i\cdot ImT_k}{(\overline{T}_k-z)^2}e^{-ia_k}\overline{\psi_k(\tau_k)}d\bar{\tau}_k\right\} \tag{72}$$

$$B_1\{L_k\}\psi_k(\tau_k) = \frac{1}{2\pi}\int_{L_k}\left\{\frac{-(1-A)\overline{T}_k-(A-B)T_k}{(T_k-z)^2}e^{i\alpha_k}\psi_k(\tau_k)d\tau_k + \frac{1-A}{T_k-z}e^{-i\alpha_k}\overline{\psi_k(\tau_k)}d\bar{\tau}_k\right\} \tag{73}$$

$$B_2\{L_k\}\psi_k(\tau_k) = \frac{1}{2\pi}\int_{L_k}\left\{\left[\frac{A}{(\overline{T}_k-z)^2}-\frac{1}{(T_k-z)^2}\right]\overline{T_k}e^{ia_k}\psi_k(\tau_k)d\tau_k + \left[\frac{1}{T_k-z}-\frac{B}{\overline{T_k}-z}-2AiImT_k\frac{\overline{T_k}+z}{(\overline{T_k}-z)^3}\right]e^{-i\alpha_k}\overline{\psi_k(\tau_k)}d\bar{\tau}_k\right\} \tag{74}$$

where

$$T_k = \tau_k e^{i\alpha_k} + z_k^0, \quad B = 1 - \frac{\mu_1(\kappa_2+1)}{\mu_1+\kappa_1\mu_2} = \frac{\kappa_1\mu_2-\kappa_2\mu_1}{\mu_1+\kappa_1\mu_2} \tag{75}$$

Expressions of the $R_{nk}(\tau_k,t_n), S_{nk}(\tau_k,t_n)$ in Eq. (26)

$$\begin{cases} R_{nk} = K_{11}(\tau_k,t_n) - K_{12}(\tau_k,t_n) \\ S_{nk} = K_{21}(\tau_k,t_n) - K_{22}(\tau_k,t_n) \end{cases} \tag{76}$$

where

$$K_{11}(\tau_k,t_n) = \frac{e^{i\alpha_k}}{2}\left[\frac{1}{Tk-T_n'}-\frac{A}{\overline{\overline{Tk}}-T_n'}-\frac{2A\cdot i\cdot ImTk}{\left(Tk-\overline{T_n'}\right)^2}\right] \tag{77}$$

$$K_{12}(\tau_k,t_n)=\frac{e^{-i\alpha_k}}{2}\left[\frac{B}{Tk-\overline{T_n'}}-\frac{1}{\overline{Tk}-\overline{T_n'}}-2A\cdot i\cdot ImTk\frac{Tk+\overline{T_n'}-2T_n'}{\left(Tk-\overline{T_n'}\right)^3}\right]\frac{d\overline{tn}}{dtn}e^{-2i\alpha_n} \tag{78}$$

$$K_{21}(\tau_k,t_n)=\frac{e^{-i\alpha_k}}{2}\left[\frac{1}{\overline{Tk}-\overline{T_n'}}-\frac{A}{Tk-\overline{T_n'}}+2A\cdot i\cdot\frac{ImTk}{\left(\overline{Tk}-T_n'\right)^2}\right] \tag{79}$$

$$K_{22}(\tau_k,t_n)=\frac{e^{-i\alpha_k}}{2}\left[-\frac{A\left(Tk-T_n'\right)}{\left(Tk-\overline{T_n'}\right)}+\frac{Tk-T_n'}{\left(\overline{Tk}-\overline{T_n'}\right)^2}\right]\frac{d\overline{tn}}{dtn}e^{-2i\alpha_n} \tag{80}$$

Expressions of $K_{31}(\zeta,t_n)$ and $K_{32}(\zeta,t_n)$ in Eq. (27)

$$K_{31}(\zeta,t_n)=Im\frac{1+i\cdot\rho}{\zeta-T_n'}-2\cdot A\cdot b\cdot Re\frac{1+i\cdot\rho}{(\zeta-\overline{T_n'})^2} \tag{81}$$

$$K_{32}(\zeta,t_n)=$$
$$i\left[\frac{(A_0-A)\cdot(1+i\cdot\rho)}{\zeta-\overline{T_n'}}+\frac{(\zeta-T_n')(1-i\cdot\rho)}{2\left(\overline{\zeta}-\overline{T_n'}\right)^2}+\frac{A\cdot b\cdot i\cdot(1+i\cdot\rho)(\zeta+\overline{T_n'}-2T_n')}{(\zeta-\overline{T_n'})^3}\right]$$
$$\frac{d\overline{tn}}{dtn}e^{-2i\alpha_n} \tag{82}$$

Expressions of $W_2(t)$ in Eq. (31)

$$W_2(t)=\int_{\gamma_0}\left[\frac{\kappa_1(1-A)}{\zeta-t}-\frac{A_0}{\zeta-\bar{t}}\right]p_0(\zeta)\,d\zeta \tag{83}$$

Expressions of $u\{L_k\}$ in Eq. (32)

$$u\{L_k\}\psi_k(\tau_k)=\int_{L_k}[K_{41}(\tau_k,t)\psi_k(\tau_k)d\tau_k+K_{42}(\tau_k,t)\overline{\psi_k(\tau_k)}d\bar{\tau}_k] \tag{84}$$

where

$$K_{41}(\tau_k,t)=e^{i\alpha_k}\left[\frac{\kappa_1(1-B)}{Tk-t}-\frac{1-A}{\overline{Tk}-\bar{t}}\right] \tag{85}$$

$$K_{42}(\tau_k,t)=e^{-i\alpha_k}2\cdot i\cdot\frac{(1-A)ImTk-b(1-B)}{\left(\overline{Tk}-\bar{t}\right)^2} \tag{86}$$

Expressions of $Q(\tau_1,t_1)$, $U(\zeta_1,t_1)$, $V(\tau_1,x)$ in Eq. (55) and Eq. (56)

$$Q(\tau_1,t_1)=\frac{1}{\tau_1-t_1}+\frac{A+B}{2}\frac{1}{\tau_1+t_1+2h}-\frac{2A(\tau_1+h)}{(\tau_1+t_1+2h)^2}+\frac{4A(\tau_1+h)(t_1+h)}{(\tau_1+t_1+2h)^3} \tag{87}$$

$$U(\zeta_1,t_1)=Re\frac{A_0-A}{2(\zeta-\overline{T_1'})}+Re\frac{\zeta-T_1'}{2\left(\overline{\zeta}-\overline{T_1'}\right)^2}-A\cdot b\cdot Im\frac{\zeta+\overline{T_1'}-2T_1'}{(\zeta-\overline{T_1'})^3}$$

$$T_1'=d_1+i\cdot(t_1+h)\quad \zeta=\zeta_1+i\cdot b \tag{88}$$

$$V(\tau_1,x)=Im\left[\frac{k_1(1-B)}{d_1-x-i(\tau_1+h+b)}-\frac{1-A}{d_1-x+i(\tau_1+h+b)}\right]+2Re\frac{(1-A)(\tau_1+h)+b(1-B)}{[d_1-x+i(\tau_1+h+b)]^2} \tag{89}$$