A Lie-Group Shooting Method Estimating Nonlinear Restoring Forces in Mechanical Systems

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Abstract: For an inverse vibration problem of nonlinear mechanical system to estimate displacement- and velocity-dependent restoring force, we transform the equation of motion into a parabolic type partial differential equation (PDE). Then by a semi-discretization of the PDE, the inverse vibration problem is formulated as a multi-dimensional two-point boundary value problem with unknown sources, allowing a closed-form estimation through a Lie-group shooting method to construct the restoring force surface over phase plane. Only one set of displacements measured at sampling time points is used in the estimation. The new method does not require to assume a priori the functional form of unknown restoring force, and more importantly, it is free of iteration. The estimated results are very accurate for stably identifying the restoring force under noise, which can be well used in the engineering of vibrational mechanics.

Keyword: Inverse vibration problem, Nonlinear mechanical system, Restoring force, Lie-group shooting method

1 Introduction

Structural dynamics is to analyze and determine the responses of a given structure subject to various external loading conditions. Based on the results analyzed, structural engineers are able to check whether a proposed structural design meets the requirements of adequate resistance to a combination of loading conditions and, if necessary, to revise a proposed design until all such requirements are satisfied. Experimental testing and system identification play a key role in structural dynamics, because they help us to reconcile numerical predictions with experimental investigations. System identification is referred to as a mathematical procedure for a direct extraction of information about structures from experimental data; see, e.g., Chao, Chen and Lin (2001), and Hunag and Shih (2007).

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The dissipation of energy in a mechanical structure is often described by a viscous damping term, while the conservative part is described by a nonlinear spring element. The resulting equation of vibration is attractive because it can be mathematically treated. However, sometimes we may encounter the problem that the viscoelastic properties of structure or the external force are not yet known, and then the resulting problem is an inverse vibration problem. It is concerned with the estimations of those properties such as damping coefficient [Adhikari and Woodhouse (2001a); Adhikari and Woodhouse (2001b); Ingman and Suzdalnitsky (2001); Liang and Feeny (2006)], stiffness [Huang (2001); Shiguemori, Chiwiacowsky and de Campos Velho (2005)], as well as external force [Huang (2005); Feldman (2007)]. With the aid of measurable vibration data, such as frequency, mode shape, displacement or velocity at different time, the researchers are interesting to estimate those properties [Kerschen, Worden, Vakakis and Golinval (2006)].

For the inverse vibration problems of linear structures by estimating constant damping or stiffness coefficients there were many papers, for example, Gladwell (1986), Gladwell and Movahhedy (1995), Lancaster and Maroulas (1987), Starek and Inman (1991, 1995, 1997), and Starek, Inman and Kress (1992). However, when the coefficients are time-dpendent the inverse vibration problems are nonlinear and they are more difficult to solve. Huang (2001) has employed a conjugate gradient method to solve the nonlinear inverse vibration problem for the estimation of time-dependent stiffness coefficient. Recently, Liu (2008a, 2008b) has developed a Lie-group method to study the inverse vibration problem for estimating both the time-dependent damping and stiffness coefficients.

In the realm of nonlinear system identification of structural dynamics, Kerschen, Worden, Vakakis and Golinval (2006) have given a very comprehensive review of the developments of some useful methods. Parameter identification is a major step towards the establishment of a structural model with good predictive accuracy. Among many existent methods, the restoring force surface method (or force state mapping method) is a simple procedure allowing a direct identification of restoring force for nonlinear mechanical systems. The basic procedures were introduced by Masri and Caughey (1979), and then extended by Crawley and Aubert (1986), Crawley and O'Donnel (1986), and Duym, Schoukens and Guillaume (1996).

Recently, Namdeo and Manohar (2008) have developed new methods of identification of nonlinear system parameters from measured time histories of response under known excitations. Solutions are obtained by using the force state mapping technique with two alternative functional representation schemes: reproducing kernel particle method and kriging technique. They showed that their method has the capability to reproduce exactly polynomials of specified order at any point in a given domain.

The purpose of this paper is to identify the restoring force H in the following equation of motion as governed by the Newton's second law:

$$\ddot{\phi} + H(\phi, \dot{\phi}) = F(t). \tag{1}$$

Here, H can be a quite general function of displacement ϕ and velocity $\dot{\phi}$. Because H is assumed to be dependent only on ϕ and $\dot{\phi}$ it can be represented by a surface over the phase plane of $(\phi, \dot{\phi})$. A trivial rearrangement of Eq. (1) gives

$$H(\phi, \dot{\phi}) = F(t) - \ddot{\phi}. \tag{2}$$

If the time-varying excitation $F(t)$ and acceleration $\ddot{\phi}(t)$ are measurable, all the quantities on the right-hand side are known, and so is H. Usually the acceleration signal is rather irregular in time contaminated by noise, and it is a big challenge to use the above equation to reconstruct H as a function of the measured displacement and velocity. Indeed, there are a couple of issues of signal processing to treat the above problem [Worden (1990a, 1990b)]. In general, the measurement of displacements at some discretized sampling times is more easy than that to directly measure velocities and accelerations.

Denoting the measured displacement by $x_1 = g(t)$, in order to get velocity and acceleration we may face an index-three differential algebraic equations (DAEs):

$$x_1(t) = g(t),$$
$$\dot{x}_1(t) = x_2(t),$$
$$\dot{x}_2(t) = x_3(t). \tag{3}$$

How to give an effective numerical method to solve the DAEs is still an important issue requiring more study [Brenan, Campbell and Petzold (1996)].

With the above situation in mind, we will develop a new Lie-group shooting method by using only the displacement data to estimate the restoring force H, and delegate other numerical computation of Eq. (3) by a new fictitious time integration method [Liu and Atluri (2008)] into other place.

Recently, Liu (2006a, 2006b, 2006c) has made a breakthrough to extend the group preserving scheme (GPS) developed by Liu (2001) for initial value problems of ODEs to solve the boundary value problems (BVPs), namely the Lie-group shooting method (LGSM), and the numerical results revealed that the LGSM is a rather promising method to effectively solve the two-point BVPs.

In the construction of Lie-group method for the calculations of BVPs, Liu (2006a) has introduced the idea of one-step GPS by utilizing the closure property of Lie group. It needs to stress that this one-step Lie-group property cannot be shared by

other numerical methods, because those methods do not belong to the Lie-group types. This important property as first pointed out by Liu (2006d) was employed to solve the backward in time Burgers equation. After that, Liu (2006e) has used this concept to establish a one-step estimation method to estimate the temperature-dependent heat conductivity, and then extended to estimate the thermophysical properties of heat conductivity and heat capacity by Liu (2006f, 2007), and Liu, Liu and Hong (2007). Moreover, Liu (2008c) used the LGSM to estimate unknown boundary condition for thermal stress problem through an aid of an internal temperature measurement; and Liu (2008d) used the LGSM to identify time-dependent heat conductivity function by an extra measurement of temperature gradient. In addition to the above inverse problems, Chang, Chang and Liu (2008), and Liu, Chang and Chang (2008) also used the LGSM to solve the boundary layer equations in fluid mechanics. Liu (2008e) used the technique of LGSM to compute eigenvalues and eigenfunctions of Sturm-Liouville problems, and Liu (2008f) proposed an LGSM for post buckling calculations of elastica.

The Lie-group method possesses a great advantage than other numerical methods due to its group structure, and it is a powerful technique to solve direct problems and also the inverse problems of parameters identification.

This paper is arranged as follows. We introduce a novel approach of inverse vibration problem in Section 2 by transforming it into an identification problem of parabolic type PDE, and then by discretizing the PDE into a system of ODEs at the discretized times. Here we explain why a multi-dimensional two-point BVP appears naturally. In Section 3 we give a brief sketch of the GPS for ODEs for a self-content reason. Due to a good property of Lie-group, we will propose an integration technique, such that the one-step GPS can be used to identify the parameters appeared in the resulted PDE. The algebraic equations are derived in Section 4 when we apply the one-step GPS to identify displacement-dependent restoring force and velocity-dependent damping force. In Section 5 numerical examples are examined to test the Lie-group shooting method (LGSM). In Section 6 we extend the LGSM to identify nonlinear restoring force, which is displacement-velocity-dependent, and numerical examples are also given. Finally, we draw the conclusions in Section 7.

2   Two transformations

From this section we start to develop a new method to identify the nonlinear restoring force. However, we first deal with two special cases of Eq. (1) for an easier explanation of our approach. Consider a second-order ordinary differential equa-

tion (ODE) describing the forced vibration of a nonlinear structure with

$$\ddot{\phi} + \gamma\dot{\phi} + H(\phi) = F(t), \ \ 0 \le t \le t_f, \tag{4}$$

$$\phi(0) = A_0, \tag{5}$$

$$\dot{\phi}(0) = B_0. \tag{6}$$

The direct problem is for the given initial conditions in Eqs. (5) and (6) and the given constant γ and functions $H(\phi)$ and $F(t)$ in Eq. (4) to find the displacement $\phi(t)$ in a time interval of $t \in [0, t_f]$; conversely, our present inverse vibration problem is to estimate $H(\phi)$ by using some measured data of $\phi(t)$ in a time interval of $t \in [0, t_f]$.

On the other hand, we also consider

$$\ddot{\psi} + Q(\dot{\psi}) + \alpha\psi = F(t), \ \ 0 \le t \le t_f, \tag{7}$$

$$\psi(0) = C_0, \tag{8}$$

$$\dot{\psi}(0) = D_0. \tag{9}$$

Here the inverse problem is to estimate $Q(\dot{\psi})$.

Basically these two sets of Eqs. (4)-(6) and Eqs. (7)-(9) have the similar form. So we only consider the mathematical derivations for the set of Eqs. (4)-(6), and after deriving the required equations, we can similarly apply them to Eqs. (7)-(9).

2.1 Transformation into a PDE

In the solutions of linear PDEs, a popular technique is the seperation of variables, from which the PDEs are transformed into some ODEs. In this study we reverse this process by considering

$$u(x,t) = (1+x)\phi(t), \tag{10}$$

such that Eqs. (4)-(6) can be transformed into a parabolic type PDE:

$$\frac{\partial u(x,t)}{\partial x} = \frac{\partial^2 u(x,t)}{\partial t^2} + \gamma\frac{\partial u(x,t)}{\partial t} + h(x,t) + \phi(t) - (1+x)F(t), \tag{11}$$

$$u(0,t) = \phi(t), \tag{12}$$

$$u(x,0) = A_0(1+x), \tag{13}$$

$$u(x,t_f) = \phi(t_f)(1+x), \tag{14}$$

where $\phi(t_f)$ is a measured displacement at a final time t_f, and more precisely $h(x,t) = (1+x)H(\phi(t))$. In order to identify H, we suppose that the data of $\phi(t)$ are

provided in a time interval $t \in [0, t_f]$ through measurements by a sensor mounted on the structure.

The above transformation technique was first proposed by Liu (2008g) to treat an inverse Sturm-Liouville problem by transforming an ODE into a PDE. Then, Liu (2008a, 2008b) and Liu, Chang, Chang and Chen (2008) extended this idea to develop new methods for estimating parameters in the inverse vibration problems. Liu (2008h) also employed a similar technique by transforming the obstacle problem of elliptic type into a dynamical system and then a time-marching algorithm was used to find solution.

2.2 *Transformation into a set of ODEs*

Applying a semi-discrete procedure on the PDE in Eq. (11) yields a coupled system of ODEs, where we adopt

$$\left. \frac{\partial u(x,t)}{\partial t} \right|_{t=i\Delta t} = \frac{u_{i+1}(x) - u_{i-1}(x)}{2\Delta t}, \tag{15}$$

$$\left. \frac{\partial^2 u(x,t)}{\partial t^2} \right|_{t=i\Delta t} = \frac{u_{i+1}(x) - 2u_i(x) + u_{i-1}(x)}{(\Delta t)^2}, \tag{16}$$

and $\Delta t = t_f / (n+1)$ is a uniform time increment with $u_i(x) = u(x, i\Delta t)$ for a simple notation. So, Eq. (11) can be approximated by

$$u_i'(x) = \frac{1}{(\Delta t)^2} [u_{i+1}(x) - 2u_i(x) + u_{i-1}(x)] + \frac{\gamma}{2\Delta t} [u_{i+1}(x) - u_{i-1}(x)] + h_i(x)$$
$$+ \phi_i - (1+x)F_i, \quad (17)$$

where $h_i(x) = (1+x)H(\phi_i)$ with $\phi_i = \phi(t_i)$ and $F_i = F(t_i)$, $i = 1, \ldots, n$.

When $i = 1$ the term $u_0(x)$ is replaced by the boundary condition (13) with $u_0(x) = A_0(1+x)$. Similarly, when $i = n$ the term $u_{n+1}(x)$ is replaced by the boundary condition (14) with $u_{n+1}(x) = \phi_{n+1}(1+x) = \phi(t_f)(1+x)$. Eq. (17) has totally n coupled ODEs for the n variables $u_i(x), i = 1, \ldots, n$.

Now the problem becomes a two-point BVP with Eq. (17) not only subject to an initial condition $u_i(0) = \phi_i$ and also subject to a final condition $u_i(x_f) = (1+x_f)\phi_i$ obtained from Eq. (10) by inserting $x = x_f$, where x_f is a new constant chosen by the user. Therefore, we have overspecified conditions for the n-dimensional ODEs system (17); because the source terms h_i are unknown, we attempt to use this two-point BVP formulation to find h_i. Below, we will develop a Lie-group shooting method to solve this problem.

3 GPS for differential equations system

3.1 Group-preserving scheme

Upon letting $\mathbf{u} = (u_1, \ldots, u_n)^T$ and denoting by $\mathbf{f}$ the right-hand side of Eq. (17) we can write it as a vector form:

$$\mathbf{u}' = \mathbf{f}(\mathbf{u}, x), \quad \mathbf{u} \in \mathbb{R}^n, \quad x \in \mathbb{R}. \tag{18}$$

Liu (2001) has embedded Eq. (18) into an augmented differential equations system:

$$\frac{d}{dx} \begin{bmatrix} \mathbf{u} \\ \|\mathbf{u}\| \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{n \times n} & \frac{\mathbf{f}(\mathbf{u}, x)}{\|\mathbf{u}\|} \\ \frac{\mathbf{f}^T(\mathbf{u}, x)}{\|\mathbf{u}\|} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \|\mathbf{u}\| \end{bmatrix}. \tag{19}$$

It is obvious that the first row in Eq. (19) is the same as the original equation (18), but the inclusion of the second row in Eq. (19) gives us a Minkowskian structure of the augmented state variables of $\mathbf{X} := (\mathbf{u}^T, \|\mathbf{u}\|)^T$, which satisfies the cone condition:

$$\mathbf{X}^T \mathbf{g} \mathbf{X} = 0, \tag{20}$$

where

$$\mathbf{g} := \begin{bmatrix} \mathbf{I}_n & \mathbf{0}_{n \times 1} \\ \mathbf{0}_{1 \times n} & -1 \end{bmatrix} \tag{21}$$

is a Minkowski metric, $\mathbf{I}_n$ is the identity matrix of order n, and the superscript T stands for the transpose. In terms of $(\mathbf{u}, \|\mathbf{u}\|)$, Eq. (20) becomes

$$\mathbf{X}^T \mathbf{g} \mathbf{X} = \mathbf{u} \cdot \mathbf{u} - \|\mathbf{u}\|^2 = \|\mathbf{u}\|^2 - \|\mathbf{u}\|^2 = 0, \tag{22}$$

where the dot between two vectors denotes the inner product.

Consequently, we have an $n+1$-dimensional augmented system:

$$\mathbf{X}' = \mathbf{A}\mathbf{X} \tag{23}$$

with a constraint (20), where

$$\mathbf{A} := \begin{bmatrix} \mathbf{0}_{n \times n} & \frac{\mathbf{f}(\mathbf{u}, x)}{\|\mathbf{u}\|} \\ \frac{\mathbf{f}^T(\mathbf{u}, x)}{\|\mathbf{u}\|} & 0 \end{bmatrix} \tag{24}$$

is a Lie algebra $so(n, 1)$ of the proper orthochronous Lorentz group $SO_o(n, 1)$, because of $\mathbf{A}$ satisfying

$$\mathbf{A}^T \mathbf{g} + \mathbf{g} \mathbf{A} = \mathbf{0}. \tag{25}$$

This fact prompts us to devise the group-preserving scheme (GPS), whose discretized mapping $\mathbf{G}$ must exactly preserve the following properties:

$$\mathbf{G}^{\mathrm{T}}\mathbf{g}\mathbf{G} = \mathbf{g}, \tag{26}$$

$$\det \mathbf{G} = 1, \tag{27}$$

$$G_0^0 > 0, \tag{28}$$

where G_0^0 is the 00th component of $\mathbf{G}$.

Although the dimension of the new system is raising by one, it has been shown that the new system permits a GPS given as follows [Liu (2001)]:

$$\mathbf{X}_{\ell+1} = \mathbf{G}(\ell)\mathbf{X}_\ell, \tag{29}$$

where $\mathbf{X}_\ell$ denotes the numerical value of $\mathbf{X}$ at x_ℓ, and $\mathbf{G}(\ell) \in SO_o(n,1)$ is the group value of $\mathbf{G}$ at x_ℓ. If $\mathbf{G}(\ell)$ satisfies the properties in Eqs. (26)-(28), then $\mathbf{X}_\ell$ satisfies the cone condition in Eq. (20).

The Lie group can be generated from $\mathbf{A} \in so(n,1)$ by an exponential mapping,

$$\mathbf{G}(\ell) = \exp[\Delta x \mathbf{A}(\ell)] = \begin{bmatrix} \mathbf{I}_n + \frac{(a_\ell-1)}{\|\mathbf{f}_\ell\|^2}\mathbf{f}_\ell\mathbf{f}_\ell^{\mathrm{T}} & \frac{b_\ell\mathbf{f}_\ell}{\|\mathbf{f}_\ell\|} \\ \frac{b_\ell\mathbf{f}_\ell^{\mathrm{T}}}{\|\mathbf{f}_\ell\|} & a_\ell \end{bmatrix}, \tag{30}$$

where

$$a_\ell := \cosh\left(\frac{\Delta x\|\mathbf{f}_\ell\|}{\|\mathbf{u}_\ell\|}\right), \tag{31}$$

$$b_\ell := \sinh\left(\frac{\Delta x\|\mathbf{f}_\ell\|}{\|\mathbf{u}_\ell\|}\right). \tag{32}$$

Substituting Eq. (30) for $\mathbf{G}(\ell)$ into Eq. (29), we obtain

$$\mathbf{u}_{\ell+1} = \mathbf{u}_\ell + \eta_\ell\mathbf{f}_\ell, \tag{33}$$

$$\|\mathbf{u}_{\ell+1}\| = a_\ell\|\mathbf{u}_\ell\| + \frac{b_\ell}{\|\mathbf{f}_\ell\|}\mathbf{f}_\ell \cdot \mathbf{u}_\ell, \tag{34}$$

where

$$\eta_\ell := \frac{b_\ell\|\mathbf{u}_\ell\|\|\mathbf{f}_\ell\| + (a_\ell-1)\mathbf{f}_\ell \cdot \mathbf{u}_\ell}{\|\mathbf{f}_\ell\|^2}. \tag{35}$$

3.2 One-step GPS

Throughout this paper the superscript f denotes the value at $x = x_f$, while the superscript 0 denotes the value at $x = 0$. Assume that the total length x_f is divided by K steps, that is, the stepsize we use in the GPS is $\Delta x = x_f/K$.

Starting from $\mathbf{X}^0 = \mathbf{X}(0)$ we want to calculate the value $\mathbf{X}^f$ at $x = x_f$. By Eq. (29) we can obtain

$$\mathbf{X}^f = \mathbf{G}_K(\Delta x)\cdots\mathbf{G}_1(\Delta x)\mathbf{X}^0. \tag{36}$$

However, let us recall that each $\mathbf{G}_i$, $i = 1, \ldots, K$, is an element of the Lie group $SO_o(n,1)$, and by the closure property of Lie group, $\mathbf{G}_K\cdots\mathbf{G}_1$ is also a Lie group element of $SO_o(n,1)$ denoted by $\mathbf{G}$. Hence, we have

$$\mathbf{X}^f = \mathbf{G}\mathbf{X}^0. \tag{37}$$

This is a one-step Lie-group transformation from $\mathbf{X}^0$ to $\mathbf{X}^f$.

3.2.1 A generalized mid-point rule

We can calculate $\mathbf{G}$ by a generalized mid-point rule, which is obtained from an exponential mapping of $\mathbf{A}$ by taking the values of the argument variables of $\mathbf{A}$ at a generalized mid-point. The Lie group generated from such an $\mathbf{A} \in so(n,1)$ is known as a proper orthochronous Lorentz group, which admits a closed-form representation:

$$\mathbf{G} = \begin{bmatrix} \mathbf{I}_n + \frac{(a-1)}{\|\hat{\mathbf{f}}\|^2}\hat{\mathbf{f}}\hat{\mathbf{f}}^{\mathrm{T}} & \frac{b\hat{\mathbf{f}}}{\|\hat{\mathbf{f}}\|} \\ \frac{b\hat{\mathbf{f}}^{\mathrm{T}}}{\|\hat{\mathbf{f}}\|} & a \end{bmatrix}, \tag{38}$$

where

$$\hat{\mathbf{u}} = r\mathbf{u}^0 + (1-r)\mathbf{u}^f, \tag{39}$$

$$\hat{\mathbf{f}} = \mathbf{f}(\hat{\mathbf{u}}, \hat{x}), \tag{40}$$

$$a = \cosh\left(\frac{x_f\|\hat{\mathbf{f}}\|}{\|\hat{\mathbf{u}}\|}\right), \tag{41}$$

$$b = \sinh\left(\frac{x_f\|\hat{\mathbf{f}}\|}{\|\hat{\mathbf{u}}\|}\right). \tag{42}$$

Here, we use the initial $\mathbf{u}^0$ and the final $\mathbf{u}^f$ through a suitable weighting factor r to calculate $\mathbf{G}$, where $0 < r < 1$ is a parameter and $\hat{x} = (1-r)x_f$. The above method applied a generalized mid-point rule on the calculation of $\mathbf{G}$, and the resultant is a single-parameter Lie group element $\mathbf{G}(r)$. After developing the LGSM, we can determine the best r by matching the given final condition.

3.2.2 *A Lie group mapping between two points on the cone*

Let us define a new vector

$$\mathbf{F} := \frac{\hat{\mathbf{f}}}{\|\hat{\mathbf{u}}\|}, \tag{43}$$

such that Eqs. (38), (41) and (42) can also be expressed as

$$\mathbf{G} = \begin{bmatrix} \mathbf{I}_n + \frac{a-1}{\|\mathbf{F}\|^2}\mathbf{F}\mathbf{F}^{\mathrm{T}} & \frac{b\mathbf{F}}{\|\mathbf{F}\|} \\ \frac{b\mathbf{F}^{\mathrm{T}}}{\|\mathbf{F}\|} & a \end{bmatrix}, \tag{44}$$

$$a = \cosh(x_f \|\mathbf{F}\|), \tag{45}$$

$$b = \sinh(x_f \|\mathbf{F}\|). \tag{46}$$

From Eqs. (37) and (44) it follows that

$$\mathbf{u}^f = \mathbf{u}^0 + \eta\mathbf{F}, \tag{47}$$

$$\|\mathbf{u}^f\| = a\|\mathbf{u}^0\| + b\frac{\mathbf{F}\cdot\mathbf{u}^0}{\|\mathbf{F}\|}, \tag{48}$$

where

$$\eta := \frac{(a-1)\mathbf{F}\cdot\mathbf{u}^0 + b\|\mathbf{u}^0\|\|\mathbf{F}\|}{\|\mathbf{F}\|^2}. \tag{49}$$

Substituting Eq. (47), written as

$$\mathbf{F} = \frac{1}{\eta}(\mathbf{u}^f - \mathbf{u}^0), \tag{50}$$

into Eq. (48) and dividing both the sides by $\|\mathbf{u}^0\|$ we can obtain

$$\frac{\|\mathbf{u}^f\|}{\|\mathbf{u}^0\|} = a + b\frac{(\mathbf{u}^f - \mathbf{u}^0)\cdot\mathbf{u}^0}{\|\mathbf{u}^f - \mathbf{u}^0\|\|\mathbf{u}^0\|}, \tag{51}$$

where

$$a = \cosh\left(\frac{x_f\|\mathbf{u}^f - \mathbf{u}^0\|}{\eta}\right), \tag{52}$$

$$b = \sinh\left(\frac{x_f\|\mathbf{u}^f - \mathbf{u}^0\|}{\eta}\right) \tag{53}$$

are obtained from Eqs. (45) and (46) by inserting Eq. (50) for $\mathbf{F}$.

Let

$$\cos\theta := \frac{[\mathbf{u}^f - \mathbf{u}^0]\cdot\mathbf{u}^0}{\|\mathbf{u}^f - \mathbf{u}^0\|\,\|\mathbf{u}^0\|}, \tag{54}$$

$$S := x_f\|\mathbf{u}^f - \mathbf{u}^0\|, \tag{55}$$

and from Eqs. (51)-(53) it follows that

$$\frac{\|\mathbf{u}^f\|}{\|\mathbf{u}^0\|} = \cosh\left(\frac{S}{\eta}\right) + \cos\theta\sinh\left(\frac{S}{\eta}\right). \tag{56}$$

Defining

$$Z := \exp\left(\frac{S}{\eta}\right), \tag{57}$$

and from Eq. (56) we obtain a quadratic equation for Z:

$$(1+\cos\theta)Z^2 - \frac{2\|\mathbf{u}^f\|}{\|\mathbf{u}^0\|}Z + 1 - \cos\theta = 0. \tag{58}$$

The solution is found to be

$$Z = \frac{\frac{\|\mathbf{u}^f\|}{\|\mathbf{u}^0\|} + \sqrt{\left(\frac{\|\mathbf{u}^f\|}{\|\mathbf{u}^0\|}\right)^2 - 1 + \cos^2\theta}}{1+\cos\theta}, \tag{59}$$

and then from Eqs. (57) and (55) we can obtain

$$\eta = \frac{x_f\|\mathbf{u}^f - \mathbf{u}^0\|}{\ln Z}. \tag{60}$$

Therefore, between any two points $(\mathbf{u}^0, \|\mathbf{u}^0\|)$ and $(\mathbf{u}^f, \|\mathbf{u}^f\|)$ on the cone, there exists a Lie group element $\mathbf{G} \in SO_o(n, 1)$ mapping $(\mathbf{u}^0, \|\mathbf{u}^0\|)$ onto $(\mathbf{u}^f, \|\mathbf{u}^f\|)$, which is given by

$$\begin{bmatrix} \mathbf{u}^f \\ \|\mathbf{u}^f\| \end{bmatrix} = \mathbf{G}\begin{bmatrix} \mathbf{u}^0 \\ \|\mathbf{u}^0\| \end{bmatrix}, \tag{61}$$

where $\mathbf{G}$ is uniquely determined by $\mathbf{u}^0$ and $\mathbf{u}^f$ through the following equations:

$$\mathbf{G} = \begin{bmatrix} \mathbf{I}_n + \frac{a-1}{\|\mathbf{F}\|^2}\mathbf{F}\mathbf{F}^{\mathrm{T}} & \frac{b\mathbf{F}}{\|\mathbf{F}\|} \\ \frac{b\mathbf{F}^{\mathrm{T}}}{\|\mathbf{F}\|} & a \end{bmatrix}, \tag{62}$$

$$a = \cosh(x_f\|\mathbf{F}\|), \tag{63}$$

$$b = \sinh(x_f\|\mathbf{F}\|), \tag{64}$$

$$\mathbf{F} = \frac{1}{\eta}(\mathbf{u}^f - \mathbf{u}^0). \tag{65}$$

4 Identifying $H(\phi)$ and $Q(\dot{\psi})$ by the LGSM

In this section we begin to estimate nonlinear spring force $H(\phi)$ and nonlinear damping force $Q(\dot{\psi})$. From Eqs. (43) and (47) follow a very useful equation:

$$\mathbf{u}^f = \mathbf{u}^0 + \eta\frac{\hat{\mathbf{f}}}{\|\hat{\mathbf{u}}\|}, \tag{66}$$

where by using Eq. (10) we have

$$u_i^f = (1+x_f)u_i^0 = (1+x_f)\phi_i. \tag{67}$$

Thus the vector $\mathbf{u}^f$ with components u_i^f is proportional to $\mathbf{u}^0$ with components u_i^0 by a multiplier $1+x_f$ larger than 1. Under this condition we have $\cos\theta = 1$ by Eq. (54), and from Eqs. (58) and (67) it follows that

$$Z = 1 + x_f. \tag{68}$$

Moreover, by Eqs. (60) and (67) we have

$$\eta = \frac{x_f^2\|\mathbf{u}^0\|}{\ln(1+x_f)}, \tag{69}$$

and by Eqs. (39) and (67) we have

$$\|\hat{\mathbf{u}}\| = x_r\|\mathbf{u}^0\|, \tag{70}$$

where

$$x_r := 1 + \hat{x} = r + (1-r)(1+x_f). \tag{71}$$

Substituting Eqs. (69) and (70) into Eq. (66) we have

$$\mathbf{u}^f = \mathbf{u}^0 + \eta_0\hat{\mathbf{f}}, \tag{72}$$

where

$$\eta_0 = \frac{x_f^2}{x_r \ln(1+x_f)}. \tag{73}$$

By applying Eq. (72) to Eq. (17) we obtain

$$u_i^f = u_i^0 + \frac{\eta_0}{(\Delta t)^2}(\hat{u}_{i+1} - 2\hat{u}_i + \hat{u}_{i-1}) + \frac{\eta_0 \gamma}{2\Delta t}(\hat{u}_{i+1} - \hat{u}_{i-1}) + \eta_0 x_r H_i + \eta_0 \phi_i - \eta_0 x_r F_i, \tag{74}$$

where

$$\hat{u}_i = x_r \phi_i, \quad i = 1, \dots, n. \tag{75}$$

After inserting Eq. (75) for $\hat{u}_i$, Eq. (73) for η_0, and Eq. (67) for u_i^f with $u_i^0 = \phi_i$, it is not difficult to rewrite Eq. (74) as

$$H_i = \frac{\phi_i \ln(1+x_f)}{x_f} - \frac{1}{(\Delta t)^2}(\phi_{i+1} - 2\phi_i + \phi_{i-1}) - \frac{\phi_i}{x_r} - \frac{\gamma}{2\Delta t}(\phi_{i+1} - \phi_{i-1}) + F_i. \tag{76}$$

Because of Eq. (71), the above estimating equation depends on r. Now, the problem is how to choose a suitable r. The numerical procedures for determining r are described as follows. In the range of $r \in (0,1)$ we insert each r into the above equation to obtain H_i, and we can exactly integrate Eq. (17) from $x = 0$ to $x = x_f$ by noting Eq. (10). Then, u_i^f is given by

$$u_i^f = (1+x_f)\phi_i$$
$$+ \frac{1}{2}x_f(2+x_f)\left[\frac{1}{(\Delta t)^2}(\phi_{i+1} - 2\phi_i + \phi_{i-1}) + \frac{\gamma}{2\Delta t}(\phi_{i+1} - \phi_{i-1}) + H_i - F_i\right]. \tag{77}$$

By comparing the above u_i^f with the target given exactly by Eq. (67), we can pick up the best r by satisfying

$$\min_{r \in (0,1)} \sqrt{\sum_{i=1}^n [u_i^f - (1+x_f)\phi_i]^2}. \tag{78}$$

When r is selected we can insert it into Eq. (76) to calculate H_i.

Similarly, for Eq. (7) we can derive

$$Q_i = \frac{\psi_i \ln(1+x_f)}{x_f} - \frac{1}{(\Delta t)^2}(\psi_{i+1} - 2\psi_i + \psi_{i-1}) - \frac{\psi_i}{x_r} - \alpha \psi_i + F_i, \tag{79}$$

where $\psi_i = \psi(t_i)$. We can select the best r by a similar process , such that the above equation can be used to calculate Q_i.

5   Numerical examples

5.1   Example 1

Let us consider

$$\gamma = 0.3,$$
$$H(\phi) = \phi^3 - \phi. \tag{80}$$

In order to identify the restoring force H as a function of ϕ we require ϕ to be a monotonic function of t. Here we suppose that ϕ is given by $\phi(t) = t^2 - 8$. To obtain this ϕ the external force is given by

$$F(t) = 10 + 2\gamma t + (t^2 - 8)^3 - t^2. \tag{81}$$

We use the vibration data of displacement $\phi_i = \phi(t_i)$ as the inputs in Eq. (76) to estimate H_i. In this calculation we have fixed $\Delta t = 4/500$ and $x_f = 1.2 \times 10^{-5}$, and found that $r = 1/2$ is the best one. The computed profile of $H(\phi)$ is plotted in Fig. 1(a) by the dashed line, which is compared with the exact one plotted by the solid line. The maximum estimation error of H as shown in Fig. 1(b) is smaller than 10^{-10}.

In order to test the stability of the present method we also consider

$$\hat{\phi}_i = \phi_i[1 + sR(i)] \tag{82}$$

as inputs in the estimation equations, where R_i are random numbers in $[-1, 1]$, and s is a level of noise. Under a noise with $s = 0.01$ the computed profile of $H(\phi)$ is plotted in Fig. 1(a) by the dashed-dotted line, which is compared with the exact one plotted by the solid line. In this calculation we have fixed $\Delta t = 4/40$ and $x_f = 0.001$. It can be seen that our estimation method is well against the noisy disturbances.

5.2   Example 2

Next we consider a Rayleigh damping oscillator given by

$$\alpha = 1,$$
$$Q(\dot{\psi}) = \dot{\psi}^3 - \dot{\psi}. \tag{83}$$

In order to identify the damping force Q as a function of $\dot{\psi}$ we require $\dot{\psi}$ to be a monotonic function of t. Here we suppose that $\dot{\psi}$ is given by $\dot{\psi}(t) = t^2 - 8$, and ψ is given by $\psi(t) = t^3/3 - 8t$. To obtain this ψ the external force is given by

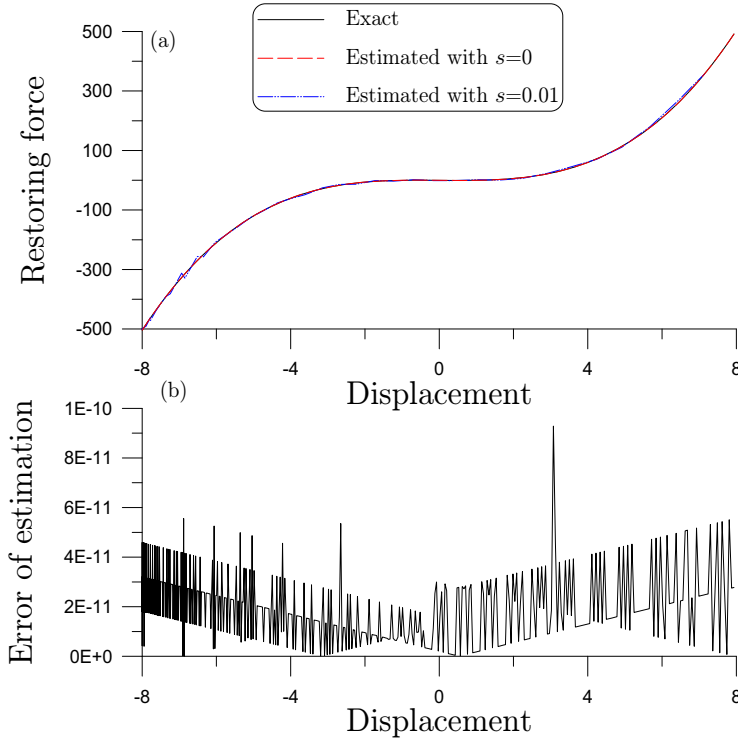$$F(t) = 8 + (t^2 - 8)^3 - t^2 + t^3/3 - 6t. \tag{84}$$

Figure 1: For Example 1: (a) comparing estimated and exact restoring forces, and (b) displaying the error of estimation.

We use the vibration data of displacement ψ_i as the inputs in Eq. (79) to estimate Q_i. In this calculation we have fixed $\Delta t = 4/300$ and $x_f = 9 \times 10^{-6}$, and found that $r = 1/2$. The computed profile of $Q(\dot{\psi})$ is plotted in Fig. 2(a) by the dashed line, which is compared with the exact one plotted by the solid line. The maximum estimation error of Q as shown in Fig. 2(b) is smaller than 10^{-10}.

Under a noise with $s = 0.01$ the computed profile of $Q(\dot{\phi})$ is plotted in Fig. 2(a) by the dashed-dotted line, which is compared with the exact one plotted by the solid line. In this calculation we have fixed $\Delta t = 4/40$ and $x_f = 0.001$. It can be seen that the estimated result is acceptable, and is well against the noisy disturbances.

6 Identifying $H(\phi, \dot{\phi})$ by the LGSM

In this section we directly estimate the nonlinear restoring force function. For this purpose, let us consider a second-order ODE describing the forced vibration of a
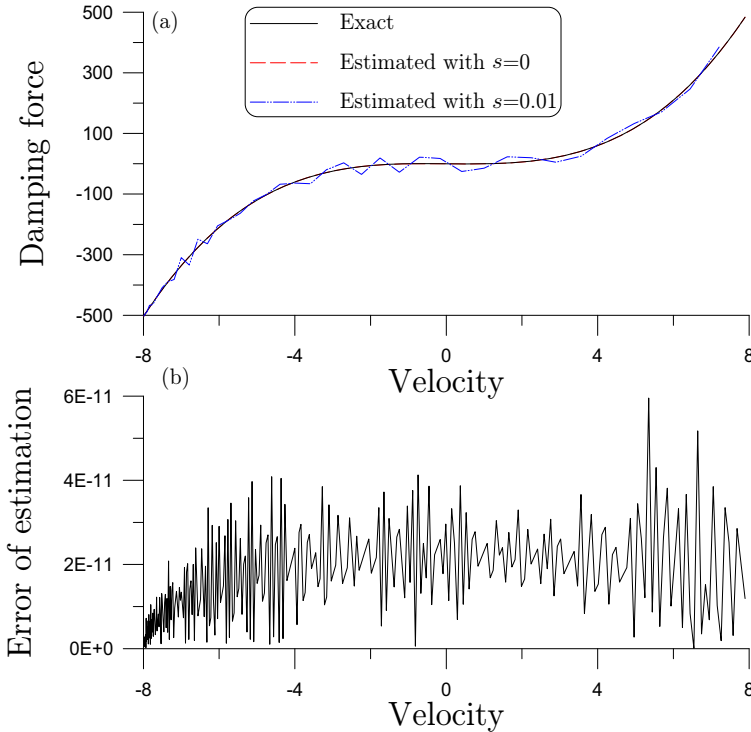
Figure 2: For Example 2: (a) comparing estimated and exact damping forces, and (b) displaying the error of estimation.

nonlinear structure by

$$\ddot{\phi} + H(\phi, \dot{\phi}) = F(t), \ \ 0 \le t \le t_f, \tag{85}$$

$$\phi(0) = A_0, \tag{86}$$

$$\dot{\phi}(0) = B_0. \tag{87}$$

6.1 *Estimation of* $H(\phi, \dot{\phi})$

By considering Eq. (10), Eqs. (85)-(87) can be changed to a parabolic type PDE:

$$\frac{\partial u(x,t)}{\partial x} = \frac{\partial^2 u(x,t)}{\partial t^2} + h(x,t) + \phi(t) - (1+x)F(t), \tag{88}$$

$$u(0,t) = \phi(t), \tag{89}$$

$$u(x,0) = A_0(1+x), \tag{90}$$

$$u(x,t_f) = \phi(t_f)(1+x), \tag{91}$$

where $h(x,t) = (1+x)H(\phi(t), \dot{\phi}(t))$. By utilizing a finite difference on Eq. (88) we obtain

$$u_i'(x) = \frac{1}{(\Delta t)^2}[u_{i+1}(x) - 2u_i(x) + u_{i-1}(x)] + h_i(x) + \phi_i - (1+x)F_i, \tag{92}$$

where $h_i(x) = (1+x)H(\phi_i, \dot{\phi}_i) = (1+x)H_i$ with $\phi_i = \phi(t_i)$, $\dot{\phi}_i = \dot{\phi}(t_i)$ and $F_i = F(t_i)$, $i = 1, \ldots, n$.

By applying Eq. (72) to Eq. (92) we obtain

$$u_i^f = u_i^0 + \frac{\eta_0}{(\Delta t)^2}(\hat{u}_{i+1} - 2\hat{u}_i + \hat{u}_{i-1}) + \eta_0 x_r H_i + \eta_0 \phi_i - \eta_0 x_r F_i. \tag{93}$$

After inserting Eq. (75) for $\hat{u}_i$, Eq. (73) for η_0, and Eq. (67) for u_i^f with $u_i^0 = \phi_i$, it is not difficult to rewrite Eq. (93) as

$$H_i = \frac{\phi_i \ln(1+x_f)}{x_f} - \frac{1}{(\Delta t)^2}(\phi_{i+1} - 2\phi_i + \phi_{i-1}) - \frac{\phi_i}{x_r} + F_i. \tag{94}$$

Eq. (94) can be written as

$$H_i = \left[\frac{\ln(1+x_f)}{x_f} - \frac{1}{1+(1-r)x_f}\right]\phi_i - \frac{1}{(\Delta t)^2}(\phi_{i+1} - 2\phi_i + \phi_{i-1}) + F_i, \tag{95}$$

which can be viewed as a modification of a standard central finite difference of Eq. (85), because the above equation reduces to a central finite difference equation when $x_f = 0$. The best r seems $r = 1/2$ because the target u_i^f is proportional to u_i^0 (numerical examples in Section 5 also support this assertion). Thus, by fixing $r = 1/2$ we get

$$H_i = \left[\frac{\ln(1+x_f)}{x_f} - \frac{2}{2+x_f}\right]\phi_i - \frac{1}{(\Delta t)^2}(\phi_{i+1} - 2\phi_i + \phi_{i-1}) + F_i. \tag{96}$$

Now the problem is how to determine x_f, of which the numerical procedures are described as follows. In the range of $x_f \in (0, x_u)$, where x_u is a maximum distance of the target determined by the user, we insert each x_f into the above equation to obtain H_i, and we can exactly integrate Eq. (92) from $x = 0$ to $x = x_f$ by noting Eq. (10). Then, u_i^f is given by

$$u_i^f = (1+x_f)\phi_i + \frac{1}{2}x_f(2+x_f)\left[\frac{1}{(\Delta t)^2}(\phi_{i+1} - 2\phi_i + \phi_{i-1}) + H_i - F_i\right]. \tag{97}$$

By comparing the above u_i^f with the target given exactly by Eq. (67), we can pick up the best x_f by satisfying

$$\min_{x_f \in (0, x_u)} \sqrt{\sum_{i=1}^{n} [u_i^f - (1 + x_f)\phi_i]^2}. \tag{98}$$

When x_f is selected we can insert it into Eq. (96) to calculate H_i.

6.2 Example 3

We consider a complex one with

$$H(\phi, \dot{\phi}) = \phi + (1 - \dot{\phi}^2 + 0.01\dot{\phi}^3)\dot{\phi}. \tag{99}$$

In order to identify the nonlinear restoring force H as a function of ϕ and $\dot{\phi}$ we require ϕ and $\dot{\phi}$ both to be monotonic functions of t. Here we suppose that ϕ is given by $\phi(t) = A_0 + t^3/3 - 8t$, and $\dot{\phi}$ is given by $\dot{\phi}(t) = t^2 - 8$.

To obtain this ϕ the external force is

$$F(t) = 2t + H(\phi(t), \dot{\phi}(t)) \tag{100}$$

by inserting $\phi(t) = A_0 + t^3/3 - 8t$ and $\dot{\phi}(t) = t^2 - 8$ into the above equation.

We use the vibration data of displacement at discretized time by inserting t_i into the given function $\phi_i = \phi(t_i) = A_0 + t_i^3/3 - 8t_i$ as the inputs in Eq. (96) to estimate H_i. We can obtain the surface of the function $H(\phi, \dot{\phi})$ by selecting different A_0. In this calculation we have fixed $\Delta t = 4/500$ and $x_u = 10^{-6}$. The maximum estimation error of H as shown in Fig. 3 is smaller than 10^{-8}.

6.3 Example 4

Let us estimate $H(\phi, \dot{\phi})$ of the Van der Pol oscillator given by

$$H(\phi, \dot{\phi}) = \phi + (\phi^2 - 1)\dot{\phi}. \tag{101}$$

Here we also suppose that ϕ is given by $\phi(t) = A_0 + t^3/3 - 8t$, and $\dot{\phi}$ is given by $\dot{\phi}(t) = t^2 - 8$.

In this calculation we have fixed $\Delta t = 4/500$ and $x_u = 10^{-6}$. The maximum estimation error of H as shown in Fig. 4 is smaller than 10^{-8}.

6.4 Example 5

As a last example to estimate $H(\phi, \dot{\phi})$ we consider a combination of the nonlinear restoring forces of Duffing and Rayleigh, that is,

$$H(\phi, \dot{\phi}) = \phi^3 - \phi^2 + \dot{\phi}^3 - \dot{\phi}. \tag{102}$$

The maximum estimation error of H as shown in Fig. 5 is smaller than 10^{-8}.
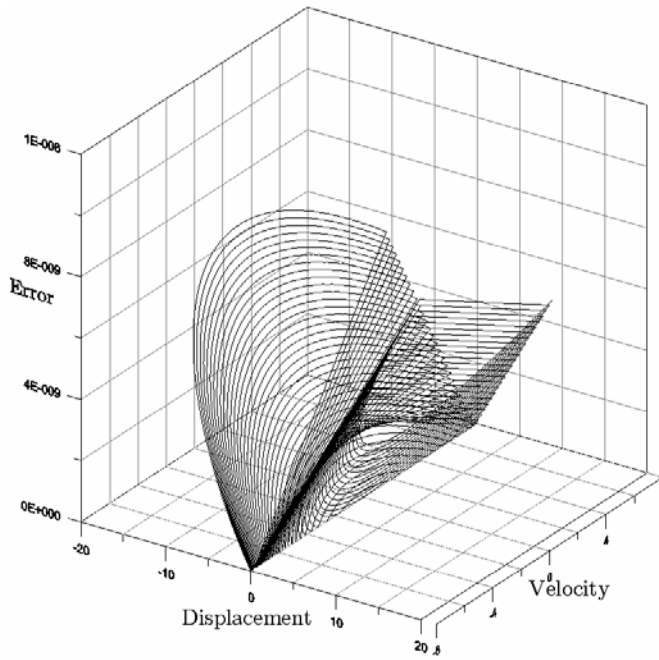
Figure 3: For Example 3 showing the error of estimation.
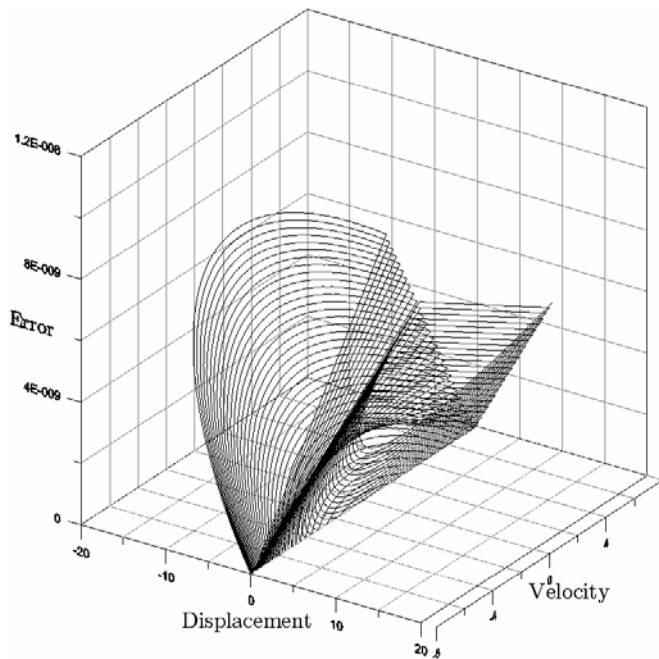

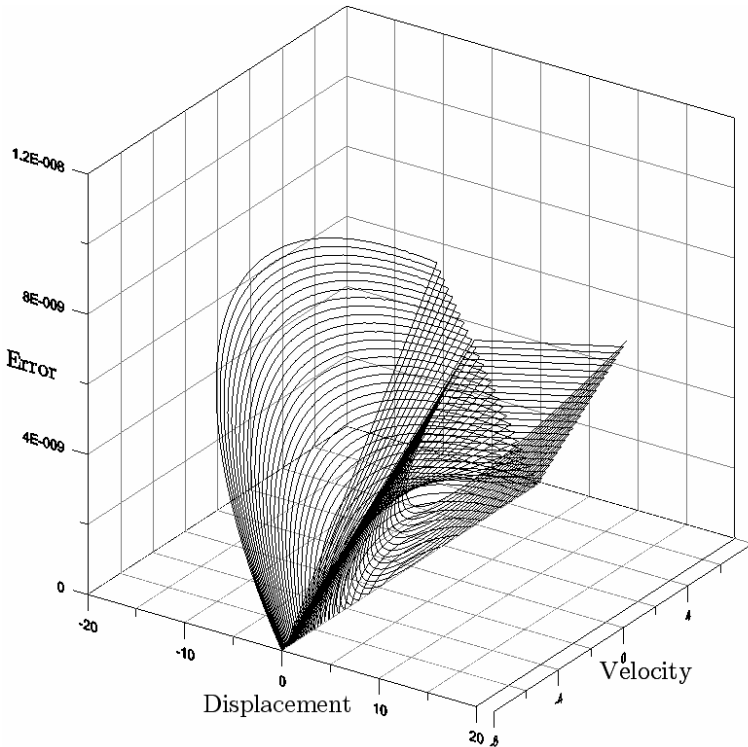Figure 4: For Example 4 showing the error of estimation.

Figure 5: For Example 5 showing the error of estimation.

7 Conclusions

The inverse vibration problem of estimating nonlinear restoring force for nonlinear mechanical system is rather difficult. However, the present paper could offer very accurate and simple method without any iteration to estimate restoring force, which is represented as a surface on the phase plane of displacement and velocity. Data of velocity and acceleration are derivative quantities, which are hard directly inserting into the equation of motion to obtain accurate restoring force values. To overcome this difficulty we have only used the displacement data as our formulation variables in the estimation equation. A two-point BVP formulation basing on a fictitious time concept as well as an establishment of the Lie-group shooting method led us a closed-form estimating equation, which is highly effective and time saving even in the estimation of restoring forces under noised displacement data. The estimation accuracy assessed by using the absolute error can be controlled within the eighth decimal point.

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