Copyright © 2008 Tech Science Press CMES, vol.35, no.2, pp.91-111, 2008

A Highly Accurate MCTM for Inverse Cauchy Problems of Laplace Equation in Arbitrary Plane Domains

Chein-Shan Liu[1]

Abstract: We consider the inverse Cauchy problems for Laplace equation in simply and doubly connected plane domains by recoverning the unknown boundary value on an inaccessible part of a noncircular contour from overspecified data. A modified Trefftz method is used directly to solve those problems with a simple collocation technique to determine unknown coefficients, which is named a modified collocation Trefftz method (MCTM). Because the condition number is small for the MCTM, we can apply it to numerically solve the inverse Cauchy problems without needing of an extra regularization, as that used in the solutions of direct problems for Laplace equation. So, the computational cost of MCTM is very saving. Numerical examples show the effectiveness of the new method in providing an excellent estimate of unknown boundary data, even by subjecting the given data to a large noise.

Keyword: Inverse Cauchy problem, Modified Trefftz method, Laplace equation, Modified collocation Trefftz method (MCTM)

1 Inverse Cauchy problems

The inverse Cauchy problem is difficult to solve, since its solution does not depend continuously on the given data. Because of this ill-posedness, the errors in measured data will be enlarged in the solution by a numerical treatment, if we do not take this trouble into account. Therefore, we must tackle this type problem with a suitable numerical algorithm, which compromises accuracy and stability. Chang, Yeih and Shieh (2001) have shown that neither the traditional Tikhonov's regularization method nor the singular value decomposition method can yield acceptable numerical result for the inverse Cauchy problem of Laplace equation, when the influence matrix is highly ill-posed.

[1] Department of Mechanical and Mechatronic Engineering, Department of Harbor and River Engineering, Taiwan Ocean University, Keelung, Taiwan. E-mail: csliu@mail.ntou.edu.tw

Lesnic, Elliott and Ingham (1997), and Mera, Elliott, Ingham and Lesnic (2000) have applied the boundary element method for the solutions of inverse Cauchy problems. Their methods are inevitably required an iterative process to adjust the solution. In this paper, we begin with a modified Trefftz method proposed by Liu (2007a, 2007b), and leave the unknown coefficients determined by partial but over-specified boundary conditions from a collocation method. The collocation method is very useful in the computations of direct problems in engineering, because the algebraic equations can be easily derived. However, it is seldom used in the inverse problems because the ill-posedness is inherent.

The method of fundamental solutions (MFS) utilizes the fundamental solutions as basis functions to expand the solution, which is another popularly used meshless method. While Jin and Zheng (2006) have applied the MFS to solve the inverse problem of Helmholtz equation, Marin and Lesnic (2005) have applied the MFS to solve the inverse Cauchy problem associated with a two-dimensional biharmonic equation. In order to tackle of the ill-posedness of MFS and the inherent ill-posed property of the inverse Cauchy problems, those authors proposed new numerical schemes with the regularization parameters determined by the L-curve method. Ling and Takeuchi (2008) have combined the MFS and boundary control technique to solve the inverse Cauchy problem of Laplace equation. Liu and Atluri (2008a) reformulated the inverse Cauchy problem of Laplace equation in a rectangle as an optimization problem, and applied a fictitious time integration method [Liu and Atluri (2008b)] to solve an algebraic equations system to obtain the data on an unspecified portion of boundary. When an extension to nonlinear inverse Cauchy problem is concerned with, they showed that good result can be obtained by using their method.

Our starting point employs a meshless method of Trefftz type, and a new modification is required in order to get a non ill-posed linear equations system. Furthermore, our method does not need regularization even the input data is noised seriously. Recently, Liu (2008a) has applied a similar technique by modifying the traditional Trefftz method for biharmonic equation, and showed that both the direct and inverse problems can be unifiedly solved very well by the same modified collocation Trefftz method (MCTM).

The detection of corrosion inside a pipe is very important in engineering applications. In this paper we consider a mathematical modeling of this problem and give an effective numerical algorithm for a method to detect the corrosion by an electrical field in the pipe. Given the Cauchy data $u(x,y)$ and Neumann data $\partial u/\partial n(x,y)$ at the point $(x,y) \in \mathbb{R}^2$ with unit outward normal $n(x,y)$ on the accessible external part Γ_1 of a noncircular contour, we consider an inverse Cauchy problem of the Laplace equation $\Delta u(x,y) = 0$ in two dimensions to find the unknown function $u(x,y)$ on an inaccessible part Γ_2 of $\Gamma = \Gamma_1 \cup \Gamma_2$ as schematically shown in Fig. 1(a). This is the first type inverse Cauchy problem considered in the present paper.

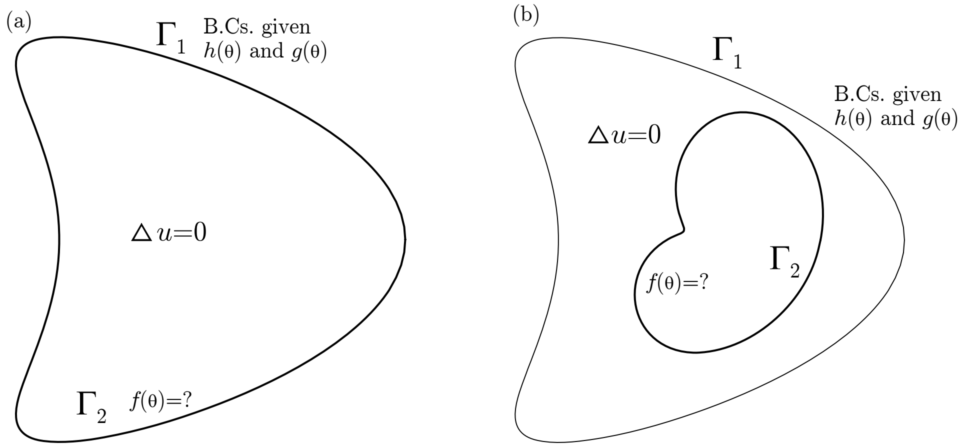

Figure 1: The inverse Cauchy problems are schematically shown in (a) for the first type in a simply-connected domain, and (b) for the second type in a doubly-connected domain.

On the other hand, this paper also addresses the second type inverse Cauchy problem in a doubly-connected plane domain as schematically shown in Fig. 1(b), where we want to recover the unknown boundary data on an inner part Γ_2 by giving overspecified data on an outer part Γ_1. This problem setting can be used in the electrostatic image of the inverse problem in human electro-cardiography [Johnston (2001)]. The use of electrostatic image in the nondestructive testing of metallic plates leads to an inverse boundary value problem for Laplace equation in two-dimensions. In order to detect the unknown shape of the inclusion within a conducting metal, the overspecified Cauchy data, for example the voltage and current, are imposed on the accessible exterior boundary [Akduman and Kress (2002); Inglese (1997); Kaup, Santosa and Vogelius (1996)]. This amounts to solving an inverse Cauchy problem from available data on a partial part of the boundary. Because this problem is well known to be highly ill-posed since the work of Hadamard, there have been many studies of this type problems, e.g., Andrieux, Baranger and Ben Abda (2006), Aparicio and Pidcock (1996), Ben Belgacem and El Fekih (2005), Berntsson and Eldén (2001), Bourgeois (2005, 2006); Chapko and Kress (2005), Kress (2004), Mera, Elliott, Ingham and Lesnic (2000), and Slodicka and Van Keer (2004).

Let us consider

$$\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \tag{1}$$

$$u(\rho,\theta) = h(\theta), \;\; 0 \leq \theta \leq \beta\pi, \tag{2}$$

$$u_n(\rho,\theta) = g(\theta), \;\; 0 \leq \theta \leq \beta\pi, \tag{3}$$

where $h(\theta)$ and $g(\theta)$ are given functions. $\beta \leq 1$ for the first type inverse Cauchy problem, while $\beta = 2$ for the second type inverse Cauchy problem. Here, $r = \rho(\theta)$ is a given contour describing the boundary shape of an interior domain Ω. The contour Γ in the polar coordinates is described by $\Gamma = \{(r,\theta)|r = \rho(\theta),\ 0 \leq \theta \leq 2\pi\}$. For the second type inverse Cauchy problem, while $r = \rho$ is used to describe the outer contour, $r = \kappa(\theta)$ is used to describe the inner contour.

The present inverse Cauchy problem is given as follows:

Inverse problem. *To seek an unknown function $f(\theta)$ on the unspecified part of the boundaries under Eqs. (1)-(3).*

This problem is for solving the Laplace equation under an overspecified Cauchy data on a partial noncircular boundary. Usually, it requires $\beta \geq 1$, which is specified by Mera, Elliott and Ingham (2003) as a necessary condition for the numerically identifiable of the first type inverse Cauchy problem. However, in the present study we can let $\beta < 1$ without losing much accuracy.

When the contour is circular, Liu (2008b) has applied a modified Trefftz method to recover the unknown boundary data for the first type inverse Cauchy problem, but needs to consider a regularization technique by truncating the higher-mode components of the given data. In this paper we extend the study to arbitrary plane domain, without needing of regularization technique. In Sections 2 and 3 we give numerical method and numerical examples for the first type inverse Cauchy problem, while in Sections 4 and 5 we give numerical method and numerical examples for the second type inverse Cauchy problem. The latter type is more difficult to solve than the first type.

2 A modified collocation Trefftz method

In this section we derive a new numerical method to solve the first type inverse Cauchy problem, and numerical examples are given in the next section.

Liu (2007a, 2007b, 2007c) has proposed a modified Trefftz method by supposing that

$$u(r,\theta) = a_0 + \sum_{k=1}^{m} \left[a_k \left(\frac{r}{R_0}\right)^k \cos k\theta + b_k \left(\frac{r}{R_0}\right)^k \sin k\theta \right], \tag{4}$$

where

$$R_0 \geq \rho_{\max} = \max_{\theta \in [0,2\pi]} \rho(\theta) \tag{5}$$

is a number greater than the characteristic length of the problem domain we consider. Besides, m is a positive integer chosen by the user, and $a_0, a_k, b_k,\ k = 1,\ldots,m$ are unknown coefficients to be determined below. Recently, Chen, Liu and Chang (2008) have applied the MCTM to discontinuous boundary value problem, singular problem and degenarate scale problem of Laplace equation by using much higher-order terms with m larger than 100, and they showed that the MCTM is more powerful and robust against noise than other numerical methods.

Through some effort we can derive (see the Appendix)

$$u_n(\rho,\theta) = \eta(\theta)\left[\frac{\partial u(\rho,\theta)}{\partial \rho} - \frac{\rho'}{\rho^2}\frac{\partial u(\rho,\theta)}{\partial \theta}\right], \tag{6}$$

where

$$\eta(\theta) = \frac{\rho(\theta)}{\sqrt{\rho^2(\theta) + [\rho'(\theta)]^2}}. \tag{7}$$

From Eqs. (6) and (4) it follows that

$$u_n(\rho,\theta) = \sum_{k=1}^{m} \gamma^k \left[\left\{\frac{ka_k}{\rho} - \frac{kb_k\rho'}{\rho^2}\right\}\cos k\theta + \left\{\frac{kb_k}{\rho} + \frac{ka_k\rho'}{\rho^2}\right\}\sin k\theta\right], \tag{8}$$

where

$$\gamma(\theta) := \frac{\rho(\theta)}{R_0}. \tag{9}$$

By imposing conditions (2) and (3) on Eqs. (4) and (8) we can obtain

$$a_0 + \sum_{k=1}^{m}[a_kC_k + b_kD_k] = h(\theta),\ \ 0 \leq \theta \leq \beta\pi, \tag{10}$$

$$\sum_{k=1}^{m}[a_kE_k + b_kF_k] = g(\theta),\ \ 0 \leq \theta \leq \beta\pi, \tag{11}$$

where

$$C_k := \gamma^k \cos k\theta, \tag{12}$$

$$D_k := \gamma^k \sin k\theta, \tag{13}$$

$$E_k := \eta\gamma^k\left[\frac{k}{\rho}\cos k\theta + \frac{k\rho'}{\rho^2}\sin k\theta\right], \tag{14}$$

$$F_k := \eta\gamma^k\left[\frac{k}{\rho}\sin k\theta - \frac{k\rho'}{\rho^2}\cos k\theta\right]. \tag{15}$$

Eqs. (10) and (11) are then imposed at $m+1$ different collocated points $(\rho(\theta_i),\theta_i)$ in an interval of $0 \le \theta_i \le \beta\pi$:

$$a_0 + \sum_{k=1}^{m} [a_k C_k^i + b_k D_k^i] = h(\theta_i), \tag{16}$$

$$\sum_{k=1}^{m} [a_k E_k^i + b_k F_k^i] = g(\theta_i), \tag{17}$$

where for simple notations we use $C_k^i = C_k(\theta_i)$, etc.

It can be seen that the idea behind the collocation method is rather simple, and it has a great advantage of the flexibility to apply to different geometric shapes, and the simplicity for a computer program.

Let

$$\theta_i = i\Delta\theta, \;\; i = 1,\ldots,m, \tag{18}$$

where $\Delta\theta = \beta\pi/(m+1)$, and $(\rho(\theta_i),\theta_i)$ are the collocated points on the partial noncircular contour. When the index i in Eqs. (16) and (17) runs from 1 to m we obtain a linear equations system with dimensions $n = 2m+1$:

$$\begin{bmatrix} 1 & C_1^0\cos\theta_0 & D_1^0\sin\theta_0 & \ldots & C_m^0\cos(m\theta_0) & D_m^0\sin(m\theta_0) \\ 1 & C_1^1\cos\theta_1 & D_1^1\sin\theta_1 & \ldots & C_m^1\cos(m\theta_1) & D_m^1\sin(m\theta_1) \\ 0 & E_1^1\cos\theta_1 & F_1^1\sin\theta_1 & \ldots & E_m^1\cos(m\theta_1) & F_m^1\sin(m\theta_1) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & C_1^m\cos\theta_m & D_1^m\sin\theta_m & \ldots & C_m^m\cos(m\theta_m) & D_m^m\sin(m\theta_m) \\ 0 & E_1^m\cos\theta_m & F_1^m\sin\theta_m & \ldots & E_m^m\cos\theta_m & F_m^m\sin(m\theta_m) \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ b_1 \\ \vdots \\ a_m \\ b_m \end{bmatrix} = \begin{bmatrix} h(\theta_0) \\ h(\theta_1) \\ g(\theta_1) \\ \vdots \\ h(\theta_m) \\ g(\theta_m) \end{bmatrix}. \tag{19}$$

Corresponding to the uniformly distributed collocated points on the upper partial contour, $\theta_0 = 0$ is a single collocated point which is supplemented to provide the n-th equation.

We denote the above equation by

$$\mathbf{Rc} = \mathbf{b}_1,$$

where $\mathbf{c} = [a_0, a_1, b_1, \cdots, a_m, b_m]^{\mathrm{T}}$ is the vector of unknown coefficients. The conjugate gradient method can be used to solve the following normal equation:

$$\mathbf{A}\mathbf{c} = \mathbf{b}, \tag{20}$$

where

$$\mathbf{A} := \mathbf{R}^{\mathrm{T}}\mathbf{R}, \ \ \mathbf{b} := \mathbf{R}^{\mathrm{T}}\mathbf{b}_1. \tag{21}$$

Inserting the calculated $\mathbf{c}$ into Eq. (4) we thus have a semi-analytical solution of $u(r,\theta)$,

$$u(r,\theta) = c_1 + \sum_{k=1}^{m} \left(\frac{r}{R_0}\right)^k [c_{2k}\cos k\theta + c_{2k+1}\sin k\theta], \tag{22}$$

and thus the data to be recovered is given by

$$f(\theta) = c_1 + \sum_{k=1}^{m} \left(\frac{\rho(\theta)}{R_0}\right)^k [c_{2k}\cos k\theta + c_{2k+1}\sin k\theta]. \tag{23}$$

3 Numerical examples for the first type inverse Cauchy problem

Before embarking a numerical testing of the new method, we are concerned with the stability of modified collocation Trefftz method (MCTM), in the case when the boundary data are contaminated by random noise, which is investigated by adding different levels of random noise on the boundary data. We use the function RANDOM_NUMBER given in Fortran to generate the noisy data $R(i)$, which are random numbers in $[-1,1]$. Hence we use the simulated noisy data given by

$$\hat{h}(\theta_i) = h(\theta_i) + sR(i) \tag{24}$$

as the inputs in Eq. (19), where $\theta_i = i\beta\pi/(m+1)$, $i = 0,1,\ldots,m+1$, and s is the level of noise. Similarly, we do this for the Neumann boundary data $g(\theta)$.

3.1 Example 1

We first consider a simple example with the exact solution

$$u = x^2 - y^2 = r^2\cos(2\theta). \tag{25}$$

Therefore, the data on the upper contour are given by

$$h(\theta) = \rho^2\cos(2\theta), \ \ 0 \le \theta \le \beta\pi, \tag{26}$$

$$g(\theta) = \eta(\theta)[2\rho\cos(2\theta) + 2\rho'\sin(2\theta)], \ \ 0 \le \theta \le \beta\pi, \tag{27}$$

where the contour is described by an epitrochoid boundary shape,

$$\rho(\theta) = \sqrt{(a+b)^2 + 1 - 2(a+b)\cos(a\theta/b)} \tag{28}$$

with $a = 4$ and $b = 1$.

We can apply the MCTM to this example with very high accuracy as shown in Fig. 2(a) by comparing the numerical solution with exact solution, and by displaying the absolute error in Fig. 2(b), where $m = 3$, $R_0 = 10$, $\beta = 1$ and $s = 0$ were used. The L^2 error is about 2.9×10^{-12} for this case. We also plot the numerical errors in Fig. 2(b) for other cases with $m = 100$, $R_0 = 2000$, $\beta = 0.8$ and $s = 0$, and $m = 50$, $R_0 = 1000$, $\beta = 0.9$ and $s = 0.01$. Upon comparing the dashed-dotted line in Fig. 2(a) with the solid line, it can be seen that the noise disturbs the numerical solution little from the exact one. The present method is robust against the noise.

For comparison purpose we also apply the collocation Trefftz method (CTM) without a modification, i.e., $R_0 = 1$, to the calculation of this case by fixing $\beta = 1$ and $m = 8$ (when $m > 8$, the CTM is not applicable). As shown by the solid line filled with solid circles in Fig. 2(a), the CTM produces a bad result even no noise is added.

The inaccuracy of CTM may result from its numerical instability. In order to observe this phenomenon we plot the condition number of $\mathbf{A}$ with respect to the number of bases in Fig. 3, which is defined by

$$\text{Cond}(\mathbf{A}) = \|\mathbf{A}\|\|\mathbf{A}^{-1}\|. \tag{29}$$

The norm used for $\mathbf{A}$ is the Frobenius norm. Therefore, we have

$$\frac{1}{n}\text{Cond}(\mathbf{A}) \le \frac{\lambda_{\max}}{\lambda_{\min}} \le \text{Cond}(\mathbf{A}), \tag{30}$$

where λ is the eigenvalue of $\mathbf{A}$. Conventionally, $\lambda_{\max}/\lambda_{\min}$ is used to define the condition number of $\mathbf{A}$. For the present study we use Eq. (29) to define the condition number of $\mathbf{A}$.

It can be seen that the present method can greatly reduce the condition number about twenty orders under $m = 18$ as shown in Fig. 3. No matter which m is selected, the condition number of the presently modified Trefftz method is always much smaller than that of the original Trefftz method. Therefore, when the new method is very accurate, the Trefftz method leads to a bad numerical result as shown in Fig. 2(a).

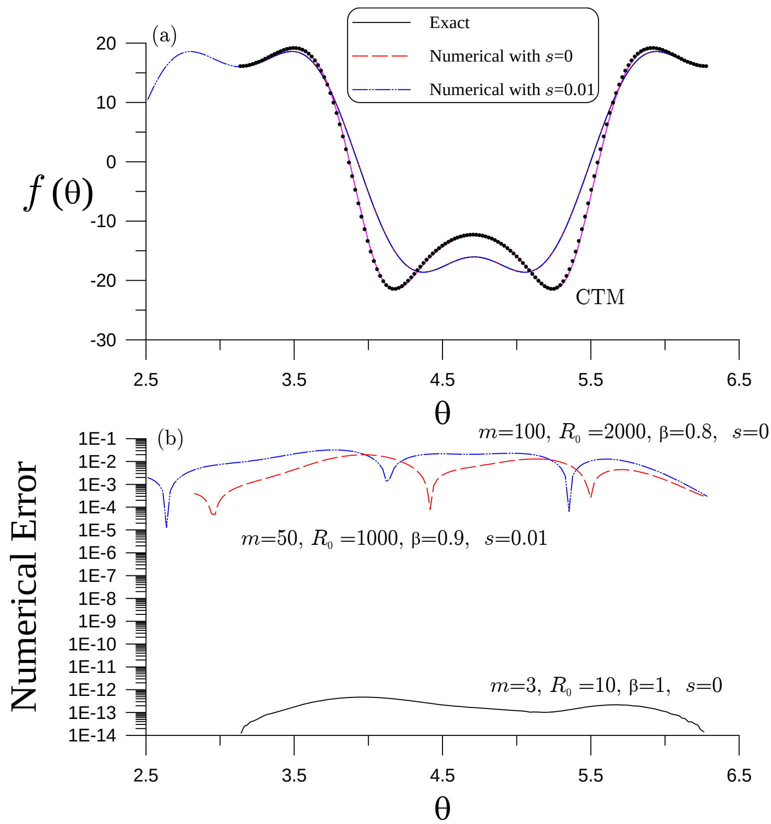

Figure 2: For Example 1 in Section 3: (a) comparing numerical and exact solutions, and (b) showing the numerical errors for different parameters.

3.2 Example 2

In this example a complex amoeba-like irregular shape is shown in the inset of Fig. 4:

$$\rho(\theta) = \exp(\sin\theta)\sin^2(2\theta) + \exp(\cos\theta)\cos^2(2\theta). \tag{31}$$

We consider the following analytical solution:

$$u(x,y) = \cos x \cosh y + \sin x \sinh y, \tag{32}$$

from which the exact boundary data can be derived.

In Fig. 4(a), by comparing the numerical solution with exact solution, and displaying the absolute error for $m = 20$, $R_0 = 10$, $\beta = 1$ and $s = 0$ in Fig. 4(b), we can

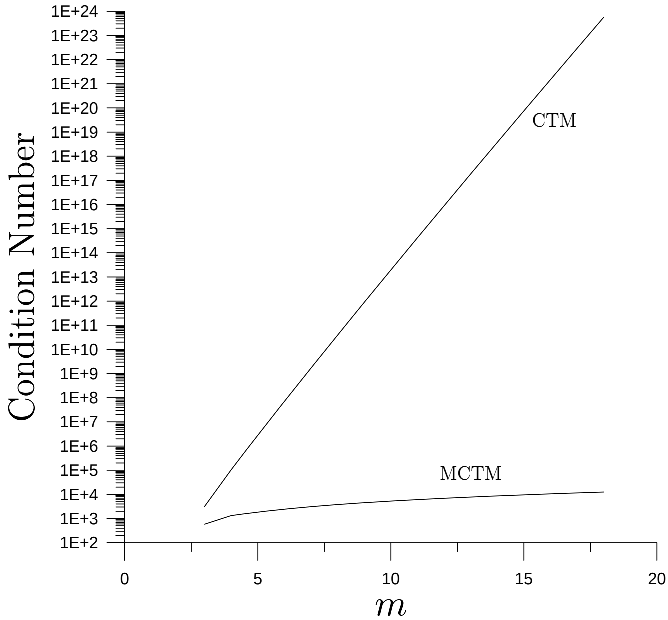

Figure 3: For Example 1 in Section 3 the condition numbers of CTM and MCTM are plotted with respect to m.

see that the present MCTM is a good mathematical tool to recover the unknown boundary data. We also plotted the numerical error in Fig. 4(b) for the noised case with $m = 90$, $R_0 = 40$, $\beta = 0.9$ and $s = 0.005$.

Upon comparing with the numerical results in Hayashi, Ohura and Onishi (2002) and Delvare, Cimetiére and Pons (2000), we can say that the present method is much better than the numerical methods in those papers.

For comparison purpose we also apply the CTM without a modification, i.e., $R_0 = 1$, to the calculation of this case by fixing $\beta = 1$ and $m = 10$ (when $m > 10$, the CTM is not applicable). As shown by the dashed-dotted line in Fig. 4(b), the CTM produces a bad result even no noise is added.

4 Inverse Cauchy problem in doubly-connected domain

In this section we derive a new numerical method to solve the second type inverse Cauchy problem, and numerical examples are given in the next section.

For the Laplace equation in a doubly-connected domain, Liu (2008c) has proposed

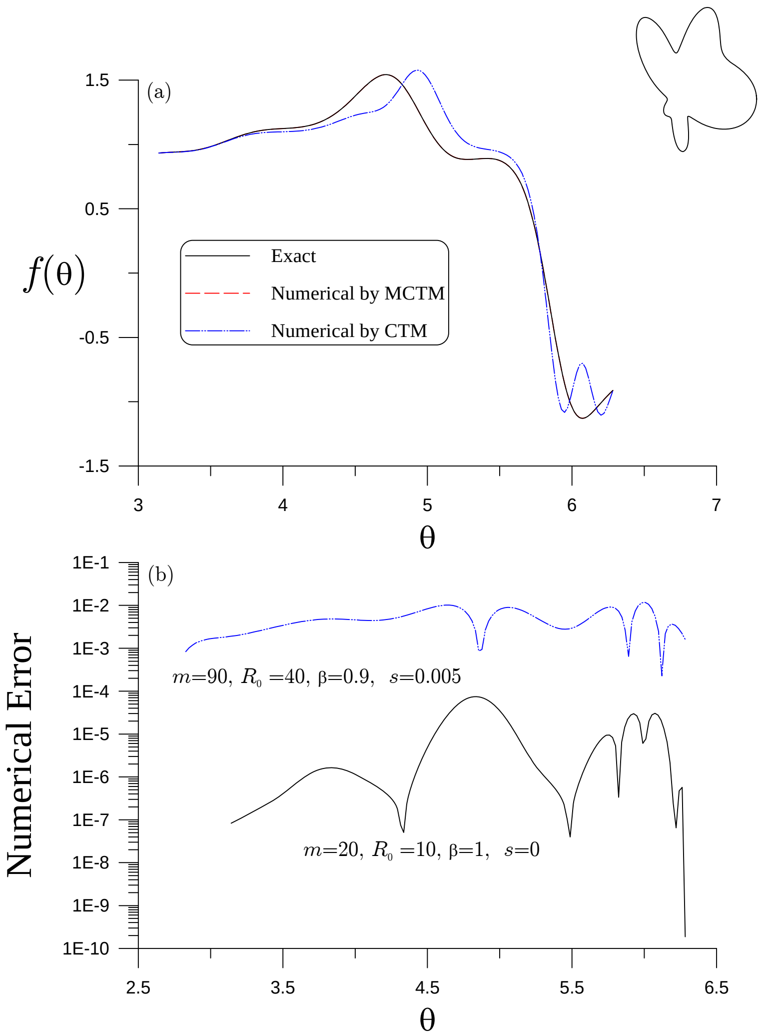

Figure 4: For Example 2 in Section 3: (a) comparing numerical and exact solutions, and (b) showing the numerical errors for different parameters.

a modified Trefftz method by supposing that

$$u(r,\theta) = a_0 + b_0 \ln r + \sum_{k=1}^{m} \left[\left(a_k \left(\frac{r}{R_{01}} \right)^k + b_k \left(\frac{R_{02}}{r} \right)^k \right) \cos k\theta \right.$$
$$\left. + \left(c_k \left(\frac{r}{R_{01}} \right)^k + d_k \left(\frac{R_{02}}{r} \right)^k \right) \sin k\theta \right], \tag{33}$$

where

$$R_{01} \geq \max_{\theta \in [0,2\pi]} \rho(\theta), \tag{34}$$

$$R_{02} \leq \min_{\theta \in [0,2\pi]} \kappa(\theta), \tag{35}$$

and ρ and κ are respectively the outer and inner contours Γ_1 and Γ_2 of the doubly-connected domain as schematically shown in Fig. 1(b).
We require that

$$\begin{aligned}\frac{\partial u(r,\theta)}{\partial r} =& \frac{b_0}{r} + \sum_{k=1}^{m} \left[\left(\frac{ka_k}{r} \left(\frac{r}{R_{01}} \right)^k - \frac{kb_k}{r} \left(\frac{R_{02}}{r} \right)^k \right) \cos k\theta \right. \\ &\left. + \left(\frac{kc_k}{r} \left(\frac{r}{R_{01}} \right)^k - \frac{kd_k}{r} \left(\frac{R_{02}}{r} \right)^k \right) \sin k\theta \right], \end{aligned} \tag{36}$$

$$\begin{aligned}\frac{\partial u(r,\theta)}{\partial \theta} =& \sum_{k=1}^{m} \left[- \left(ka_k \left(\frac{r}{R_{01}} \right)^k + kb_k \left(\frac{R_{02}}{r} \right)^k \right) \sin k\theta \right. \\ &\left. + \left(kc_k \left(\frac{r}{R_{01}} \right)^k + kd_k \left(\frac{R_{02}}{r} \right)^k \right) \cos k\theta \right]. \end{aligned} \tag{37}$$

Inserting these two equations into Eq. (6) we can obtain

$$u_n(\rho,\theta) = \frac{\eta b_0}{\rho} + \sum_{k=1}^{m} [E_k a_k + F_k b_k + G_k c_k + H_k d_k], \tag{38}$$

where

$$E_k = \eta \left(\frac{\rho}{R_{01}} \right)^k \left[\frac{k\cos k\theta}{\rho} + \frac{k\rho' \sin k\theta}{\rho^2} \right], \tag{39}$$

$$F_k = \eta \left(\frac{R_{02}}{\rho} \right)^k \left[\frac{k\rho' \sin k\theta}{\rho^2} - \frac{k\cos k\theta}{\rho} \right], \tag{40}$$

$$G_k = \eta \left(\frac{\rho}{R_{01}} \right)^k \left[\frac{k\sin k\theta}{\rho} - \frac{k\rho' \cos k\theta}{\rho^2} \right], \tag{41}$$

$$H_k = -\eta \left(\frac{R_{02}}{\rho} \right)^k \left[\frac{k\rho' \cos k\theta}{\rho^2} + \frac{k\sin k\theta}{\rho} \right]. \tag{42}$$

By imposing conditions (2) and (3) with $\beta = 2$ on Eqs. (33) and (38) leads to

$$a_0 + b_0 \ln \rho + \sum_{k=1}^{m} [A_k a_k + B_k b_k + C_k c_k + D_k d_k] = h(\theta), \tag{43}$$

$$\frac{\eta b_0}{\rho} + \sum_{k=1}^{m} [E_k a_k + F_k b_k + G_k c_k + H_k d_k] = g(\theta), \tag{44}$$

where

$$A_k = \left(\frac{\rho}{R_{01}}\right)^k \cos k\theta, \tag{45}$$

$$B_k = \left(\frac{R_{02}}{\rho}\right)^k \cos k\theta, \tag{46}$$

$$C_k = \left(\frac{\rho}{R_{01}}\right)^k \sin k\theta, \tag{47}$$

$$D_k = \left(\frac{R_{02}}{\rho}\right)^k \sin k\theta. \tag{48}$$

There are totally $4m+2$ unknown coefficients, and Eqs. (43) and (44) are then imposed at $2m+1$ different collocated points $(\rho(\theta_i), \theta_i)$ in the interval of $0 \le \theta_i \le 2\pi$:

$$a_0 + b_0 \ln \rho(\theta_i) + \sum_{k=1}^{m} [A_k^i a_k + B_k^i b_k + C_k^i c_k + D_k^i d_k] = h(\theta_i), \tag{49}$$

$$\frac{b_0 \eta(\theta_i)}{\rho(\theta_i)} + \sum_{k=1}^{m} [E_k^i a_k + F_k^i b_k + G_k^i c_k + H_k^i d_k] = g(\theta_i), \tag{50}$$

where for simple notations we use $A_k^i = A_k(\theta_i)$, etc.
When the index i in Eqs. (49) and (50) runs from 1 to $2m+1$ we obtain a linear

equations system with dimensions $n = 4m+2$:

$$\begin{bmatrix} 1 & \ln\rho(\theta_1) & A_1^1 & B_1^1 & C_1^1 & D_1^1 & \cdots & A_m^1 & B_m^1 & C_m^1 & D_m^1 \\ 0 & \eta(\theta_1)/\rho(\theta_1) & E_1^1 & F_1^1 & G_1^1 & H_1^1 & \cdots & E_m^1 & F_m^1 & G_m^1 & H_m^1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \ln\rho(\theta_{2m+1}) & A_1^{2m+1} & B_1^{2m+1} & C_1^{2m+1} & D_1^{2m+1} & \cdots & A_m^{2m+1} & B_m^{2m+1} & C_m^{2m+1} & D_m^{2m+1} \\ 0 & \eta(\theta_{2m+1})/\rho(\theta_{2m+1}) & E_1^{2m+1} & F_1^{2m+1} & G_1^{2m+1} & H_1^{2m+1} & \cdots & E_m^{2m+1} & F_m^{2m+1} & G_m^{2m+1} & H_m^{2m+1} \end{bmatrix} \begin{bmatrix} a_0 \\ b_0 \\ a_1 \\ b_1 \\ c_1 \\ d_1 \\ \vdots \\ a_m \\ b_m \\ c_m \\ d_m \end{bmatrix} = \begin{bmatrix} h(\theta_1) \\ g(\theta_1) \\ \vdots \\ h(\theta_{2m+1}) \\ g(\theta_{2m+1}) \end{bmatrix}. \tag{51}$$

Inserting the calculated $\mathbf{e} = [a_0, b_0, a_1, b_1, c_1, d_1, \cdots, a_m, b_m, c_m, d_m]^{\mathrm{T}}$ into Eq. (33) we thus have a semi-analytical solution of $u(r,\theta)$:

$$\begin{aligned} u(r,\theta) = & e_1 + e_2 \ln r + \sum_{k=1}^{m} \left[\left(e_{4k-1} \left(\frac{r}{R_{01}} \right)^k + e_{4k} \left(\frac{R_{02}}{r} \right)^k \right) \cos k\theta \right. \\ & \left. + \left(e_{4k+1} \left(\frac{r}{R_{01}} \right)^k + e_{4k+2} \left(\frac{R_{02}}{r} \right)^k \right) \sin k\theta \right], \end{aligned} \tag{52}$$

where $(e_1, \ldots, e_n)$ are the components of $\mathbf{e}$. Therefore, the boundary data on the inner contour is calculated by

$$\begin{aligned} f(\theta) = & e_1 + e_2 \ln \chi(\theta) + \sum_{k=1}^{m} \left[\left(e_{4k-1} \left(\frac{\chi(\theta)}{R_{01}} \right)^k + e_{4k} \left(\frac{R_{02}}{\chi(\theta)} \right)^k \right) \cos k\theta \right. \\ & \left. + \left(e_{4k+1} \left(\frac{\chi(\theta)}{R_{01}} \right)^k + e_{4k+2} \left(\frac{R_{02}}{\chi(\theta)} \right)^k \right) \sin k\theta \right]. \end{aligned} \tag{53}$$

5 Numerical examples for the second type inverse Cauchy problem

5.1 Example 1

We consider a kite-shape outer boundary with its parameterization given by

$$\rho(\theta) = 2\sqrt{(0.6\cos\theta + 0.3\cos 2\theta - 0.2)^2 + (0.6\sin\theta)^2}, \tag{54}$$

while the inner boundary is an apple shape described by

$$\kappa(\theta) = \frac{0.5 + 0.2\cos\theta + 0.1\sin 2\theta}{1.5 + 0.7\cos\theta}. \tag{55}$$

The above two curves are shown in the inset of Fig. 5.

In order to test our method we consider an exact solution as that given by Eq. (25), which however led to very complicated boundary conditions overspecified on the outer boundary.

Under the parameters $R_{01} = 20$, $R_{02} = 0.1$ and $m = 14$, we solve this problem by the method in Section 4, whose numerical result along the inner boundary is shown in Fig. 5(a) by the dashed line. It is almost coincident with the exact one as shown by the solid line in Fig. 5(a), and the numerical error as shown in Fig. 5(b) is smaller than 10^{-6}. A highly accurate result is obtained as compared with exact solution.

When we apply the Trefftz method to this problem by using $R_{01} = 1$, $R_{02} = 1$ and $m = 14$, we find that the solution is unstable as shown in Fig. 5(a) by the dashed-dotted line.

All the computations in this paper are carried out in a PC-586 with pentium-100. For this example, the computations both by the Trefftz method and our new method spent the CPU time smaller than one second. Basically, the most time is spent in the solution of Eq. (51). Under the same stopping criterion 10^{-15}, through 387 iterations the solution of $\mathbf{e}$ by Eq. (51) is obtained for the MCTM, but the CTM requires 2088 iterations. Due to this, the computational time of MCTM is less than that charged by the CTM.

5.2 Example 2

We consider

$$u(r,\theta) = e^x \cos y = e^{r\cos\theta}\cos(r\sin\theta), \tag{56}$$

under the boundaries:

$$\rho(\theta) = \frac{5 + 2\cos\theta + \sin 2\theta}{1.5 + 0.7\cos\theta}, \tag{57}$$

$$\kappa(\theta) = 2\sqrt{(0.6\cos\theta + 0.3\cos 2\theta - 0.2)^2 + (0.6\sin\theta)^2}. \tag{58}$$

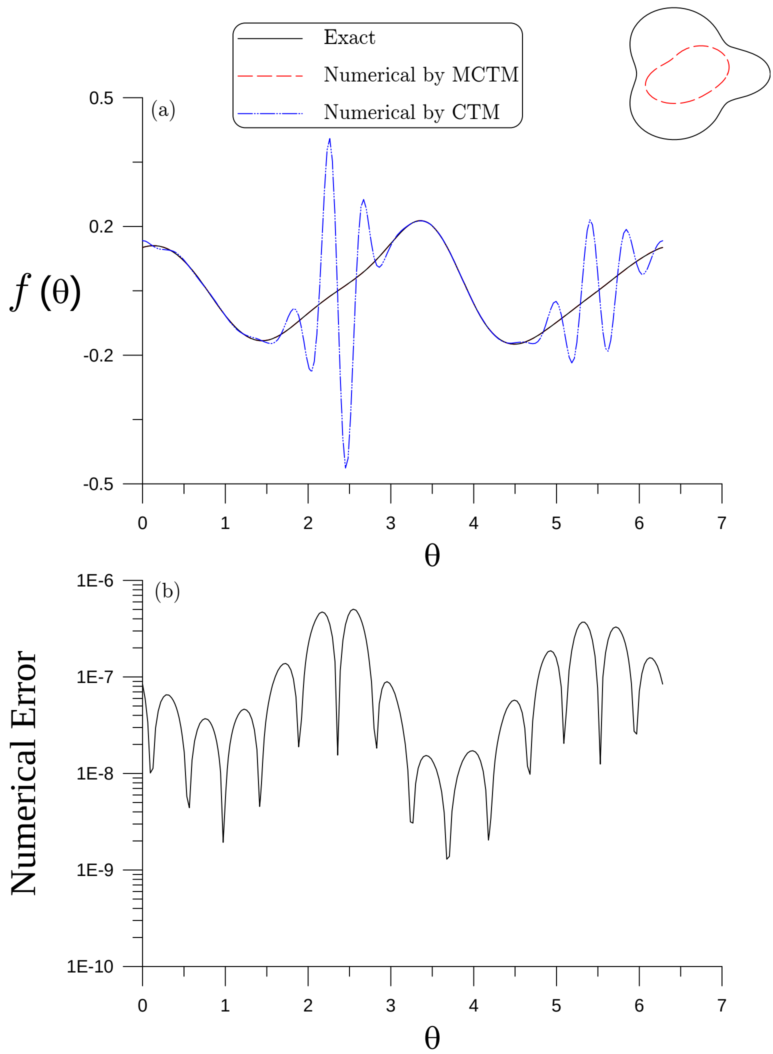

Figure 5: For Example 1 in Section 5: (a) comparing numerical and exact solutions, and (b) showing the numerical error for MCTM.

The condition number heavily depends on the values of R_{01} and R_{02}. For this problem $\rho_{\max}$ is about 4.6. In Fig. 6 we plot the variation of condition numbers with respect to R_{01} by fixing $R_{02} = 0.1$ and $m = 20$. It can be seen that when $R_{01} < \rho_{\max}$, the condition number increases rapidly, and for the case of Trefftz method, i.e. $R_{01} = 1$ used in the CTM, the condition number reaches its maximum about in the order of 10^{14}. Therefore, we can see that the Trefftz method is essentially unstable.

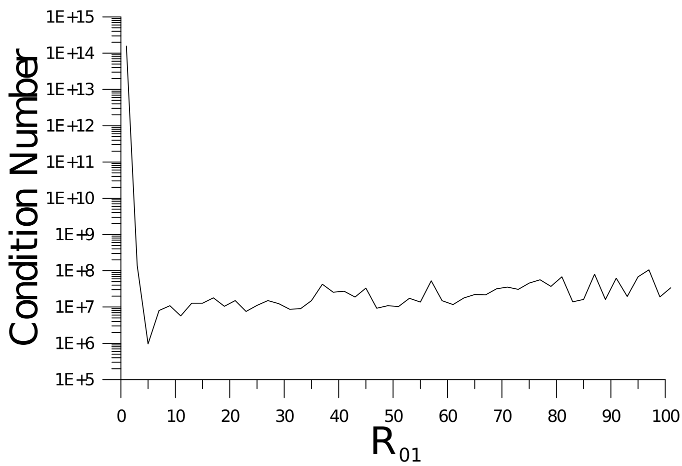

Figure 6: For Example 2 in Section 5 the condition number of MCTM is plotted with respect to R_{01}.

In Fig. 7(a) we compare the numerical solutions of CTM and MCTM with exact solution, where $R_{01} = 1$, $R_{02} = 1$ and $m = 10$ for CTM, and $R_{01} = 6$, $R_{02} = 0.1$ and $m = 30$ for MCTM. We find that the solution by CTM is rather unstable by using other m. The numerical errors are shown in Fig. 7(b), from which it is obvious that MCTM is much accurate than CTM. Even under a large noise with $s = 0.01$, the numerical error of MCTM as shown in Fig. 7(b) by the dashed line is still smaller than 0.02.

6 Conclusions

We have employed a new idea to treat the inverse Cauchy problems in arbitrary plane domains by a modified collocation Trefftz method. The new method can provide a semi-analytical solution in terms of the modified Trefftz basis functions, which renders a rather compendious numerical implementation to solve the inverse Cauchy problems without needing of any iteration and any regularization. The new methods were found accurate, effective and stable. These points are very diiferent from other numerical methods for the inverse Cauchy problems. The author asserted that a good numerical method is not only applicable to the direct problems but also to the inverse problems. That the conventional method is not applicable to the inverse problems is due to its inherent ill-conditioned property. If the ill-conditioned property can be gotten rid of, it is not only applicable to the direct

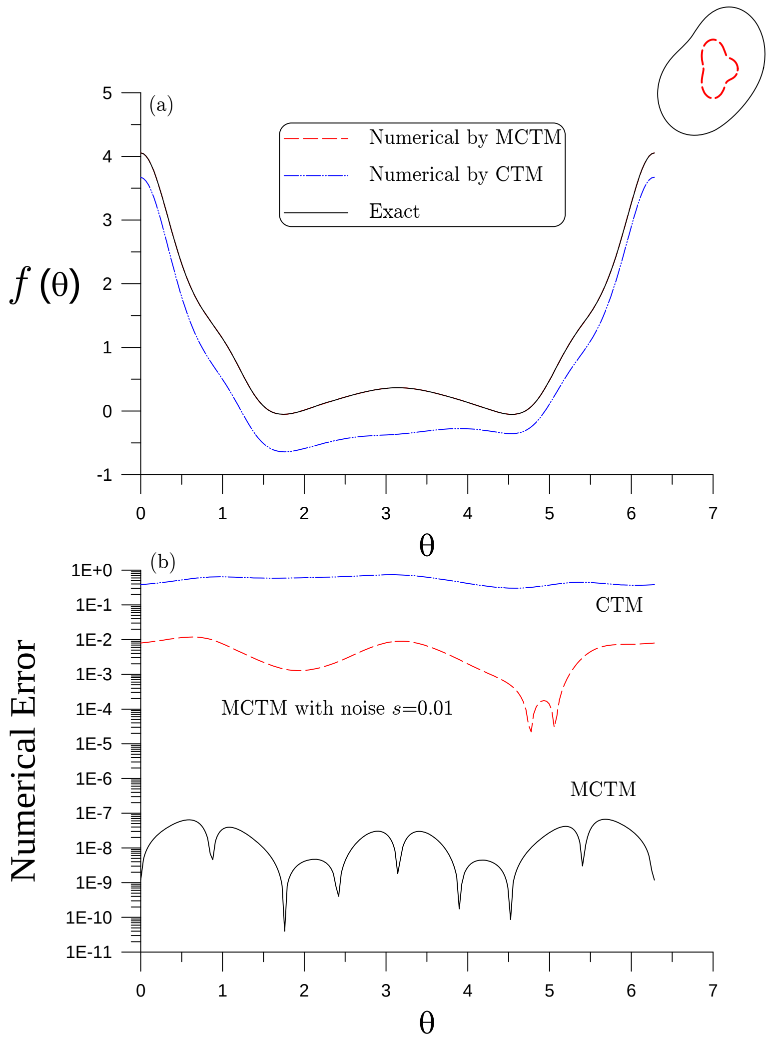

Figure 7: For Example 2 in Section 5: (a) comparing numerical and exact solutions, and (b) showing the numerical errors for different parameters.

problems, but also applicable to the inverse problems. Without needing of any regularization technique, the present MCTM is very time saving to solve the inverse Cauchy problems.

Acknowledgement: Taiwan's National Science Council projects NSC-96-2221-E-019-027-MY3 and NSC-97-2221-E-019-009-MY3 granted to the author are highly

appreciated.

7 References

Akduman, I.; Kress, R. (2002): Electrostatic imaging via conformal mapping. *Inverse Problems*, vol. 18, pp. 1659-1672.

Andrieux, S.; Baranger, T. N.; Ben Abda, A. (2006): Solving Cauchy problems by minimizing an energy-like functional. *Inverse Problems*, vol. 22, pp. 115-133.

Aparicio, N. D.; Pidcock, M. K. (1996): The boundary inverse problem for the Laplace equation in two dimensions. *Inverse Problems*, vol. 12, pp. 565-577.

Ben Belgacem, F.; El Fekih, H. (2005): On Cauchy's problem. I. a variational Steklov-Poincaré theory. *Inverse Problems*, vol. 21, pp. 1915-1936.

Berntsson, F.; Eldén, L. (2001): Numerical solution of a Cauchy problem for the Laplace equation. *Inverse Problems*, vol. 17, pp. 839-854.

Bourgeois, L. (2005): A mixed formulation of quasi-reversibility to solve the Cauchy problem for Laplace's equation. *Inverse Problems*, vol. 21, pp. 1087-1104.

Bourgeois, L. (2006): Convergence rates for the quasi-reversibility method to solve the Cauchy problem for Laplace's equation. *Inverse Problems*, vol. 22, pp. 413-430.

Chang, J. R.; Yeih, W.; Shieh, M. H. (2001): On the modified Tikhonov's regularization method for the Cauchy problem of the Laplace equation. *J. Mar. Sci. Tech.*, vol. 9, pp. 113-121.

Chapko, R.; Kress, R. (2005): A hybrid method for inverse boundary value problems in potential theory. *J. Inv. Ill-Posed Problems*, vol. 13, pp. 27-40.

Chen, Y. W.; Liu, C.-S.; Chang, J. R. (2008): Applications of the modified Trefftz method for the Laplace equation. *Engng. Anal. Bound. Elem.*, vol. 33, pp. 137-146.

Delvare, F.; Cimetiére, A.; Pons, F. (2000): Two iterative boundary element methods for inverse Cauchy problems. In Brebbia, C. A. and Power, H., Eds., Boundary Elements, vol. XXII, pp. 211-20, WIT Press.

Hayashi, K.; Ohura, Y.; Onishi, K. (2002): Direct method of solution for general boundary value problem of the Laplace equation. *Engng. Anal. Bound. Elem.*, vol. 26, pp. 763-771.

Inglese, G. (1997): An inverse problem in corrosion detection. *Inverse Problems*, vol. 13, pp. 1425-1438.

Jin, B.; Zheng, Y. (2006): A meshless method for some inverse problems associated with the Helmholtz equation. *Comp. Meth. Appl. Mech. Engng.*, vol. 195, pp.

2270-2288.

Johnston, P. (2001): Computational Inverse Problems in Electrocardiography, Advances in Computational Bioengineering, Vol. 5, Johnston, P. Ed., WIT Press.

Kaup, P. G.; Santosa, F.; Vogelius, M. (1996): Method for imaging corrosion damage in thin plates from electrostatic data. *Inverse Problems*, vol. 12, pp. 279-293.

Kress, R. (2004): Inverse Dirichlet problem and conformal mapping. *Math. Comp. Simul.*, vol. 66, pp. 255-265.

Lesnic, D.; Elliott, L.; Ingham, D. B. (1997): An iterative boundary element method for solving numerically the Cauchy problem for the Laplace equation. *Engng. Anal. Bound. Elem.*, vol. 20, pp. 123-133.

Ling, L.; Takeuchi, T. (2008): Boundary control for inverse Cauchy problems of the Laplace equations. *CMES: Computer Modeling in Engineering & Sciences*, vol. 29, pp. 45-54.

Liu, C.-S. (2007a): A modified Trefftz method for two-dimensional Laplace equation considering the domain's characteristic length. *CMES: Computer Modeling in Engineering & Sciences*, vol. 21, pp. 53-66.

Liu, C.-S. (2007b): A highly accurate solver for the mixed-boundary potential problem and singular problem in arbitrary plane domain. *CMES: Computer Modeling in Engineering & Sciences*, vol. 20, pp. 111-122.

Liu, C.-S. (2007c): An effectively modified direct Trefftz method for 2D potential problems considering the domain's characteristic length. *Engng. Anal. Bound. Elem.*, vol. 31, pp. 983-993.

Liu, C.-S. (2008a): A highly accurate MCTM for direct and inverse problems of biharmonic equation in arbitrary plane domains. *CMES: Computer Modeling in Engineering & Sciences*, vol. 30, pp. 65-75.

Liu, C.-S. (2008b): A modified collocation Trefftz method for the inverse Cauchy problem of Laplace equation. *Engng. Anal. Bound. Elem.*, vol. 32, pp. 778-785.

Liu, C.-S. (2008c): A highly accurate collocation Trefftz method for solving the Laplace equation in the doubly-connected domains. *Numer. Meth. Partial Diff. Eq.*, vol. 24, pp. 179-192.

Liu, C.-S.; Atluri, S. N. (2008a): A fictitious time integration method (FTIM) for solving mixed complementarity problems with applications to non-linear optimization. *CMES: Computer Modeling in Engineering & Sciences*, in press.

Liu, C.-S.; Atluri, S. N. (2008b): A novel time integration method for solving a large system of non-linear algebraic equations. *CMES: Computer Modeling in Engineering & Sciences*, vol. 31, pp. 71-83.

Marin, L.; Lesnic, D. (2005): The method of fundamental solutions for inverse boundary value problems associated with the two-dimensional biharmonic equation. *Math. Comp. Model.*, vol. 42, pp. 261-278.

Mera, N. S.; Elliott, L.; Ingham, D. B.; Lesnic, D. (2000): An iterative boundary element method for the solution of a Cauchy steady state heat conduction problem. *CMES: Computer Modeling in Engineering* & *Sciences*, vol. 1, pp. 101-106.

Mera, N. S.; Elliott, L.; Ingham, D. B.; Lesnic, D. (2000): The boundary element solution of the Cauchy steady heat conduction problem in an anisotropic medium. *Int. J. Num. Meth. Engng.*, vol. 49, pp. 481-499.

Mera, N. S.; Elliott, L.; Ingham, D. B. (2003): On the use of genetic algorithms for solving ill-posed problems. *Inverse Prob. Engng.*, vol. 11, pp. 105-121.

Slodička, M.; Van Keer, R. (2004): A numerical approach for the determination of a missing boundary data in elliptic problems. *Appl. Math. Comp.*, vol. 147, pp. 569-580.

Appendix

In this appendix we derive Eq. (6). The plane curve of Γ has an arc length s as a parameter. The unit normal vector $\mathbf{n}$ along Γ is given by

$$\mathbf{n} = \left(\frac{dy}{ds}, -\frac{dx}{ds}\right). \tag{A1}$$

The gradient of $u(r,\theta)$ is given by

$$\nabla u = \left(\frac{\partial u}{\partial r}\cos\theta - \frac{\partial u}{\partial \theta}\frac{\sin\theta}{r}, \frac{\partial u}{\partial r}\sin\theta + \frac{\partial u}{\partial \theta}\frac{\cos\theta}{r}\right). \tag{A2}$$

By using $x = r\cos\theta$ and $y = r\sin\theta$, one has

$$dx = (r'(\theta)\cos\theta - r(\theta)\sin\theta)d\theta, \;\; dy = (r'(\theta)\sin\theta + r(\theta)\cos\theta)d\theta, \tag{A3}$$

when $(x,y) \in \Gamma$. In the above, $r = r(\theta)$ is the curve of Γ, and r' denotes the derivative with respect to θ.

Therefore, by $ds = \sqrt{r^2 + r'^2}d\theta$ and Eq. (A1) we have

$$\mathbf{n} = \frac{r}{\sqrt{r^2+r'^2}}\left(\cos\theta + \frac{r'}{r}\sin\theta, \sin\theta - \frac{r'}{r}\cos\theta\right). \tag{A4}$$

Inserting Eqs. (A4) and (A2) into

$$u_n = \mathbf{n}\cdot\nabla u, \tag{A5}$$

and through some manipulations we can derive Eq. (6).