 CMES, vol.33, no.1, pp.1-15, 2008

Large Rotations and Nodal Moments in Corotational Elements

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Abstract: This paper deals with the parameterisation of large rotations in corotational beam and shell elements. Several alternatives, presented in previous articles, are summarised, completed and compared to each other. The implementation of applied external moments and eccentric forces, consistent with the different parameterisations, is also considered.

Keyword: Large rotations, rotational vector, Euler parameters, beam elements, shell elements.

1 Introduction

One difficulty in the implementation of non-linear beam and shell elements is the treatment of large finite rotations. Since the initial work of Argyris (1982), several computational procedures have been proposed by different authors, see e.g. the works of Spring (1986), Iura and Atluri (1988) and the works of Ibrahimbegovic (1997) and Liu (2006) for a review.

Concerning corotational elements, formulations based on non-additive spatial rotations and additive rotational vectors have been proposed by Pacoste (1998) for shell elements and Battini and Pacoste (2002) for beam elements. In particular, it has been shown that the second parameterisation is obtained from the first one through a change of variables. Recently, the same approach has been used in Battini (2007) to obtain a parameterisation based on three of the four Euler parameters (quaternion).

The first purpose of this paper is to summarise these different approaches (in Sections 2 to 5) and to present their respective advantages and drawbacks, see Section 6. The second purpose concerns the definition of the internal moments at the nodes which is different for the different parameterisations. As a consequence, see Sec-

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tion 7, applied moments and eccentric forces require a special treatment. Finally, two numerical applications are presented in Section 8.

2 Rotational parameters

The finite rotation at a node of a beam or shell element is defined by 3×3 orthogonal matrix $\mathbf{R}$ involving nine components. However, due to the orthonormality, it can be described in terms of three independent parameters.

The first alternative studied in this paper is based on the "rotational vector" defined by

$$\Psi = \begin{bmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{bmatrix} = \mathbf{u}\,\psi \qquad \psi = \sqrt{\Psi_1^2 + \Psi_2^2 + \Psi_3^2} \tag{1}$$

The geometrical significance of this definition is that any finite rotation can be represented by a unique rotation with an angle ψ about an axis defined by the unit vector $\mathbf{u}$. In terms of Ψ, the matrix $\mathbf{R}$ is given by

$$\mathbf{R} = \mathbf{I} + \frac{\sin\psi}{\psi}\tilde{\Psi} + \frac{1}{2}\left[\frac{\sin(\psi/2)}{\psi/2}\right]^2 \tilde{\Psi}^2 \tag{2}$$

or

$$\mathbf{R} = \mathbf{I} + \tilde{\Psi} + \frac{1}{2}\tilde{\Psi}^2 + \cdots = \exp(\tilde{\Psi}) \tag{3}$$

with

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \tilde{\Psi} = \begin{bmatrix} 0 & -\Psi_3 & \Psi_2 \\ \Psi_3 & 0 & -\Psi_1 \\ -\Psi_2 & \Psi_1 & 0 \end{bmatrix} \tag{4}$$

It can be noted that the relation between the orthogonal matrix $\mathbf{R}$ and the rotational vector Ψ is a bijection only for angles ψ less than 2π.

The second alternative studied in this paper is based on the Euler parameters (quaternion) defined by

$$\mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} \tag{5}$$

In terms of $\mathbf{q}$, the rotational matrix $\mathbf{R}$ is given by

$$\mathbf{R} = 2\begin{bmatrix} q_0^2+q_1^2-\frac{1}{2} & q_1q_2-q_3q_0 & q_1q_3+q_2q_0 \\ q_2q_1+q_3q_0 & q_0^2+q_2^2-\frac{1}{2} & q_2q_3-q_1q_0 \\ q_3q_1-q_2q_0 & q_3q_2+q_1q_0 & q_0^2+q_3^2-\frac{1}{2} \end{bmatrix} \tag{6}$$

with

$$q_0 = \sqrt{1-q_1^2-q_2^2-q_3^2} \tag{7}$$

The proposed parameterisation based on Euler parameters requires only three parameters (q_1,q_2,q_3) instead of four (q_1,q_2,q_3,q_0) as usually found in the literature. This is possible only for rotations characterised by an angle ψ less than π. For such cases, the indeterminacy associated to the fact that both q_0,q_1,q_2,q_3 and $-q_0,-q_1,-q_2,-q_3$ represent the same orthogonal matrix $\mathbf{R}$ can be be avoided by replacing the equation

$$q_0 = \pm\sqrt{1-q_1^2-q_2^2-q_3^2} \tag{8}$$

by the expression (7). Then, for angles ψ less than π, the relation between the matrix $\mathbf{R}$ and the vector $\mathbf{q}$ is a bijection and $\mathbf{q}$ is related to the rotational vector by

$$\mathbf{q} = \sin(\psi/2)\,\mathbf{u} \tag{9}$$

3 Non-additive formulation

Corotational beam and shell formulations are derived using the principal of virtual work for which infinitesimal displacements and rotations are required. The infinitesimal variation $\delta\mathbf{R}$ of the orthogonal matrix $\mathbf{R}$ can be obtained in the following way, see Fig. 1. The rotation at a node is given by the orthogonal cartesian frame $(\mathbf{t}_1,\mathbf{t}_2,\mathbf{t}_3)$ and the rotational matrix $\mathbf{R}$ such as

$$\mathbf{t}_i = \mathbf{R}\,\mathbf{e}_i \qquad i=1,2,3 \tag{10}$$

where $(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3)$ is the global cartesian system for the structure. An incremental rotation carries the moving frame $(\mathbf{t}_1,\mathbf{t}_2,\mathbf{t}_3)$ into a new position $(\mathbf{t}'_1,\mathbf{t}'_2,\mathbf{t}'_3)$, through a spatial rotation $\mathbf{R}_s = \exp(\tilde{\theta})$ applied to $(\mathbf{t}_1,\mathbf{t}_2,\mathbf{t}_3)$, i.e.

$$\mathbf{t}'_i = \mathbf{R}_s\,\mathbf{t}_i = \mathbf{R}_s\,\mathbf{R}\,\mathbf{e}_i = \mathbf{R}_n\,\mathbf{e}_i = \exp(\tilde{\theta})\,\mathbf{R}\,\mathbf{e}_i \tag{11}$$

If the rotation $\mathbf{R}_s$ is infinitesimal, Eq. (11) can be rewritten using Eq. (3) as

$$\mathbf{R}_n = \mathbf{R}+\delta\mathbf{R} = \exp(\delta\tilde{\theta})\,\mathbf{R} = (\mathbf{I}+\delta\tilde{\theta})\,\mathbf{R} \tag{12}$$

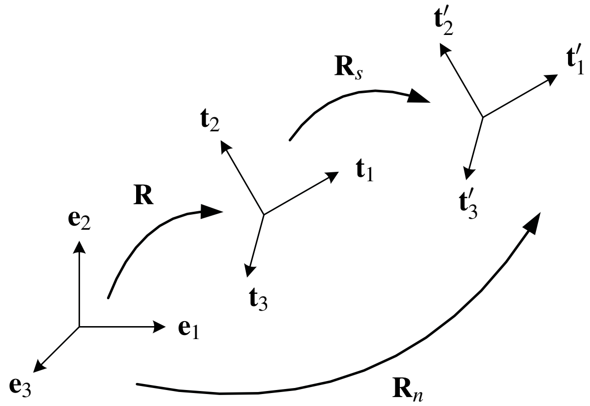


Figure 1: Successive finite rotations

which gives

$$\delta\mathbf{R} = \delta\tilde{\theta}\,\mathbf{R} \tag{13}$$

The derivation of non-linear corotational beam and shell elements is done using Eq. (13), which implies that obtained internal force vectors $\mathbf{f}_\theta$ and tangent stiffness matrices $\mathbf{K}_\theta$ are consistent with infinitesimal spatial rotations $\delta\theta$ as defined in Eq. (11). For a 3D beam element with two nodes an six degrees of freedom at each node (i.e. without warping degree of freedom), it gives

$$\mathbf{f}_\theta = \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{m}_{\theta 1} \\ \mathbf{f}_2 \\ \mathbf{m}_{\theta 2} \end{bmatrix} \qquad \delta\mathbf{p}_\theta = \begin{bmatrix} \delta\mathbf{u}_1 \\ \delta\theta_1 \\ \delta\mathbf{u}_2 \\ \delta\theta_2 \end{bmatrix} \qquad \delta\mathbf{f}_\theta = \mathbf{K}_\theta\,\delta\mathbf{p}_\theta \tag{14}$$

where $\mathbf{u}_1$ and $\mathbf{u}_2$ are the nodal displacements. The internal virtual work produced by $\mathbf{f}_\theta$ is expressed by

$$W_i = \delta\mathbf{p}_\theta^{\mathrm{T}}\,\mathbf{f}_\theta \tag{15}$$

Using Eq. (11), the updating after each Newton-Raphson iteration of the rotational matrix at each node is performed according to

$$\mathbf{R}^{\mathrm{n}} = \exp(\mathrm{d}\tilde{\theta})\,\mathbf{R}^{\mathrm{o}} \tag{16}$$

The superscripts o and n denote the “old" and “new" quantities whereas $\mathrm{d}\theta$ are the iterative spatial rotations obtained through the Newton-Raphson iteration.

4 Totally additive formulations

In order to avoid the update procedure in Eq. (16), parameterisations based on the rotational vector and the Euler parameters have been presented in Pacoste (1998) and Battini (2007). Then, the rotations become additive and can be updated at each iteration using

$$\Psi^{\mathrm{n}} = \Psi^{\mathrm{o}} + \mathrm{d}\Psi \qquad \mathbf{q}^{\mathrm{n}} = \mathbf{q}^{\mathrm{o}} + \mathrm{d}\mathbf{q} \tag{17}$$

The internal force vectors $\mathbf{f}_{\Psi}$, $\mathbf{f}_{\mathrm{q}}$ and tangent stiffness matrices $\mathbf{K}_{\Psi}$, $\mathbf{K}_{\mathrm{q}}$ consistent with Ψ, $\mathbf{q}$ are obtained from $\mathbf{f}_{\theta}$, $\mathbf{K}_{\theta}$ using a change of variables from $\delta\theta$ to $\delta\Psi$ and $\delta\theta$ to $\delta\mathbf{q}$ defined by

$$\delta\theta = \mathbf{T}_{\Psi}(\Psi)\,\delta\Psi \qquad \delta\theta = \mathbf{T}_{\mathrm{q}}(\mathbf{q})\,\delta\mathbf{q} \tag{18}$$

The virtual displacements vectors and internal forces vectors are defined by

$$\delta\mathbf{p}_{\Psi} = \begin{bmatrix} \delta\mathbf{u}_1 \\ \delta\Psi_1 \\ \delta\mathbf{u}_2 \\ \delta\Psi_2 \end{bmatrix} \qquad \delta\mathbf{p}_{\mathrm{q}} = \begin{bmatrix} \delta\mathbf{u}_1 \\ \delta\mathbf{q}_1 \\ \delta\mathbf{u}_2 \\ \delta\mathbf{q}_2 \end{bmatrix} \tag{19}$$

and

$$\mathbf{f}_{\Psi} = \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{m}_{\Psi 1} \\ \mathbf{f}_2 \\ \mathbf{m}_{\Psi 2} \end{bmatrix} \qquad \mathbf{f}_{\mathrm{q}} = \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{m}_{\mathrm{q}1} \\ \mathbf{f}_2 \\ \mathbf{m}_{\mathrm{q}2} \end{bmatrix} \tag{20}$$

By equating the virtual work in the two systems, i.e.

$$\delta\mathbf{p}_{\Psi}^{\mathrm{T}}\,\mathbf{f}_{\Psi} = \delta\mathbf{p}_{\theta}^{\mathrm{T}}\,\mathbf{f}_{\theta} \qquad \delta\mathbf{p}_{\mathrm{q}}^{\mathrm{T}}\,\mathbf{f}_{\mathrm{q}} = \delta\mathbf{p}_{\theta}^{\mathrm{T}}\,\mathbf{f}_{\theta} \tag{21}$$

and using Eqs. (18), it is obtained

$$\mathbf{f}_{\Psi} = \mathbf{B}_{\Psi}^{\mathrm{T}}\,\mathbf{f}_{\theta} \qquad \mathbf{f}_{\mathrm{q}} = \mathbf{B}_{\mathrm{q}}^{\mathrm{T}}\,\mathbf{f}_{\theta} \tag{22}$$

with

$$\mathbf{B}_{\Psi} = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{T}_{\Psi}(\Psi_1) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{T}_{\Psi}(\Psi_2) \end{bmatrix} \tag{23}$$

and

$$\mathbf{B}_{\mathrm{q}} = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{T}_{\mathrm{q}}(\mathbf{q}_1) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{T}_{\mathrm{q}}(\mathbf{q}_2) \end{bmatrix} \tag{24}$$

The expressions of the tangent stiffness matrices $\mathbf{K}_{\Psi}$ and $\mathbf{K}_{\mathrm{q}}$ are obtained by differentiation of Eqs. (22) and by introducing Eq. (14). It gives

$$\mathbf{K}_{\Psi} = \mathbf{B}_{\Psi}^{\mathrm{T}} \mathbf{K}_{\theta} \mathbf{B}_{\Psi} + \mathbf{K}_{h\Psi} \qquad \mathbf{K}_{\mathrm{q}} = \mathbf{B}_{\mathrm{q}}^{\mathrm{T}} \mathbf{K}_{\theta} \mathbf{B}_{\mathrm{q}} + \mathbf{K}_{h\mathrm{q}} \tag{25}$$

with

$$\mathbf{K}_{h\Psi} = \left.\frac{\partial(\mathbf{B}_{\Psi}^{\mathrm{T}} \mathbf{f}_{\theta})}{\partial \mathbf{p}_{\Psi}}\right|_{\mathbf{f}_{\theta}} \qquad \mathbf{K}_{h\mathrm{q}} = \left.\frac{\partial(\mathbf{B}_{\mathrm{q}}^{\mathrm{T}} \mathbf{f}_{\theta})}{\partial \mathbf{p}_{\mathrm{q}}}\right|_{\mathbf{f}_{\theta}} \tag{26}$$

The derivations and expressions of Matrices $\mathbf{T}_{\Psi}$, $\mathbf{T}_{\mathrm{q}}$, $\mathbf{K}_{h\Psi}$ and $\mathbf{K}_{h\mathrm{q}}$ can be found in Pacoste (1998) and Battini (2007).

5 Incrementally additive formulations

The operator $\mathbf{T}_{\Psi}(\Psi)$ is singular for $\psi = 2\pi$ whereas the operator $\mathbf{T}_{\mathrm{q}}(\mathbf{q})$, is not defined for $q_0 = 0$, which corresponds to a rotation $\psi = \pi$. Physically, it is due to the fact that the relations between the matrix $\mathbf{R}$ and the vectors Ψ and $\mathbf{q}$ cease to be bijections for these two angles. Consequently, the changes of variables defined in the previous section are limited to angles ψ less than 2π for the parameterisation with the rotational vector and to angles ψ less than π for the parameterisation with Euler parameters.

This implies that the procedure described in the previous section cannot be used for highly flexible structures or mechanisms which undergo very large rotations. One solution, proposed by Ibrahimbegovic (1997) is to introduce incremental rotational vectors and incremental Euler parameters. The idea is to use the change of variables presented in the previous section but to define the rotational vector and Euler parameters only within the increments. The update procedure is then performed in the following way:

$\mathbf{R}^{o}$ is the rotational matrix at the end of last step. At the beginning of the new step, the incremental rotational vector and incremental Euler parameters are zero. At each iteration i, they are updated using

$$\Psi^{i+1} = \Psi^{i} + \mathrm{d}\Psi \qquad \Psi^{0} = \mathbf{0} \tag{27}$$

$$\mathbf{q}^{i+1} = \mathbf{q}^{i} + \mathrm{d}\mathbf{q} \qquad \mathbf{q}^{0} = \mathbf{0} \tag{28}$$

and the rotational matrix is updated using

$$\mathbf{R}^{i+1} = \exp(\Psi^{i+1})\,\mathbf{R}^{o} \qquad \mathbf{R}^{i+1} = \mathbf{R}(\mathbf{q}^{i+1})\,\mathbf{R}^{o} \tag{29}$$

Hence, additive updates still apply, but only within each increment and the amplitudes of the rotations are not limited. Regarding the required changes of variables, the only difference in Eqs. (22) and (25) is that Ψ and $\mathbf{q}$ are no longer total quantities but incremental ones.

6 Comparison of the different formulations

The different parameterisations presented in Sections 3 to 5 have been implemented in both beam and shell corotational elements, see Battini and Pacoste (2002) and Battini (2007). Many numerical tests in statics have shown that all the formulations give the same numerical results and and have the same convergence properties regarding the Newton-Raphson iterations. It is therefore not possible to state that one approach is better than another one. However, the following aspects can be considered in the choice of the parameterisation:

- If the amplitudes of the rotations are larger than 2π, the rotational vector can not be used; if the amplitudes of the rotations are larger than π, the Euler parameters can not be used.

- With the formulations based on totally additive variables, the rotations are handled in the same way as displacements at the structural level and only three rotational parameters at each node are required. With the other formulations, the rotational matrix and three rotational parameters (for the incrementally additive formulations) are necessary at each node at the structural level. This requires more memory space, especially for large scale problems.

- For corotational beam elements, the tangent stiffness matrix is not symmetric. However, for the formulations based on additive variables (totally or incrementally), the tangent stiffness matrix can be symmetrised without affecting the convergence properties of the Newton-Raphson iterations. This is not possible for the formulation based on non-additive spatial rotations. For corotational shell elements, the tangent stiffness matrix is symmetric if additive (totally or incrementally) variables are used and the tangent stiffness matrix can be symmetrised if spatial rotations are used. This aspect is particularly interesting for large scale problems.

- As pointed out by Ibrahimbegovic (1997), the totally or incrementally additive formulations are very interesting in dynamics. As a matter of fact, classical Newmark time-stepping algorithms based on additive variables can be directly used. If spatial rotations have to be used, many difficulties are encountered. The choice of the additive variables seems to be still open. As example, Cardona and Gerardin (1988) used the rotational vector to develop a beam element and Spring (1986) showed that Euler parameters are also a very good alternative.

- The numerical tests performed on small case problems have shown that the formulation based on spatial rotations requires the least computational time. The slowest formulation is the one based on the incremental rotational vector and requires about 20% more computational time. This aspect is illustrated in the examples.

- All these formulations are based on different parameterisations of the nodal rotations and consequently different definitions of the internal moments at the nodes. This is not a problem but must be carefully considered if external moments or eccentric forces are applied. This is the purpose of the next section and will be illustrated in Example 2.

7 Applied moments and eccentric forces

7.1 Applied moments

Consider that at a node i, an external moment $\mathbf{m}_\theta$ is applied and that the formulation based on spatial rotations is used. This moment performs the external virtual work

$$W_e = \delta\theta_i^{\mathrm{T}}\,\mathbf{m}_\theta \tag{30}$$

Consider now that the same problem is to be analysed using the formulations based on the rotational vector and Euler parameters. The applied external moments $\mathbf{m}_\Psi$ or $\mathbf{m}_{\mathrm{q}}$ must perform the same virtual work as in Eq. (30),

$$\delta\Psi_i^{\mathrm{T}}\,\mathbf{m}_\Psi = \delta\theta_i^{\mathrm{T}}\,\mathbf{m}_\theta \qquad \delta\mathbf{q}_i^{\mathrm{T}}\,\mathbf{m}_{\mathrm{q}} = \delta\theta_i^{\mathrm{T}}\,\mathbf{m}_\theta \tag{31}$$

which, introducing Eqs. (18), gives

$$\mathbf{m}_\Psi = \mathbf{T}_\Psi^{\mathrm{T}}(\Psi_i)\,\mathbf{m}_\theta \qquad \mathbf{m}_{\mathrm{q}} = \mathbf{T}_{\mathrm{q}}^{\mathrm{T}}(\mathbf{q}_i)\,\mathbf{m}_\theta \tag{32}$$

Differentiation of the above equations gives the 3×3 moment-correction stiffness terms

$$\mathbf{K}_{c\Psi} = \left.\frac{\partial(\mathbf{T}_{\Psi}^{\mathrm{T}}(\Psi_i)\,\mathbf{m}_\theta)}{\partial\Psi_i}\right|_{\mathbf{m}_\theta} \qquad \mathbf{K}_{c\mathrm{q}} = \left.\frac{\partial(\mathbf{T}_{\mathrm{q}}^{\mathrm{T}}(\mathbf{q}_i)\,\mathbf{m}_\theta)}{\partial\mathbf{q}_i}\right|_{\mathbf{m}_\theta} \tag{33}$$

which must be subtracted from the tangent stiffness matrices $\mathbf{K}_\Psi$ and $\mathbf{K}_\mathrm{q}$.

If the parameterisations in terms of incremental rotational vector and incremental Euler parameters are adopted, the same transformations are required. The importance of these transformations will be shown in Example 2. In particular, it will be shown that if these transformations are not applied, the different formulations produce different results.

7.2 Eccentric forces

The physical interpretation of the different applied moments is not trivial. As example, let us consider the case

$$\mathbf{m}_\theta = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \Psi_i = \begin{bmatrix} \pi/4 \\ \pi/4 \\ \pi/4 \end{bmatrix} \tag{34}$$

Using Eq. (32), it is obtained

$$\mathbf{m}_\Psi = \begin{bmatrix} 0.813 \\ 0.429 \\ -0.242 \end{bmatrix} \tag{35}$$

Both $\mathbf{m}_\theta$ and $\mathbf{m}_\Psi$ are associated to rotations in radians and represent the same physical moment. Then, why are they completely different?

One way to obtain a physical interpretation is to consider that these moments are due to eccentric forces applied to the ends of rigid links, see Izzuddin (2001).

Let us consider that the constant external force vector $\mathbf{f}_a$ is applied at the point a, and that point a is rigidly connected to node i through the rigid link defined by the vector $\mathbf{v}_o$ such as

$$\mathbf{x}_a = \mathbf{x}_i + \mathbf{v}_o \tag{36}$$

where $\mathbf{x}_\mathrm{a}$ and $\mathbf{x}_i$ denote the initial positions of a and i. The displacement vector of the point a is given by

$$\mathbf{u}_a = \mathbf{u}_i + (\mathbf{R}_i - \mathbf{I})\,\mathbf{v}_o \qquad \mathbf{R}_i = \mathbf{R}_a \tag{37}$$

$\mathbf{f}_i$ and $\mathbf{m}_i$, the external load and moment vectors at node i, can be obtained by considering that the applied loads at a and i must perform the same external virtual work,

$$\delta \mathbf{u}_a^{\mathrm{T}} \mathbf{f}_a = \delta \mathbf{u}_i^{\mathrm{T}} \mathbf{f}_i + \delta \theta_i^{\mathrm{T}} \mathbf{m}_i \tag{38}$$

Differentiation of Eq. (37) gives

$$\delta \mathbf{u}_a = \delta \mathbf{u}_i + \delta \mathbf{R}_i \mathbf{v}_o \tag{39}$$

which, using Eq. (13), can be rewritten as

$$\delta \mathbf{u}_a = \delta \mathbf{u}_i + \delta \tilde{\theta}_i \mathbf{v} = \delta \mathbf{u}_i - \tilde{\mathbf{v}}\, \delta \theta_i \qquad \mathbf{v} = \mathbf{R}_i \mathbf{v}_o \tag{40}$$

and finally as

$$\delta \mathbf{u}_a^{\mathrm{T}} = \delta \mathbf{u}_i^{\mathrm{T}} + \delta \theta_i^{\mathrm{T}} \tilde{\mathbf{v}} \tag{41}$$

Then, from Eqs. (38) and (41), it is obtained

$$\mathbf{f}_i = \mathbf{f}_a \qquad \mathbf{m}_i = \tilde{\mathbf{v}} \mathbf{f}_a = \mathbf{v} \times \mathbf{f}_a \tag{42}$$

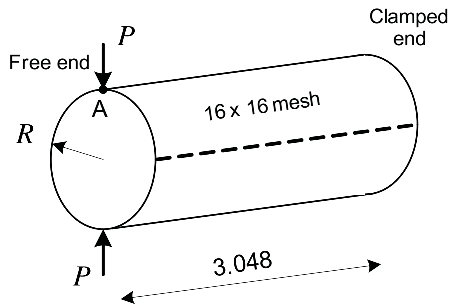
Eq. (42) shows that the components of $\mathbf{m}_i$ are the moments around the global axes $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ created by $\mathbf{f}_a$ at point i. Since $\mathbf{m}_i$ is associated to spatial rotations $\delta\theta_i$, it can then be deduced that the components of the vector $\mathbf{m}_\theta$, see Eq. (30), are moments around the fixed global cartesian axes. To find physical interpretations of $\mathbf{m}_\Psi$ and $\mathbf{m}_q$ seems much more difficult.
Since $\mathbf{m}_i$ is not constant, a 3×3 moment-correction stiffness term $\mathbf{K}_c$ defined by

$$\delta \mathbf{m}_i = \mathbf{K}_c\, \delta \theta_i \tag{43}$$

must be be subtracted from the tangent stiffness matrix $\mathbf{K}_\theta$. Differentiation of Eq. (42) gives

$$\delta \mathbf{m}_i = \delta \tilde{\mathbf{v}} \mathbf{f}_a = -\tilde{\mathbf{f}}_a\, \delta \mathbf{v} = -\tilde{\mathbf{f}}_a\, \delta \tilde{\theta}_i \mathbf{v} = \tilde{\mathbf{f}}_a \tilde{\mathbf{v}}\, \delta \theta_i \tag{44}$$

Finally, $\mathbf{m}_i$ and $\mathbf{K}_\theta$ are consistent with infinitesimal spatial rotations $\delta\theta$. If the formulations based on the (incremental) rotational vector or (incremental) Euler parameters are to be used, the transformations in Sections 7.1 and 4,5 must be performed.



$$
\begin{aligned}
E &= 20685000 \\
\nu &= 0.3 \\
R &= 1.016 \\
t &= 0.03
\end{aligned}
$$

Figure 2: Example 1: Pinched cantilever cylinder

Table 1: Example 1 – numerical performances

Formulation	spat. rot.	rot. v.	inc. rot. v.	Euler	inc. Eul.
cpu time	119	135	139	124	126

8 Numerical examples

8.1 Example 1: Pinched cantilever cylinder

The purpose of this first example, depicted in Fig. 2, is to compare the numerical performances of the different formulations. The cylinder, which is clamped at one end and free at the other, is subjected to two opposite loads P. Using symmetry, only one quarter of the cylinder is modelled with a 16×16 mesh of corotational shell elements. The analysis is performed using 16 steps, up to a displacement of about $1.6R$, ignoring the physically occuring contact. All the formulations give exactly the same numerical results, see Fig. 3 and required the same number of Newton-Raphson iterations (102). The numerical performances given Table 1 show that the fastest formulation (spatial rotations) is 17% faster than the slowest one (incremental rotational vector).

8.2 Example 2: L-shaped cantilever frame

The second example, see Fig. 4, is loaded by a moment at the free end A. As noted by Gendy and Saleeb (1994), the buckling load and post-buckling response depend on the method used to generate the applied moment. Two cases are studied here: in the first case, a moment vector $[0\,0\,m]^{\mathrm{T}}$ is applied; in the second case, two opposite and conservative forces $F/2$ are applied at the ends of rigid links of unit length.

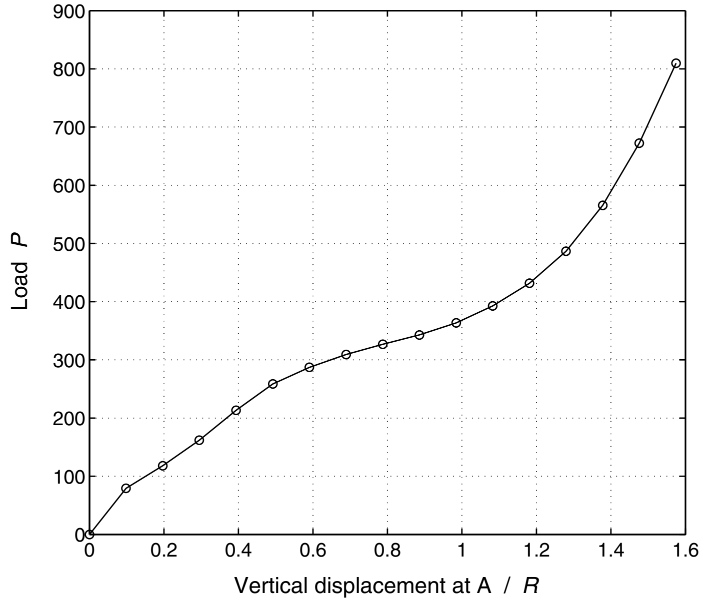


Figure 3: Example 1: Load-displacement diagram

Table 2: Example 2 – case 2 – numerical performances

Formulation	spat. rot.	rot. v.	inc. rot. v.	Euler	inc. Eul.
nb. iterations	150	156	155	156	157
cpu time	4.01	4.98	5.01	4.29	4.41

In both cases, small imperfections are introduced by applying at A a force of magnitude $m/1000$ and $F/1000$ in z-direction. For case 1, the results, see Fig. 5, are obtained without the moment transformation defined in Section 7.1, which means that a moment $[0\,0\,m]^{\mathrm{T}}$ is applied for the three considered parametrisations. As expected, the obtained results are different since the applied moments are physically different. For case 2, both transformations in sections 7.1 and 4,5 have been applied and all the parameterisations give exactly the same results. The numerical performances given in Table 2 show that the fastest formulation (spatial rotations) is 25% faster than the slowest one (incremental rotational vector).

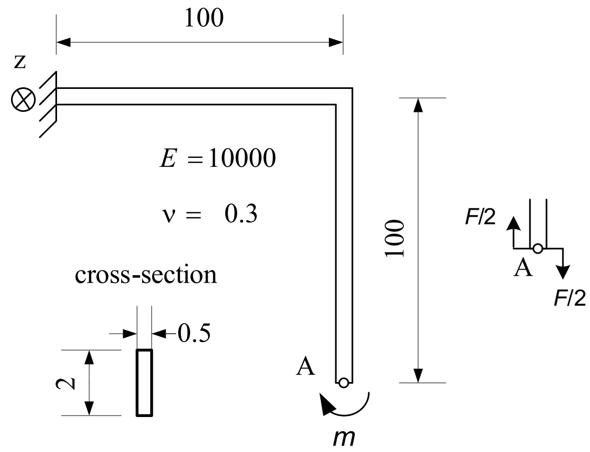


Figure 4: Example 2: L-shaped cantilever frame

9 Conclusion

This paper has presented and compared different approaches to parameterise finite rotations in corotational beam and shell elements. The first one, based on spatial rotations, is obtained naturally during the derivation of the element. The rotational parameters are non-additive and a special updating procedure is required. For the other approaches, based on the rotational vector and the Euler parameters, the rotational variables are totally or incrementally additive. These parameterisations are obtained from the first one through a change of rotational variables which is independent on the element formulation. As a consequence, this method is not limited to corotational elements but can also be used for other non-linear beam and shell elements.

Many tests in statics, see also Section 8, have shown that all the formulations give the same numerical results and have the same convergence properties regarding the Newton-Raphsons iterations. However, as explained in Section 6, all these formulations have advantages and drawbacks and the choice of an efficient formulation depends on the problem that must be studied.

Finally, if moments or eccentric forces are applied, transformations of the moments and moment-correction stiffness terms, consistent with the parameterisation of the rotations, are required.

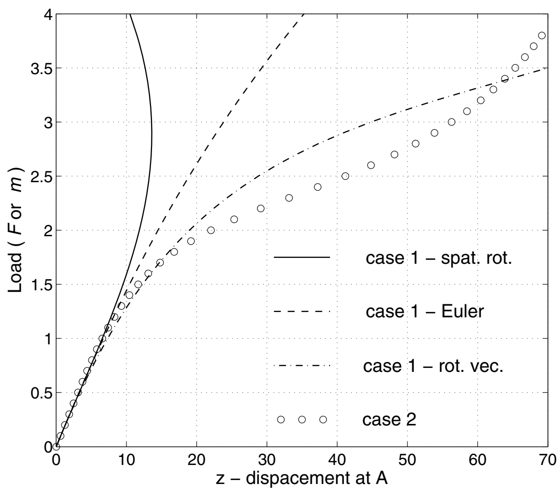


Figure 5: Example 2: Load-displacement diagrams

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