 CMES, vol.32, no.3, pp.175-184, 2008

Nonlinear Dynamical Analysis of Cavitation in Anisotropic Incompressible Hyperelastic Spheres under Periodic Step Loads

X.G. Yuan[1,2] **and H.W. Zhang**[1]

Abstract: In this paper, a dynamic problem that describes void formation and motion in an incompressible hyperelastic solid sphere composed of a transversely isotropic Valanis-Landel material is examined, where the sphere is subjected to a class of periodic step tensile loads on its surface. A motion equation of void is derived. On analyzing the dynamical properties of the motion equation and examining the effect of material anisotropy on void formation and motion in the sphere, we obtain some new and interesting results. Firstly, under a constant surface tensile load, it is proved that a void would form in the sphere as the tensile load exceeds a certain critical value and that the motion of the formed void with time would present a class of singular period oscillations, the oscillation center is also determined. Secondly, under periodic step tensile loads, the existence conditions for periodic oscillation of the formed void are presented.

Keyword: Incompressible hyperelastic material; motion equation of void; nonlinear periodic vibration; constant load; periodic step loads

1 Introduction

In practice, it is commonly recognized that the sudden formation and growth of voids (cavitation) occur in engineering materials as precursors to failure. These phenomena are mainly due to instability of materials, and thus prediction of void formation and its motion rule has long attracted extensive attention. Since Gent and Linderly (1958) discovered experimentally the phenomenon of sudden void formation in tensioning the cylinder composed of vulcanized rubber for the first time in 1958, many similar experiments have been made, see the review article by Gent (1990) on cavitation in rubber up to 1990. The impetus of the theoretical work was supplied by Ball (1982), in which void formation and growth were described as a class of static bifurcation problems in the context of nonlinear elasticity. Thereafter, many significant works have been made for static cavitation problems. Horgan and Polignone (1995) presented a comprehensive review of results up to 1995 for hyperelastic materials. Further references on static formation and growth of void for both incompressible and compressible hyperelastic materials in recent years may be found in [Polignone and Horgan (1993); Murphy and Biwa (1997); Ren and Cheng (2002a, b); Shang and Cheng (2001); Yuan, et. al. (2004a, 2005)]. On the other hand, many numerical methods were employed to solve the differential equations which may be helpful, such as Atluri, et. al. (2006), Ling and Atluri (2008), and so on.

However, while the static formation of voids in hyperelastic materials is well understood, the analogous dynamic problem is relatively unexplored due to the complexity of the governing equations. Chou-Wang and Horgan (1989) investigated dynamical cavitation in an isotropic incompressible neo-Hookean sphere in the context of elastodynamics, moreover, the authors concluded that a void would form at the center of the sphere as the surface tensile load exceeds a certain critical value, and that the motion of the formed void would present a nonlinear periodic oscillation. Based on the work of Chou-Wang and Horgan, the similar dynamic problems were respec-

[1] Department of Engineering Mechanics, Faculty of Vehicle Engineering and Mechanics, State Key Laboratory of Structural Analysis for Industrial Equipment, Dalian University of Technology, Dalian 116024, China, yxg1971@163.com, zhanghw@dlut.edu.cn
[2] School of Science, Dalian Nationalities University, Dalian 116600, Liaoning, P.R. China

tively studied by Ren and Cheng (2003) for the transversely isotropic incompressible Ogden material, by Yuan, et. al. (2004b) for a class of generalized incompressible neo-Hookean materials.

The earlier investigations are all focused on the constant tensile load which is independent of time, however, the loading types acting on structures are always dynamic loads in practice, for example, periodic load, step loads relating to time, and so on. Yuan, et. al. (2006) studied problems of cavity formation and motion in the incompressible neo-Hookean sphere under a class of prescribed periodic step tensile loads, which is also based on the work of Chou-Wang and Horgan (1989), in particular, the existence conditions for periodic oscillation of the formed cavity are determined in Yuan, et. al. (2006). To better understand the problem of cavitation, Yuan, et. al. (2007) studied the periodic motion of a pre-existing micro-void in the interior of the incompressible hyperelastic spheres.

The purpose of this paper is to examine the dynamical bifurcation problem of void formation and motion in an incompressible hyperelastic solid sphere composed of a generalized Valanis-Landel material which is transversely isotropic about the radial direction, where the sphere is subjected to a class of periodic step tensile loads on its surface. The mathematical model of the problem is formulated in the context of nonlinear elastodynamics. The motion equation of void of the dynamic problem is derived by using the incompressibility constraint and the boundary conditions. The dynamical properties of the motion equation are examined in detail, and then some new and interesting results are obtained by using and improving the theory of nonlinear dynamics. Firstly, under a constant radial tensile load which is independent of time, the classical periodic solution and two kinds of generalized periodic solutions are defined as the motion equation satisfies different initial conditions. It is proved that the nonzero solutions of the motion equation are all periodic solutions for any nonzero initial conditions and for any given parameters presented in the equation. In particular, by using the method of energy analysis, that the nonzero solutions belong to which kind of periodic solutions to the end are discussed in detail as the material parameters take different values. Correspondingly, if the solution of the motion equation satisfies the zero initial conditions, it is shown that a void would form at the center of the sphere as the surface dead-load exceeds a certain critical value and that the motion of the formed void with respect to time presents a class of singular period oscillations. Secondly, under periodic step tensile loads which are related to time, the existence conditions of periodic solutions are determined by using the phase diagrams of the motion equation. In each section, numerical examples are also carried out.

2 Formulation and solutions

For a solid sphere with radius R_0 composed of an incompressible hyperelastic material, we are concerned with the radially symmetric motion of the sphere under a class of surface tensile loads $\hat{p}(t)$ depending on time t, where $\hat{p}(t)$ is a step function of period $T = 2t_0 + 2t_1$, $(k = 0, 1, 2, \cdots)$:

$$\hat{p}(t) = \begin{cases} p_0, & t \in [2kT,\ 2kT + t_0), \\ p_0 + \varepsilon, & t \in [2kT + t_0,\ 2kT + t_0 + 2t_1), \\ p_0, & t \in [2kT + t_0 + 2t_1,\ 2(k+1)T]. \end{cases} \tag{1}$$

Under the assumption of spherical deformation, the position (R, Θ, Φ) of the particle in the undeformed configuration moves to the position (r, θ, ϕ) at time $t \geq 0$, where $r = r(R,t) \geq 0$, $(0 < R < R_0)$ is a radial deformation function to be determined, and $\Theta = \theta, \Phi = \phi$. The deformation gradient tensor is given by

$$\begin{aligned} \mathbf{F} &= diag\left(\frac{\partial r(R,t)}{\partial R}, \frac{r(R,t)}{R}, \frac{r(R,t)}{R}\right) \\ &= diag\left(\lambda_1, \lambda_2, \lambda_3\right), \end{aligned} \tag{2}$$

where λ_i, $(i = 1, 2, 3)$ are the principal stretches. The differential equations that describe the radially symmetric motion of the sphere, in the absence of body force, reduce to

$$\frac{\partial \sigma_{rr}(r,t)}{\partial R}\left(\frac{\partial r}{\partial R}\right)^{-1}+\frac{2}{r}\left[\frac{R^2}{r^2}\frac{\partial W}{\partial \lambda_1}-\frac{r}{R}\frac{\partial W}{\partial \lambda_2}\right] =\rho\frac{\partial^2 r}{\partial t^2},\quad t\geq 0,\quad (3)$$

where

$$\sigma_{rr}(r,t)=\lambda_1\frac{\partial W}{\partial \lambda_1}-p(r,t),\qquad (4)$$

is the radial Cauchy stress associated with the incompressible hyperelastic materials, $W = W(\lambda_1,\lambda_2,\lambda_3)$ is the strain energy function of the material, $p(r,t)$ is the unknown hydrostatic pressure associated with the incompressibility constraint $\lambda_1\lambda_2\lambda_3 = 1$, and ρ is the constant mass density of the material. Obviously, Eq.(3) is a class of nonlinear evolution equations of the radial deformation function $r = r(R,t)$.

In this paper, assume that the sphere is composed of a class of generalized incompressible Valanis-Landel hyperelastic materials, for which the form of the corresponding strain energy function is given by Yuan, et. al. (2004a)

$$W=W(\lambda_1,\lambda_2,\lambda_3)=2\mu\Big\{\sum_{i=1}^{3}\lambda_i(\ln\lambda_i-1) +\alpha\left[\lambda_1(\ln\lambda_1-1)\right]^2+\beta\left[\lambda_1(\ln\lambda_1-1)\right]^3\Big\},\qquad (5)$$

where $\mu > 0$ is the shear modulus for infinitesimal deformations, $\alpha,\beta\geq 0$ are the dimensionless material parameters which serve as the degree of material anisotropy. As $\alpha=\beta=0$, Eq.(5) then reduces to the isotropic incompressible hyperelastic material which is first proposed by Valanis and Landel (1967). Moreover, Ren and Cheng (2002a) investigated static cavitated bifurcation for the isotropic incompressible Valanis-Landel material. Polignone and Horgan (1993) discussed in detail the forms of the strain energy functions for anisotropic hyper-elastic materials and presented the first paper on void formation for transversely isotropic incompressible materials.

From the incompressibility constraint $\lambda_1\lambda_2\lambda_3 = 1$ and from Eq.(2) we have

$$r=r(R,t)=\left[R^3+k^3(t)\right]^{1/3},\quad t\geq 0,\qquad (6)$$

where $k(t)\geq 0$ is an undetermined function, and presents the value of the void radius at time t, namely, if $r(0+,t) = k(t) = 0$, then it means that the sphere remains solid in the current configuration; while if $r(0+,t) = k(t) > 0$, this implies that a spherical void with radius $k(t)$ forms at the center of the sphere and then motions with respect to time t. In this case, it is assumed that the void surface is traction free. On the other hand, from Eq.(6), it is easy to see that the motion of the whole sphere can be determined completely by $k(t)$.

Since the surface of the sphere is subjected to a class of periodic step loads $\hat{p}(t)$ given by Eq.(1), we have the following boundary condition

$$\sigma_{rr}(r(R_0,t),t)=\hat{p}(t)\left[\frac{R_0}{r(R_0,t)}\right]^2,\quad t\geq 0.\qquad (7)$$

At the center of the sphere, the boundary condition requires that

$$r(0+,t)\sigma_{rr}(r(0+,t),t)=0,\quad t\geq 0,\qquad (8)$$

which means that if no void forms in the interior of the sphere, then $r(0+,t) = 0$; where as $r(0+,t) = k(t) > 0$, namely, a void with radius $k(t)$ forms at the center of the sphere at time t, for the traction free void, we have $\sigma_{rr}(r(0+,t),t) = 0$.

The sphere is assumed to be in an undeformed state and at rest at time $t = 0$, so the initial conditions are given by $r(R,0) = R, \partial r(R,0)/\partial t = 0$ and from Eq.(6) we have

$$k(0)=0,\dot{k}(0)=0.\qquad (9)$$

Note. Dots over all letters in this paper denote the derivative with respect to time.

Further, it is not difficult to show that

$$\frac{\partial^2 r}{\partial t^2}=\frac{\partial\left(-r^{-1}(2k\dot{k}^2+k^2\ddot{k})+r^{-4}k^4\dot{k}^2/2\right)}{\partial r}.\qquad (10)$$

For convenience, we rewrite Eq.(6) as $R = \left[r^3-k^3(t)\right]^{1/3}$, and let $\eta=\eta(r,k)=(1-\frac{k^3}{r^3})^{1/3}$, so $\lambda_i,(i=1,2,3)$ can be denoted by $\lambda_1=\eta^2, \lambda_2=\lambda_3=\eta^{-1}$.

Based on the above notations, we integrate Eq.(3) with respect to r from k to r, and then obtain

$$\sigma_{rr}(r,t)-\sigma_{rr}(k,t)+4\mu\int_k^r H(\eta(\xi,k),\alpha,\beta)\frac{d\xi}{\xi} = \rho\left[\left(\frac{k^4}{2r^4}-\frac{2k}{r}+\frac{3}{2}\right)\left(\dot{k}\right)^2+k\left(1-\frac{k}{r}\right)\ddot{k}\right], \tag{11}$$

where $\eta(\xi,k)=(1-\frac{k^3}{\xi^3})^{1/3}$ and

$$H(\eta,\alpha,\beta)=\eta^2\ln\eta\left(2+4\alpha\eta^2(2\ln\eta-1)+6\beta\eta^4(2\ln\eta-1)^2+\eta^{-3}\right). \tag{12}$$

Multiplying both sides of Eq.(11) by $r(0+,t)$, and using the boundary conditions (7) and (8), we have

$$k\hat{p}(t)\left(\frac{R_0}{S}\right)^2+4\mu k\int_k^S H(\eta(\xi,k),\alpha,\beta)\frac{d\xi}{\xi} = \rho k\left[\left(\frac{k^4}{2S^4}-\frac{2k}{S}+\frac{3}{2}\right)\left(\dot{k}\right)^2+k\left(1-\frac{k}{S}\right)\ddot{k}\right], \tag{13}$$

where $S=r(R_0,t)=(R_0^3+k(t)^3)^{1/3}$.

In what follows, it is convenient to introduce the following quantities

$$x(t)=\frac{k(t)}{R_0},\quad \dot{x}(t)=\frac{\dot{k}(t)}{R_0},\quad P(t)=\frac{\hat{p}(t)}{\mu}. \tag{14}$$

In this case, the initial conditions (9) become

$$x(0)=0,\quad \dot{x}(0)=0. \tag{15}$$

Further, from Eq.(14) and the relationship between η and ξ in Eq.(12), Eq.(13) can be written as

$$\frac{\rho R_0^2}{\mu}x\left[\left(x-\frac{x^2}{(1+x^3)^{1/3}}\right)\ddot{x} + \left(\frac{x^4}{2(1+x^3)^{4/3}}-\frac{2x}{(1+x^3)^{1/3}}+\frac{3}{2}\right)(\dot{x})^2\right] = P(t)x(1+x^3)^{-2/3} - 4x\int_0^{(1+x^3)^{-1/3}} H(\eta,\alpha,\beta)\frac{\eta^2 d\eta}{\eta^3-1}. \tag{16}$$

Eq.(16) is a second-order nonlinear differential equation with respect to the dimensionless void radius $x(t)$ at time t, and describes the exact relation between $P(t)$ and $x(t)\geq 0$ for the generalized incompressible Valanis-Landel materials. Thus Eq.(16) is called the **formation and motion equation of void**.

3 Nonlinear dynamical properties of solutions of Eq.(16)

Obviously, $x(t)\equiv 0$ is a solution of Eq.(16), and it corresponds to the homogeneous deformation of the sphere, i.e., $r(R,t)=R$, and thus it is called the trivial solution of Eq.(16). Next we discuss the existence conditions and the qualitative properties of the nontrivial solutions, $x(t)\geq 0$, of Eq.(16).

3.1 Constant tensile load case: $\varepsilon=0$

Let $\varepsilon=0$ in Eq.(1), we then have $P(t)\equiv P$ (or $\hat{p}(t)\equiv p_0$), namely, the sphere is subjected to a constant tensile load.

For the initial condition $x(0)=0$, however, if we set $t\to 0+$, from Eq.(16) it is easy to show that

$$\dot{x}(0+)=\pm\left(\frac{2\mu(P-G(0,\alpha,\beta))}{3\rho b^2}\right)^{1/2}, \tag{17}$$

that is to say, the first-order derivative of $x(t)$ of Eq.(16) occurs discontinuous at the initial time $t=0$ for $P\neq G(0,\alpha,\beta)$, and thus Eq.(16) is called the singular second-order nonlinear differential equation for the initial condition $x(0)=0$. On the other hand, if the solutions satisfy the initial condition $x(0)=x_0\neq 0$, to determine the solutions of Eq.(16) completely, another initial condition $\dot{x}(0)=\dot{x}_0$ must also be presented.

Let $y=\dot{x}$, then Eq.(16) is equivalent to the following first-order differential equations:

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=\begin{pmatrix}y\\ F(x,y)\end{pmatrix}, \tag{18}$$

where

$$F(x,y)=\frac{(1+x^3)^{1/3}}{x(1+x^3)^{1/3}-x^2}\left\{\frac{\mu\left[P(1+x^3)^{-2/3}-G(x,\alpha,\beta)\right]}{\rho R_0^2}-y^2\left[\frac{x^4}{2(1+x^3)^{4/3}}-\frac{2x}{(1+x^3)^{1/3}}+\frac{3}{2}\right]\right\},\quad(19)$$

$$G(x,\alpha,\beta)=4\int_0^{(1+x^3)^{-1/3}}H(\eta,\alpha,\beta)\frac{\eta^2 d\eta}{\eta^3-1}.\quad(20)$$

Obviously, the equilibrium point of Eq.(18) is given by $(x_i,0)$, where x_i is a nonzero real root of

$$P=(1+x^3)^{2/3}G(x,\alpha,\beta).\quad(21)$$

To discuss the relation between P and the number of solutions of Eq.(21), as well as the effect of material parameters α and β on the qualitative properties of solutions of Eq.(16), we now carry out the Taylor expansion of the right hand of Eq.(21), $(1+x^3)^{2/3}G(x,\alpha,\beta)$ at $x=0$, as follows,

$$P=2.0852-1.2777\alpha+1.7421\beta+(0.0568+0.9260\alpha-1.5053\beta)x^3+O(x^4),\quad(22)$$

where

$$P_{cr}=G(0,\alpha,\beta)=2.0852-1.2777\alpha+1.7421\beta\quad(23)$$

is the critical load that describes static formation of void in the interior of the sphere composed of the generalized Valanis-Landel material. Let

$$\Omega_1=\{0.0568+0.9620\alpha-1.5053\beta>0,\ \alpha\geq 0,\beta\geq 0\},\quad(24a)$$

$$\Omega_2=\{0.0568+0.9620\alpha-1.5053\beta<0,\ \alpha\geq 0,\beta\geq 0\}.\quad(24b)$$

For the parameters (α,β) belonging to different regions, we have:

Conclusion 1 **(i)** If $(\alpha,\beta)\in\Omega_1$, the nonzero solution of Eq.(21) increases monotonously with respect to x near $x=0$; **(ii)** While if $(\alpha,\beta)\in\Omega_2$, the nonzero solution of Eq.(21) decreases monotonously near $x=0$, however, there is a secondary turning point, written as (x_n,P_n), on the nonzero solution curve. As (α,β) belongs to different regions, curves of $P\sim x$ are shown in figure 1.

Conclusion 2 **(i)** If $(\alpha,\beta)\in\Omega_1$, Eq.(21) has a unique nonzero real solution x_1 only when $P>P_{cr}$, and the corresponding equilibrium point $(x_1,0)$ of Eq.(18) (or Eq.(16)) is a center; **(ii)** While if $(\alpha,\beta)\in\Omega_2$, then **(a)** for the given $P\in(P_n,P_{cr})$, Eq.(21) has two nonzero real solutions, written as x_2 and x_3, where $x_2<x_3$, moreover, $(x_2,0)$ is a saddle point and $(x_3,0)$ is a center of Eq.(18); **(b)** for $P>P_{cr}$, Eq.(21) has a unique nonzero real solution x_4 and $(x_4,0)$ is a center.

Here we only prove the case $(\alpha,\beta)\in\Omega_2$, the proof of the case $(\alpha,\beta)\in\Omega_1$ is similar.

Proof. Consider the eingenvalues of the linearized equation of Eq.(18) about $(x_i,0)$, as follows,

$$\lambda_1=-\lambda_2=\left[\frac{-\mu G_x(x_i,\alpha,\beta)}{\rho R_0^2 x_i(1+x_i^3)^{1/3}[(1+x_i^3)^{1/3}-x_i]}\right]^{\frac{1}{2}}.\quad(25)$$

For the given $P\in(P_n,P_{cr})$, if x_2 and x_3 are nonzero solutions of Eq.(21), from $G_x(x_2,\alpha,\beta)<0$, we know that λ_1,λ_2 given by Eq.(25) are two real and equal eingenvalues with opposite sign, and thus the equilibrium point $(x_2,0)$ is a saddle point of the linearized equation of Eq.(18) and also a saddle point of Eq.(18); However, from $G_x(x_3,\alpha,\beta)>0$, we know that λ_1,λ_2 are two pure imaginary eingenvalues with opposite sign, and thus $(x_3,0)$ is a center of the linearized equation of Eq.(18). On the other hand, since $F(x,-y)=F(x,y)$ and $(x_3,0)$ is a unique stable nonzero equilibrium point of Eq.(18), we know that, from the symmetry principle, $(x_3,0)$ is a center of Eq.(18).

Similarly, it is easy to show that $(x_4,0)$ is a center of Eq.(18).

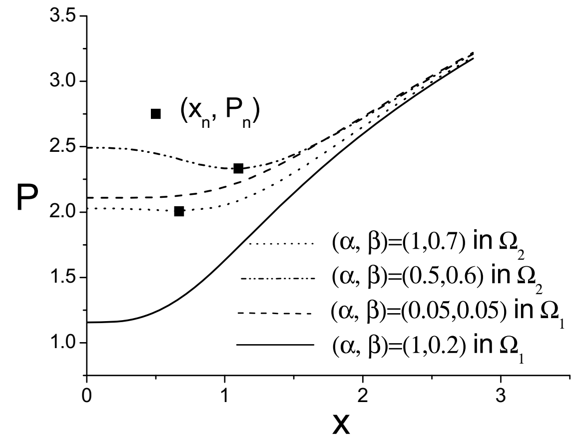

Figure 1: Curves of $P \sim x$ in Ω_1 and Ω_2

To further study the properties of the solutions of Eq.(16), we define three classes of periodic solutions, as follows,

Definition 1 If $x = x(t)$ is a periodic solution of period T, and is smoothing enough at any time t, we then call it **the classical periodic solution**.

Definition 2 If $x = x(t)$ is a periodic solution of period T, and if the left- and right-limit of $\dot{x} = dx/dt$ exist but do not equal each other at certain times, we then call it **the generalized periodic solution of the first kind**.

Definition 3 If $x = x(t)$ is a periodic solution of period T, and if at least a value of $\dot{x} = dx/dt$ does not exist at certain times, we then call it **the generalized periodic solution of the second kind**.

Multiplying both sides of Eq.(16) by $x\dot{x}$, and then integrating it with respect to t, we obtain the first integral

$$E = \frac{\rho R_0^2}{\mu} x^3 \left[1 - \frac{x}{(1+x^3)^{1/3}}\right] \dot{x}^2 - 2P\left((1+x^3)^{1/3} - 1\right) - 8\int_0^x \delta^2 \left[\int_0^{(1+\delta^3)^{-1/3}} H(\eta,\alpha,\beta)\frac{\eta^2 d\eta}{1-\eta^3}\right] d\delta, \tag{26}$$

where E is a energy constant relating to the initial conditions. Further, it is easy to show that the implicit solution of Eq.(16) is given by

$$\pm\int_{x_0}^{x} \left(\frac{\frac{\rho R_0^2}{\mu} z^3 \left(1 - \frac{z}{(1+z^3)^{1/3}}\right)}{E+E'}\right)^{1/2} dz = t - t_0, \tag{27}$$

where and $x_0 = x(t_0)$ is an arbitrary initial condition and

$$E' = 2P((1+z^3)^{1/3} - 1) + 8\int_0^z \delta^2 \left[\int_0^{(1+\delta^3)^{-1/3}} H(\eta,\alpha,\beta)\frac{\eta^2 d\eta}{1-\eta^3}\right] d\delta. \tag{28}$$

In particular, for $x(0) = 0$, we have $E = 0$. For the given $(\alpha,\beta) \in \Omega_2$, figures 2 and 3 respectively show the phase diagrams of Eq.(16) satisfying different initial conditions as $P_n \le P < P_{cr}$ and $P \ge P_{cr}$.

From the above analyses, we have the following conclusions (cf. figures 2 and 3):

Conclusion 3 If $(\alpha,\beta) \in \Omega_2$, **(i)** For the initial conditions $x(0) = x_0 \ne 0$, $\dot{x}(0) = \dot{x}_0$, it can be shown that $\dot{x} \to \infty$ as $x \to 0+$ from Eq.(26) and that the improper integration (27) is convergent as $x \to 0+$, so Eq.(16) has only the generalized periodic solutions of the second kind as $P < P_n$ and has the classical periodic solutions and the generalized periodic solutions of the second kind as $P > P_n$, where (x_n, P_n) is the secondary turning point (cf. Conclusion 1); **(ii)** For the initial condition $x(0) = 0$, Eq.(16) has only zero solution as $P < P_{cr}$ and has the classical periodic solution as $P = P_{cr}$, and has the generalized periodic solutions of the first kind as $P > P_{cr}$.

Conclusion 4 If $(\alpha,\beta) \in \Omega_1$, **(i)** For the initial conditions $x(0) = x_0 \ne 0$, $\dot{x}(0) = \dot{x}_0$, Eq.(16) has only the generalized periodic solutions of the second kind as $P < P_{cr}$ and has the classical periodic solutions and the generalized periodic solutions of the second kind as $P > P_{cr}$; **(ii)** For the initial condition $x(0) = 0$, Eq.(16) has only zero solution as $P < P_{cr}$ and has the generalized periodic solutions of the first kind as $P > P_{cr}$.

Notably, if the solution of Eq.(16) satisfies the initial condition $x(0) = 0$, then it can describe

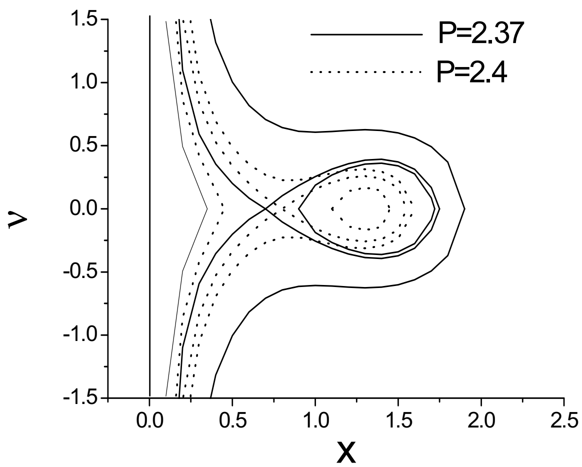

Figure 2: Phase trajectories of Eq.(16) as $P_n < P < P_{cr}$ in region Ω_2

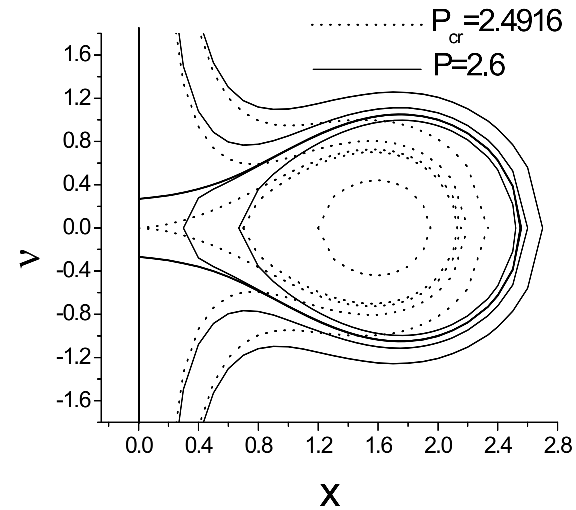

Figure 3: Phase trajectories of Eq.(16) as $P \geq P_{cr}$ in region Ω_2

void formation and motion in the sphere under the prescribed surface tensile constant load. figure 4 shows the phase diagrams of Eq.(16) satisfying the initial condition $x(0) = 0$ for $P \geq P_{cr}$ as the parameters belong to different regions. From the above Conclusions and the phase diagrams shown in fig.4, we know that, for the initial condition $x(0) = 0$, Eq.(16) has only the trivial solution $x(t) \equiv 0$ as $P \leq P_{cr}$, that is to say, the sphere remains solid; While as $P > P_{cr}$, Eq.(16) has the generalized periodic solutions of the first kind, namely, a void forms in the sphere and will expand till its radius reaches the maximum value x_m at time T_0. However, the expanding velocity $\dot{x}(t)$ of the void radius reaches directly to a positive finite value $\dot{x}(0+) = \left(\frac{2\mu(P-G(0,\alpha,\beta))}{3\rho b^2}\right)^{1/2}$ from 0 as a void forms suddenly, and will reduce to zero as the void radius reaches the maximum value at time $T/2$, i.e., $\dot{x}(T/2) = 0$. Thereafter, the void will contract and the contracting velocity will reach to $-\left(\frac{2\mu(P-G(0,\alpha,\beta))}{3\rho b^2}\right)^{1/2}$ as the void reduces to zero at time $t = 2T_0^-$, i.e., $x(T^-) = 0$, $\dot{x}(T^-) = -\left(\frac{2\mu(P-G(0,\alpha,\beta))}{3\rho b^2}\right)^{1/2}$, where $T/2$ can be obtained by setting $x_0 = x(0) = 0$ and $x = x_m$ in Eq.(27). Along with the increasing time, the expanding velocity will leap directly from $-\left(\frac{2\mu(P-G(0,\alpha,\beta))}{3\rho b^2}\right)^{1/2}$ to $\left(\frac{2\mu(P-G(0,\alpha,\beta))}{3\rho b^2}\right)^{1/2}$, and

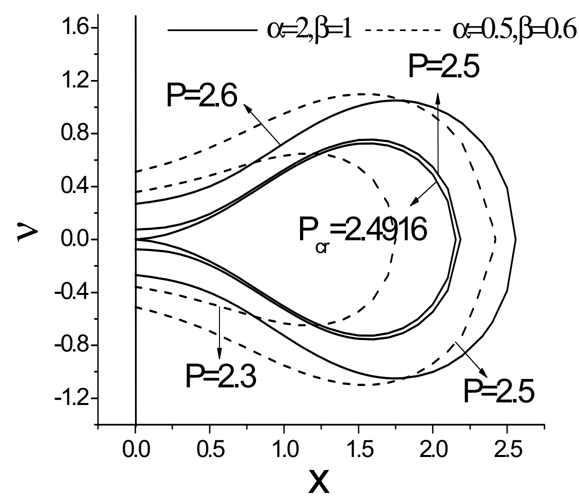

Figure 4: Phase diagrams of void oscillation as $P \geq P_{cr}$ in region Ω_1 or Ω_2

then the cycle will repeat. Thus we can say that a void forms at the center of the sphere as the surface tensile load P exceeds P_{cr}, and the motion rule of the formed void with respect to time presents a class of singular periodic oscillation.

Note. In figures.2, 3 and 4, $v = (\rho/\mu)^{1/2} R_0 y$.

It is worth pointing out that, see Conclusions 3 and 4, as the prescribed $P = P_{cr}$, (i) if $(\alpha, \beta) \in \Omega_1$, the value of x (dimensionless void radius) corresponding to P_{cr} is zero, that is to say, no void forms in the sphere and the sphere is in the critical state of void formation; (ii) however, if $(\alpha,\beta)\in\Omega_2$, there are two values of x, i.e., $0+$ and x_c corresponding to P_{cr}, since P increases continuously, we can conclude that a void has formed in the sphere at the moment, and then takes a classical nonlinear periodic oscillation, and x_c is the oscillation center.

Moreover, another interesting conclusion is that, if the parameters α,β satisfy

$$-2.0852<-1.2777\alpha+1.7421\beta<0 \quad (\text{or } -1.2777\alpha+1.7421\beta>0),$$

a void forms in the interior of the sphere composed of the generalized Valanis-Landel material (5) earlier (or later) than that for the isotropic incompressible Valanis-Landel material, and the formed void presents a class of singular nonlinear periodic oscillations.

3.2 Dynamic inflation (I): Constant pressure case

In this subsection, we only consider the existence conditions of periodic solution of Eq.(16) satisfying the initial condition $x(0)=0$, i.e., the sphere is solid at the initial time $t=0$. The periodic step loads $P(t)$ can be obtained by taking the dimensionless form of Eq.(1), and $(p_0+\varepsilon)/\mu=P+\tilde{\varepsilon}$.

From Subsection 3.1, we know that, for any given parameters α and β, the sphere remains solid as $P<P_{cr}$, however, a void occurs in the interior of the sphere as $P>P_{cr}$ and the motion of the formed void presents a class of singular periodic oscillations, the corresponding minimal positive period of periodic oscillation is denoted by $\hat{T}_1$. If $P+\tilde{\varepsilon}$ also exceeds P_{cr}, the corresponding minimal positive period is denoted by $\hat{T}_2$.

We will discuss the existence conditions of periodic oscillation of void in the following cases:

(i). For the given $P>P_{cr}$, if it is found that $t_0=m\hat{T}_1$, where m is a positive integer, that is to say, a void forms in the sphere, and then oscillates periodically m times as $t\in[0,t_0)$, moreover, we have $x(t_0)=0$. While if $P+\tilde{\varepsilon}$ where $\tilde{\varepsilon}<0$, does not exceed P_{cr}, we know that the sphere remains a solid one as $t\in[t_0,t_0+2t_1)$. In succession, as $t\in[t_0+2t_1,2t_0+2t_1]$, the tensile load is P again, a void forms, and also oscillates periodically m times. Further, in the following period T, the process will be the same as the previous process. In this case, the time t_1 can be taken as an arbitrary positive number. However, if $P+\tilde{\varepsilon}$ also exceeds P_{cr}, and if $2t_1=n\hat{T}_2$, where n is a positive integer, that is to say, the void oscillates n times with period $\hat{T}_2$ as $t\in[t_0,t_0+2t_1)$, and then oscillates m times with period $\hat{T}_1$ as $t\in[t_0+2t_1,2t_0+2t_1]$. In other words, Eq.(16) has the generalized periodic solution of the first kind satisfying the initial condition $x(0)=0$. For the given parameters α and β belonging to region Ω_1 or Ω_2, the example phase diagrams of void oscillation are shown in figure 4.

(ii). As t_0 is not equal to $m\hat{T}_1$, the types of void motion are quite different.

(a). As $m\hat{T}_1<t_0<m\hat{T}_1+\hat{T}_1/2$, namely, a void forms in the sphere and grows continuously to time t_0, Then the tensile load changes to $P+\tilde{\varepsilon}$, and the initial conditions that Eq.(16) satisfies are $x(t_0)=x_0$,$\dot{x}(t_0)=\dot{x}_0$ at the moment. Since $\dot{x}_0>0$, the void radius will not increase until $\dot{x}=0$, i.e., the void radius attains the local maximum (written as $\tilde{x}_m$, and the inequality $\tilde{x}_m<x_m$ must hold). If the motion time that can be obtained by Eqs.(26) and (27), written as t', of the void from x_0 to $\tilde{x}_m$ is exactly equal to t_1, and then the void starts contracting as $t_0+t_1<t<t_0+2t_1$. The tensile load changes to P again at time $t=t_0+2t_1$, and the corresponding initial conditions are given by $x(t_0+2t_1)=x_0$ and $\dot{x}(t_0+2t_1)=-\dot{x}_0$. The void contracts unceasingly to zero at time $t=2t_0+2t_1$. Henceforth, the motion of the void will repeat as above. In other words, the solution of Eq.(16) satisfying the initial condition $x(0)=0$ is still a generalized periodic solution of the first kind with period T. For the given $(\alpha,\beta)\in\Omega_2$, the phase diagrams are shown in figure 5. Otherwise, if $t_1\neq t'$, along with the increasing time, the solution of Eq.(16) will no longer be a periodic solution of period T.

(b). As $t_0=m\hat{T}_1+\hat{T}_1/2$, we have $x(t_0)=x_m$, $\dot{x}(t_0)=0$, and the tensile load then changes to $P+\tilde{\varepsilon}$, where $\tilde{\varepsilon}>0$, in the following time. It is noted here that, if $2t_1=n\hat{T}_2'$, where $\hat{T}_2'$ is the minimal positive period corresponding to $P+\tilde{\varepsilon}$

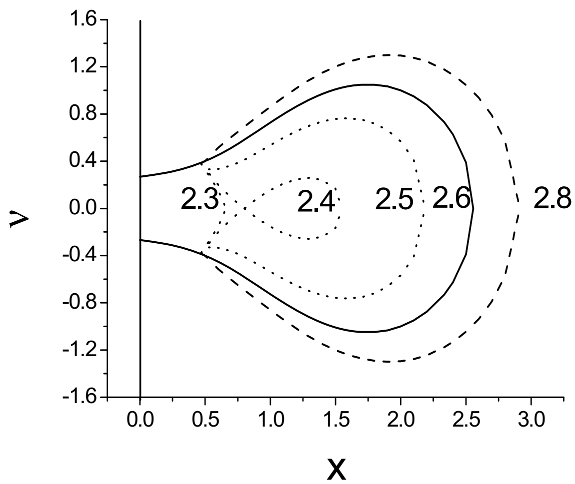

Figure 5: Phase diagrams of Eq.(16) for periodic oscillation case as $m\hat{T}_1 < t_0 < m\hat{T}_1 + \hat{T}_1/2$

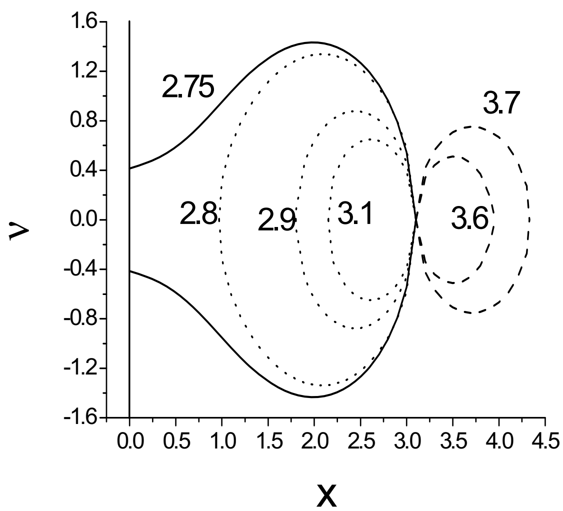

Figure 6: Phase diagrams of Eq.(16) for periodic oscillation case as $t_0 = m\hat{T}_1 + \hat{T}_1/2$

with the initial conditions $x(t_0) = x_m$, $\dot{x}(t_0) = 0$, the tensile load will change to P again as $t \in [t_0+2t_1, 2t_0+2t_1]$ and the initial conditions will be $x(t_0+2t_1) = x_m$, $\dot{x}(t_0+2t_1) = 0$. In this case, the solution of Eq.(16) is a generalized periodic solution of mix type, i.e., the first kind as $t \in [0,t_0)$ and $t \in [t_0+2t_1, 2t_0+2t_1)$, the classical kind as $t \in [t_0, t_0+2t_1)$. See the combination of solid line and dashed shown in figure 6. Otherwise, the solution of Eq.(16) will no longer be a periodic solution of period T.

(c). As $m\hat{T}_1 + \hat{T}_1/2 < t_0 < (m+1)\hat{T}_1$, in other words, the void contracts gradually and the tensile load changes to $P+\tilde{\varepsilon}$ as $t > t_0$. It is easy to show that the solution of Eq.(16) is not a periodic solution of period T for any values of t_1.

4 Conclusions

In this paper, the dependence of void formation and motion in an incompressible transversely isotropic Valanis-Landel hyperelastic solid sphere on material parameters and loading types is examined, and numerical results are also presented. In particular, under a constant radial tensile load which is independent of time, the parameters which serve as material anisotropy are divided into two regions, see Eq.(24a, b). For any given material parameters, there exists a finite critical value of tensile load, it is proved that a void would form at the center of the sphere as the surface tensile load exceeds the critical value and that the motion of the formed void with time would present a class of singular period oscillations. However, as the prescribed tensile load is exactly equal to the critical value, **(i)** for the given parameters belonging to region Ω_1, no void forms in the sphere and the sphere is in the critical state of void formation; **(ii)** while for the parameters belonging to region Ω_2, it is shown that a void has formed in the sphere at the moment, and then takes a classical nonlinear periodic oscillation. Under periodic step tensile loads which are related to time, the existence conditions for all possible periodic oscillations of the formed void are determined by using the phase diagrams of the motion equation.

Acknowledgement: This work was supported by the National Natural Science Foundation of China (10872045, 10721062, 50679013).

References

Atluri, S. N.; Liu H. T.; Han, Z. D. (2006): Meshless local Petrov-Galerkin (MLPG) mixed finite differenceMethod for solid mechanics. *CMES: Computer Modeling in Engineering & Sciences*, vol.15, pp.1-16.

Ball, J. M. (1982): Discontinuous equilibrium solutions and cavitations in nonlinear elasticity. *Philos. Trans. Roy. Soc. Lond. Ser.* A, vol.306, pp.557-611.

Chou-Wang, M-S; Horgan, C. O. (1989): Cavitation in nonlinearly elastodynamics for neo-Hookean materials. *Int J Eng Sci.*, vol.27, pp.967–973.

Gent, A. N.; Lindley, P. B. (1958): Internal rupture of bonded rubber cylinders in tension. *Proc. R. Soc. London* A, vol.249, pp.195–205.

Gent, A. N. (1990): Cavitation in rubber: a cautionary tale. *Rubber Chem Thech*, vol. 63, pp. G49-G53.

Horgan, C. O.; Polignone, D. A. (1995): Cavitation in nonlinear elastic solids: A review. *Appl. Mech. Rev.*, vol. 48, pp. 471-485.

Ling, X. W., Atluri, S. N. (2008): A hyperelastic description of single wall carbon nanotubes at moderate strains and temperatures. *CMES: Computer Modeling in Engineering & Sciences*, vol.21, pp. 81-91.

Murphy, J. G.; Biwa, S. (1997): Nonmontonic cavity growth in finite compressible elasticity. *Int. J. Solids Structures*, vol.34, pp.3859–3872.

Polignone, D. A.; Horgan, C. O. (1993): Cavitation for incompressible nonlinearly elastic spheres. *J. Elasticity*, vol. 33, pp.27–65.

Ren, J. S.; Cheng, C. J. (2002a): Cavity bifurcation for incompressible hyperelastic material. *Appl. Math. Mech.*, vol. 23, pp. 881–888

Ren, J. S.; Cheng, C. J. (2002b): Bifurcation of cavitation solutions for incompressible transversely isotropic hyper-elastic materials. *J. Engin. Math*, vol.44, pp. 245–257.

Ren, J. S., Cheng, C. J. (2003): Dynamical formation of cavity in transversely hyperelastic spheres. *Acta Mechanica Sinica*, vol.19, pp. 320–323.

Shang, X. C.; Cheng, C. J. (2001): Exact Solution for cavitated bifurcation for compressible hyperelastic materials. *Int. J. Engin. Sci.*, vol.39, pp.1101–1117.

Valanis, K. C.; Landel, R. F. (1967): The strain energy function of a hyper-elastic material in terms of the extension rations. *J. Appl. Physics*, vol.38, pp.2997–3002.

Yuan, X. G.; Zhu, Z. Y.; Cheng, C. J. (2004a): Qualitative analyses of spherical cavity nucleation and growth for incompressible generalized Valanis-Landel hyper-elastic material. *Acta Mech. Solida Sinica*, vol.17, pp.158–165.

Yuan, X. G.; Zhu, Z. Y.; Cheng, C. J. (2004b): Cavity formation and its vibration for a class of generalized incompressible hyper-Elastic materials. *Acta Mechanica Solida Sinica*, vol.17, pp.361–369.

Yuan, X. G.; Zhu, Z. Y.; Cheng, C. J. (2005): Study on Cavitated Bifurcation problems for Spheres Composed of Hyper-elastic Materials. *J. Engin. Math.*, vol.51, pp. 15-34.

Yuan, X. G.; Zhang, R. J. (2006): Nonlinear dynamical analysis in incompressible transversely isotropic nonlinearly elastic materials: Cavity formation and motion in solid sphe- res. *CMC: Computers, Materials & Continua*, vol. 3, pp. 119-129.

Yuan, X. G.; Zhu, Z. Y.; Cheng, C. J. (2007): Dynamical analysis of cavitation for a transversely isotropic incompre-ssible hyper-elastic medium: periodic motion of a pre-existing micro-void. *Int. J. Nonlinear Mech.*, vol. 42, pp. 442-449.