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# Exact Large Deflection of Beams with Nonlinear Boundary Conditions

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**Abstract:** An analytic solution method, namely the shifting function method, is developed to find the exact large static deflection of a beam with nonlinear elastic springs supports at ends for the first time. The associated mathematic system is a fourth order ordinary differential equation with nonlinear boundary conditions. It is shifted and decomposed into five linear differential equations and at most four algebra equations. After finding the roots of the algebra equations, the exact solution of the nonlinear beam system can be reconstructed. It is shown that the proposed method is valid for the problem with strong nonlinearity. Finally, examples and limiting studies are given to illustrate the analysis.

**Keyword:** beams, large deflection, nonlinear boundary conditions, shifting function method

## 1 Introduction

The applications of beam structures can be widely found in all the engineering fields. Based on the linear theory, the studies on the static and dynamic response of beam structures are tremendous [Timoshenko (1955); Meirovitch (1967); Lee and Hsu (2007); Lin, Lee and Lin (2008); Vinod, Gopalakrishnan and Ganguli (2006); Lee and Kuo (1992); Lee and Lin (1996); Huang and Shih (2007); Andreaus, Batra and Porfiri (2005); Beda (2003)]. When the physical properties of a beam structure are uniform, the associated governing differential equation is a linear fourth-order differential equation with constant coefficients. The exact solution can be found in many standard text books [Timoshenko (1955); Meirovitch (1967)]. For rotating beams [Lee and Hsu (2007); Lin, Lee and Lin (2008); Vinod Gopalakrishnan and Ganguli (2006)] or beams with non-uniform physical properties [Lee and Kuo (1992); Lee and Lin (1996)], the associated governing differential equation is a linear fourth-order differential equation with variable coefficients. Some of the exact solutions for the beams can be found in the works done by Lee and Hsu (2007), Lin, Lee and Lin (2008), Lee and Kuo (1992), Lee and Lin (1996). In addition, various kinds of numerical methods were employed to study the problems.

In the non-linear analysis, Emam and Nayfeh (2004), and Saffari, Rahgozar and Tabatabaei (2007) studied the beam problems with geometry nonlinearity. Monasa and Lewis (1983) studied the beam problems with material nonlinearity. The problems for a beam resting on nonlinear elastic foundation were examined by Kuo and Lee (1994) and Coskun (2000). Ma and Silva (2004), Turner (2004), Wolf and Gottlieb (2001), Fung and Huang (2001) and Kuang and Chen (2005) investigated the response of a beam with nonlinear elastic boundary conditions.

It is well known that, in general, the exact solutions for the nonlinear beam problems are not available. The problems were mainly solved by approximated methods such as: the perturbation method [Monasa and Lewis (1983); Kuo and Lee (1994); Wolf and Gottlieb (2001)], the iterative method [Ma and Silva (2004)], the Galerkin's method [Emam and Nayfeh (2004); Cao and Zhang (2005); Lee and Soh (1994)], the finite element method [Saffari, Rahgozar and Tabatabaei (2007); Fung and Huang (2001)] and the Ado-

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mian decomposition method [Kuang and Chen (2005)]. One exact static deflection solution for a beam with particularly designed nonlinear boundary conditions was found in the paper by Ma and Silva (2004).

From the existing literature, it can be found that a systematic analytical method to find the exact solutions for the deflection of a beam with various types nonlinear elastic boundary conditions still is not available. In this paper, a systematic analytical method which is an extension of the shifting function method developed by Lee and Lin (1996) is developed to find the exact large deflection solutions for beams with nonlinear elastically restrained end supports. The associated nonlinear mathematic system is changed and decomposed into five linear differential equations and four algebra equations. After finding the roots of the algebra equations, the exact solution of the nonlinear beam system can be reconstructed. The proposed method is valid for the problem with strong nonlinearity. Finally, examples and limiting studies are given to illustrate the method.

## 2 Mathematical Modeling of the Beam System

Consider the static deflection of a uniform Bernoulli-Euler beam resting on linear elastic foundation with nonlinear elastic boundary conditions, as shown in Figure 1.

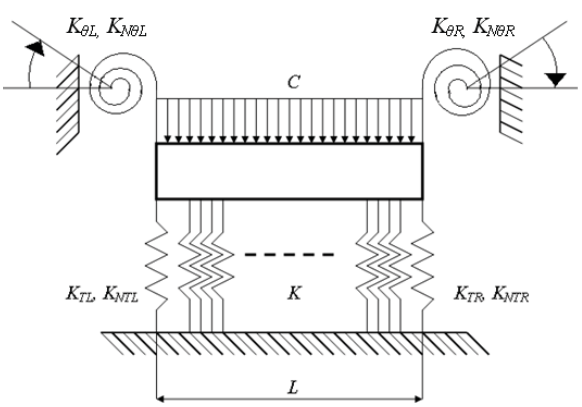


Figure 1: Geometry and coordinate system of a uniform beam with non-linear elastic boundary conditions

In terms of the following non-dimensional quantities,

$$\begin{aligned}
&\xi = \frac{x}{L}, \qquad W(\xi) = \frac{y(x)}{L}, \quad Q(x) = \frac{q(x)L^3}{EI},\\
&\beta_1 = \frac{K_{\theta L}L}{EI}, \qquad \beta_2 = \frac{K_{TL}L^3}{EI}, \quad \beta_3 = \frac{K_{\theta R}L}{EI},\\
&\beta_4 = \frac{K_{TR}L^3}{EI}, \qquad K = \frac{kL^4}{EI}, \qquad \gamma_1 = \frac{K_{N\theta L}L}{EI},
\end{aligned} \tag{1}$$

$$\gamma_2 = \frac{K_{NTL}L^5}{EI}, \qquad \gamma_3 = \frac{K_{N\theta R}L}{EI}, \quad \gamma_4 = \frac{K_{NTR}L^5}{EI},$$

the governing differential equation of the system is

$$\frac{d^4W(\xi)}{d\xi^4} + KW(\xi) = Q(\xi), \quad \xi \in (0,1), \tag{2}$$

and the associated boundary conditions are at $\xi = 0$:

$$\frac{d^2W}{d\xi^2} - \beta_1\frac{dW}{d\xi} - \gamma_1\left(\frac{dW}{d\xi}\right)^3 = 0, \tag{3}$$

$$\frac{d^3W}{d\xi^3} + \beta_2 W + \gamma_2 W^3 = 0, \tag{4}$$

at $\xi = 1$:

$$\frac{d^2W}{d\xi^2} + \beta_3\frac{dW}{d\xi} + \gamma_3\left(\frac{dW}{d\xi}\right)^3 = 0, \tag{5}$$

$$\frac{d^3W}{d\xi^3} - \beta_4 W - \gamma_4 W^3 = 0. \tag{6}$$

Here, $y(x)$ is the flexural displacement, $x$ is the space variable along the beam, $EI$ is the flexural rigidity, $k$ is the elastic foundation modulus, and $q(x)$ is the applied transverse force per unit length. $K_{TL}$, $K_{\theta L}$, $K_{TR}$ and $K_{\theta R}$ are the linear translational spring constants and the linear rotational spring constants at the left end and the right end of the beam, respectively. $K_{NTL}$, $K_{N\theta L}$, $K_{NTR}$ and $K_{N\theta R}$ are the nonlinear translational spring constants and the nonlinear rotational spring constants at the left end and the right end of the beam, respectively.

## 3 The Shifting Function Method

### 3.1 Change of variable

To find the solution for the fourth order differential equation with nonlinear elastic boundary conditions, one employs the shifting variable method developed by Lee and Lin (1996) by taking

$$W(\xi) = V(\xi) + \sum_{i=1}^{4} \overline{f}_i g_i(\xi), \tag{7}$$

where

$$\overline{f}_1 = -\gamma_1 \left(\frac{dW(0)}{d\xi}\right)^3, \tag{8}$$

$$\overline{f}_2 = -\gamma_2 (W(0))^3, \tag{9}$$

$$\overline{f}_3 = -\gamma_3 \left(\frac{dW(1)}{d\xi}\right)^3, \tag{10}$$

$$\overline{f}_4 = -\gamma_4 (W(1))^3, \tag{11}$$

and $g_i(\xi)$, $i$ =1, 2, 3, 4 are the shifting functions to be specified, $V(\xi)$ is the transformed function. Substituting equations (7-11) into equations (2-6), one has the differential equation for $V(\xi)$

$$\frac{d^4V(\xi)}{d\xi^4} + KV(\xi) = Q(\xi) - \sum_{i=1}^{4} \overline{f}_i \left\{\frac{d^4 g_i(\xi)}{d\xi^4} + Kg_i(\xi)\right\}, \quad \xi \in (0,\ 1) \tag{12}$$

and the associated boundary conditions at $\xi$ = 0:

$$\left(\frac{d^2V}{d\xi^2} - \beta_1 \frac{dV}{d\xi}\right) = -\overline{f}_1 - \sum_{i=1}^{4} \overline{f}_i \left[\left(\frac{d^2 g_i}{d\xi^2} - \beta_1 \frac{dg_i}{d\xi}\right)\right], \tag{13}$$

$$\left(\frac{d^3V}{d\xi^3} + \beta_2 V\right) = \overline{f}_2 - \sum_{i=1}^{4} \overline{f}_i \left[\left(\frac{d^3 g_i}{d\xi^3} + \beta_2 g_i\right)\right], \tag{14}$$

at $\xi$ = 1:

$$\left(\frac{d^2V}{d\xi^2} + \beta_3 \frac{dV}{d\xi}\right) = \overline{f}_3 - \sum_{i=1}^{4} \overline{f}_i \left[\left(\frac{d^2 g_i}{d\xi^2} + \beta_3 \frac{dg_i}{d\xi}\right)\right], \tag{15}$$

$$\left(\frac{d^3V}{d\xi^3} - \beta_4 V\right) = -\overline{f}_4 - \sum_{i=1}^{4} \overline{f}_i \left[\left(\frac{d^3 g_i}{d\xi^3} - \beta_4 g_i\right)\right]. \tag{16}$$

### 3.2 Shifting Functions

If the shifting functions $g_i(\xi)$, $i$ =1, 2, 3, 4 in equation (7) are chosen to satisfy the differential equation

$$\frac{d^4 g_i(\xi)}{d\xi^4} + kg_i(\xi) = 0, \tag{17}$$

and the following boundary conditions

$$-\left(\frac{d^2 g_i}{d\xi^2} - \beta_1 \frac{dg_i}{d\xi}\right)\bigg|_{\xi=0} = \delta_{ij}, \quad j = 1, \tag{18}$$

$$\left(\frac{d^3 g_i}{d\xi^3} + \beta_2 g_i\right)\bigg|_{\xi=0} = \delta_{ij}, \quad j = 2, \tag{19}$$

$$\left(\frac{d^2 g_i}{d\xi^2} + \beta_3 \frac{dg_i}{d\xi}\right)\bigg|_{\xi=1} = \delta_{ij}, \quad j = 3, \tag{20}$$

$$-\left(\frac{d^3 g_i}{d\xi^3} - \beta_4 g_i\right)\bigg|_{\xi=1} = \delta_{ij}, \quad j = 4, \tag{21}$$

where $\delta_{ij}$ is a Kronecker symbol, then the differential equation (12) and the associated boundary conditions (13 -16) can be reduced to

$$\frac{d^4V(\xi)}{d\xi^4} + KV(\xi) = Q(\xi), \tag{22}$$

at $\xi$ = 0:

$$\left(\frac{d^2V}{d\xi^2} - \beta_1 \frac{dV}{d\xi}\right) = 0,$$

$$\frac{1}{1+\beta_2}\left(\frac{\partial^3 V}{\partial \xi^3} + n\frac{\partial V}{\partial \xi} + \beta_2 V\right) = \overline{f}_2 - \overline{f}_i \left[\frac{1}{1+\beta_2}\left(\frac{\partial^3 g}{\partial \xi^3} + n\frac{\partial g}{\partial \xi} + \beta_2 g\right)\right] \tag{23}$$

$$\left(\frac{d^3V}{d\xi^3}+\beta_2 V\right)=0, \tag{24}$$

at $\xi$ = 1:

$$\left(\frac{d^2V}{d\xi^2}+\beta_3\frac{dV}{d\xi}\right)=0, \tag{25}$$

$$\left(\frac{d^3V}{d\xi^3}-\beta_4 V\right)=0. \tag{26}$$

Hence, the associated mathematic system is changed and decomposed into five linear differential equations and the associated linear boundary conditions. The exact solution form for the transformed function $V(\xi)$ can be found the paper by Lee and Kuo (1992) and the shifting functions $g_i(\xi)$, $i$ =1, 2, 3, 4 are given in the Appendix A.

Once the transformed function $V(\xi)$ and the shifting functions $g_i(\xi)$, $i$ = 1, 2, 3, 4 are determined, one substitutes these functions into equation (7). It leads to

$$\begin{aligned}W(\xi)=V(\xi)&-\gamma_1\left(\frac{dW(0)}{d\xi}\right)^3 g_1(\xi)\\&-\gamma_2\left(W(0)\right)^3 g_2(\xi)\\&-\gamma_3\left(\frac{dW(1)}{d\xi}\right)^3 g_3(\xi)-\gamma_4\left(W(1)\right)^3 g_4(\xi),\end{aligned} \tag{27}$$

where $\frac{dW(0)}{d\xi}$, $W(0)$, $\frac{dW(1)}{d\xi}$, $W(1)$ are four constants to be determined.

Differentiating equation (27) once and letting $\xi$ = 0 and $\xi$ = 1, respectively, one has the following algebra equations

$$\begin{aligned}\frac{dW(0)}{d\xi}=\frac{dV(0)}{d\xi}&-\gamma_1 S^3\frac{dg_1(0)}{d\xi}-\gamma_2 Y^3\frac{dg_2(0)}{d\xi}\\&-\gamma_3 Z^3\frac{dg_3(0)}{d\xi}-\gamma_4 U^3\frac{dg_4(0)}{d\xi},\end{aligned} \tag{28}$$

$$\begin{aligned}W(0)=V(0)&-\gamma_1 S^3 g_1(0)-\gamma_2 Y^3 g_2(0)\\&-\gamma_3 Z^3 g_3(0)-\gamma_4 U^3 g_4(0),\end{aligned} \tag{29}$$

$$\begin{aligned}\frac{dW(1)}{d\xi}=\frac{dV(1)}{d\xi}&-\gamma_1 S^3\frac{dg_1(1)}{d\xi}-\gamma_2 Y^3\frac{dg_2(1)}{d\xi}\\&-\gamma_3 Z^3\frac{dg_3(1)}{d\xi}-\gamma_4 U^3\frac{dg_4(1)}{d\xi},\end{aligned} \tag{30}$$

$$\begin{aligned}W(1)=V(1)&-\gamma_1 S^3 g_1(1)-\gamma_2 Y^3 g_2(1)\\&-\gamma_3 Z^3 g_3(1)-\gamma_4 U^3 g_4(1).\end{aligned} \tag{31}$$

After finding the roots of the four algebra equations (28 - 31), the exact solution of the nonlinear beam system can be reconstructed from equation (27).

From equations (7-11, 22-26, 27), it can be observed that total solution is the superposition of the linear and the nonlinear parts of the solution. The transformed function $V(\xi)$ is corresponding to the solution of the associated linear system. The rest of terms in equation (7) are contributed from the nonlinear parts of the boundary conditions.

From equations (17-21), one can find that the shifting functions $g_i(\xi)$, $i$ = 1, 2, 3, 4 takes the physical meanings as the non-dimensional static deflection of a general elastically restrained beam subjected to a unit non-dimensional moment and a unit non-dimensional slope of the base at the left end, a unit non-dimensional shear force and a unit non-dimensional displacement of the base at the left end, a unit non-dimensional moment and a unit non-dimensional slope of the base at the right end, a unit non-dimensional shear force and a unit non-dimensional displacement of the base at the right end, respectively.

## 4 Verification and Examples

To verify the previous analysis, the following examples and limiting studies are illustrated.

**Example 1**: Consider the beam problem discussed by Ma and Silva (2004). The governing differential equation is

$$\frac{d^4W(\xi)}{d\xi^4}=72\xi^2-\frac{2784}{61}\xi-\frac{48}{61}, \tag{32}$$

and the boundary conditions are at $\xi$ = 0:

$$\frac{dW}{d\xi}=0,\quad W=0, \tag{33}$$

at $\xi$ = 1:

$$\frac{d^2W}{d\xi^2}=0,\ \frac{d^3W}{d\xi^3}=\frac{24\sin W}{61\sin\left(\frac{48}{61}\right)}. \tag{34}$$

It can be found that one of the boundary condition at $\xi$ = 1 is nonlinear and specially designed.

To find the solution, one lets

$$W(\xi) = V(\xi) + \overline{f}_4 g_4(\xi), \tag{35}$$

where

$$\overline{f}_4 = -\frac{24\sin W(1)}{61\sin\left(\frac{48}{61}\right)}. \tag{36}$$

Here $g_4(\xi)$ is the shifting function to be specified and $V(\xi)$ is the transformed function which satisfies the differential equation

$$\frac{d^4V(\xi)}{d\xi^4} = 72\xi^2 - \frac{2784}{61}\xi - \frac{48}{61}, \tag{37}$$

and the homogeneous boundary conditions at $\xi$ = 0:

$$\frac{dV}{d\xi} = 0, \quad V = 0,$$

$$\frac{1}{1+\beta_2}\left(\frac{\partial^3 V}{\partial\xi^3} + n\frac{\partial V}{\partial\xi} + \beta_2 V\right) = \overline{f}_2 - \overline{f}_i\left[\frac{1}{1+\beta_2}\left(\frac{\partial^3 g}{\partial\xi^3} + n\frac{\partial g}{\partial\xi} + \beta_2 g\right)\right] \tag{38}$$

at $\xi$ = 1:

$$\frac{d^2V}{d\xi^2} = 0, \ \frac{d^3V}{d\xi^3} = 0. \tag{39}$$

The transformed function $V(\xi)$ is determined as

$$V(\xi) = \frac{73}{61}\xi^2 - \frac{4}{61}\xi^3 - \frac{2}{61}\xi^4 - \frac{116}{305}\xi^5 + \frac{1}{5}\xi^6. \tag{40}$$

The shifting function $g_4(\xi)$ satisfies the following differential equation and the homogeneous boundary conditions:

$$\frac{d^4 g_4(\xi)}{d\xi^4} = 0, \tag{41}$$

at $\xi$ = 0:

$$\frac{dg_4}{d\xi} = 0, \quad g_4 = 0, \tag{42}$$

at $\xi$ = 1:

$$\frac{d^2 g_4}{d\xi^2} = 0, \ \frac{d^3 g_4}{d\xi^3} = 1. \tag{43}$$

It can be found that this shifting function $g_4(\xi)$ is

$$g_4(\xi) = \frac{1}{2}\xi^2 - \frac{1}{6}\xi^3. \tag{44}$$

Substituting the transformed function $V(\xi)$, equation (40), and the shifting function $g_4(\xi)$, equation (44), back into equation (35), one has

$$W(\xi) = \frac{73}{61}\xi^2 - \frac{4}{61}\xi^3 - \frac{2}{61}\xi^4 - \frac{116}{306}\xi^5 + \frac{1}{5}\xi^6 - \frac{24\sin W(1)}{61\sin\left(\frac{48}{61}\right)}\left(\frac{1}{2}\xi^2 - \frac{1}{6}\xi^3\right), \tag{45}$$

Setting $\xi$ = 1 in equation (45) and solving the algebra equation, one obtains $W(1)$ = 48/61. As a result,

$$W(\xi) = \frac{1}{5}\xi^6 - \frac{116}{306}\xi^5 - \frac{2}{61}\xi^4 + \xi^2. \tag{46}$$

The solution is exactly the same as the one given by Ma and Silva (2004).

**Example 2**: Consider the deflection of a beam subjected to uniform distributed load $C$. The beam is clamped at the left end and is nonlinear translational spring supported at the other end.

The governing differential equation and the boundary conditions are:

$$\frac{d^4W(\xi)}{d\xi^4} = C, \tag{47}$$

at $\xi$ = 0:

$$\frac{dW}{d\xi} = 0, \quad W = 0,$$

$$\frac{1}{1+\beta_2}\left(\frac{\partial^3 V}{\partial\xi^3} + n\frac{\partial V}{\partial\xi} + \beta_2 V\right) = \overline{f}_2 - \overline{f}_i\left[\frac{1}{1+\beta_2}\left(\frac{\partial^3 g}{\partial\xi^3} + n\frac{\partial g}{\partial\xi} + \beta_2 g\right)\right] \tag{48}$$

at $\xi$ = 1:

$$\frac{d^2W(\xi)}{d\xi^2} = 0, \ \frac{d^3W(\xi)}{d\xi^3} - k_1 W(\xi) - k_2 W^3(\xi) = 0.$$

$$(49)$$

One lets

$$W(\xi) = V(\xi) + \overline{f}_4 g_4(\xi), \tag{50}$$

where

$$\overline{f}_4 = k_2 W^3(1). \tag{51}$$

Following the procedures revealed in the last section, one has

$$V(\xi) = \left(\frac{k_1+12}{16k_1+48}\right)C\xi^2 - \left(\frac{5k_1+24}{48k_1+144}\right)C\xi^3 + \frac{1}{24}C\xi^4, \tag{52}$$

and

$$g_4(\xi) = -\frac{3}{2k_1+6}\xi^2 + \frac{1}{2k_1+6}\xi^3. \tag{53}$$

After substituting the two functions above back to equation (50), one has

$$\begin{aligned} W(\xi) = &\left(\frac{k_1+12}{16k_1+48}\right)C\xi^2 - \left(\frac{5k_1+24}{48k_1+144}\right)C\xi^3 \\ &+ \frac{1}{24}C\xi^4 \\ &+ k_2W^3(1)\left(-\frac{3}{2k_1+6}\xi^2 + \frac{1}{2k_1+6}\xi^3\right). \end{aligned} \tag{54}$$

Setting $\xi = 1$ in the equation (54), one has

$$48k_2W^3(1) + (48k_1+144)W(1) = 18C. \tag{55}$$

Using the Cardano formula, one obtains

$$\begin{aligned} W(1) = &\frac{q}{(3)^{1/3}\left(81Cp^2+\sqrt{3}\sqrt{2187C^2p^4+p^3q^3}\right)^{1/3}}, \\ &+\frac{\left(81Cp^2+\sqrt{3}\sqrt{2187C^2p^4+p^3q^3}\right)^{1/3}}{(3)^{2/3}p} \end{aligned} \tag{56}$$

where $p = 48k_2$, $q = 48k_1 + 144$, $k_2 \neq 0$. After substituting it back to equation (54), one obtains the exact solution of the problem.

When $k_1 = 0$, equation (54) is reduced to

$$\begin{aligned} W(\xi) = &\frac{1}{4}C\xi^2 - \frac{1}{6}C\xi^3 + \frac{1}{24}C\xi^4 \\ &+ k_2W^3(1)\left(-\frac{1}{2}\xi^2 + \frac{1}{6}\xi^3\right), \end{aligned} \tag{57}$$

where

$$\begin{aligned} W(1) = &\frac{2(2)^{1/3}}{\left(-3Ck_2^2+\sqrt{256k_2^3+9C^2k_2^4}\right)^{1/3}} \\ &-\frac{\left(-3Ck_2^2+\sqrt{256k_2^3+9C^2k_2^4}\right)^{1/3}}{2(2)^{1/3}k_2}. \end{aligned} \tag{58}$$

When $k_2 = 0$, the system turns to be a linear problem. Equation (54) is reduced to

$$W(\xi) = \left(\frac{k_1+12}{16k_1+48}\right)C\xi^2 - \left(\frac{5k_1+24}{48k_1+144}\right)C\xi^3 + \frac{1}{24}C\xi^4. \tag{59}$$

When $k_1 = 0$ and $k_2 = 0$, equation (58) is further reduced to

$$W(\xi) = \frac{1}{4}C\xi^2 - \frac{1}{6}C\xi^3 + \frac{1}{24}C\xi^4. \tag{60}$$

It is the exact static deflection of a cantilevered beam subjected to uniformly distributed load $C$.

**Example 3**: Consider the deflection of a beam subjected to uniformly distributed load $C$ with clamped one end and nonlinear rotational spring support at the other end. The governing differential equation is

$$\frac{d^4W(\xi)}{d\xi^4} = C, \tag{61}$$

and the boundary conditions are: at $\xi = 0$:

$$\frac{dW}{d\xi} = 0, \quad W = 0,$$

$$\frac{1}{1+\beta_2}\left(\frac{\partial^3 V}{\partial \xi^3} + n\frac{\partial V}{\partial \xi} + \beta_2 V\right) =$$

$$\overline{f}_2-\overline{f}_i\left[\frac{1}{1+\beta_2}\left(\frac{\partial^3 g}{\partial\xi^3}+n\frac{\partial g}{\partial\xi}+\beta_2 g\right)\right] \quad (62)$$

at $\xi$ = 1:

$$\frac{d^2W(\xi)}{d\xi^2}-k_1\frac{dW(\xi)}{d\xi}-k_2\left(\frac{dW(\xi)}{d\xi}\right)^3=0, \quad (63)$$

$$\frac{d^3W(\xi)}{d\xi^3}=0,$$

One lets

$$W(\xi)=V(\xi)+\overline{f}_3 g_3(\xi), \quad (64)$$

where

$$\overline{f}_3=k_2\left(\frac{dW(1)}{d\xi}\right)^3. \quad (65)$$

Following the procedures revealed in the last section, one has

$$V(\xi)=\left(\frac{2k_1-3}{12k_1-12}\right)C\xi^2-\frac{1}{6}C\xi^3+\frac{1}{24}C\xi^4, \quad (66)$$

and

$$g_3(\xi)=\frac{1}{2-2k_1}\xi^2. \quad (67)$$

After substituting the two functions above back to equation (64), one has

$$W(\xi)=\left(\frac{2k_1-3}{12k_1-12}\right)C\xi^2-\frac{1}{6}C\xi^3+\frac{1}{24}C\xi^4 +k_2\left(\frac{dW(1)}{d\xi}\right)^3\left(\frac{1}{2-2k_1}\xi^2\right). \quad (68)$$

Differentiating equation (68) once and setting $\xi$ = 1 in the equation, one has

$$k_2\left(\frac{dW(1)}{d\xi}\right)^3+(k_1-1)\frac{dW(1)}{d\xi}=-\frac{1}{6}C. \quad (69)$$

Using the Cardano formula, one obtains

$$\frac{dW(1)}{d\xi}=-\left[2^{1/3}6(-1+k_1)\right]\Big/\left[\left(-972Pk_2^2+\sqrt{5038848(-1+k_1)^3k_2^3+944784P^2k_2^4}\right)^{1/3}\right] +\left[\left(-972Pk_2^2+\sqrt{5038848(-1+k_1)^3k_2^3+944784P^2k_2^4}\right)^{1/3}\right]\Big/\left[2^{1/3}18k_2\right], \quad (70)$$

where $k_2 \neq 0$. After substituting it back to equation (68), one obtains the exact solution of the problem.

When $k_2$ = 0, the system turns to be a linear problem. Equation (68) is reduced to

$$W(\xi)=\left(\frac{2k_1-3}{12k_1-12}\right)C\xi^2-\frac{1}{6}C\xi^3+\frac{1}{24}C\xi^4. \quad (71)$$

Following the same procedures, one can easily develop the deflection of a nonlinear spring supported beam subjected to various kinds of loads. The nonlinear deflection of a beam subjected to two concentrated loads with clamped one end and nonlinear translational spring support at the other end is given in the Appendix B.

## 5 Conclusions

In this paper, an analytic solution method, namely the shifting function method, is developed to find the exact large deflection of a beam structure with nonlinear elastic springs supports at ends for the first time. It is shown that the method can be applied to a wide class of problems with nonlinear boundary conditions. The proposed method is also valid for the problem with strong nonlinearity. It will be interesting to extend the proposed solution method to study the large deflection of a nonlinear Timoshenko beam, the dynamic response of nonlinear beams and the nonlinear response of non-homogeneous beams.

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**Appendix A: The shifting functions $g_i(\xi)$**

The four shifting functions are

$$g_i(\xi) = C_{i,1}G_1(\xi) + C_{i,2}G_2(\xi) + C_{i,3}G_3(\xi) + C_{i,4}G_4(\xi), \quad i = 1,2,3,4 \quad \text{(A1)}$$

where $C_{i,1}$, $C_{i,2}$, $C_{i,3}$ and $C_{i,4}$ are constants to be determined and the four fundamental solutions of differential equation (17), $G_j(\xi)$, $j$= 1,2 3, 4 are

$$G_1 = \frac{1}{2}[\cosh \alpha x + \cos \alpha x], \quad \text{(A2)}$$

$$G_2 = \frac{1}{2\alpha}[\sinh \alpha x + \sin \alpha x], \quad \text{(A3)}$$

$$G_3 = \frac{1}{2\alpha^2}[\cosh \alpha x - \cos \alpha x], \quad \text{(A4)}$$

$$G_4 = \frac{1}{2\alpha^3}[\sinh \alpha x - \sin \alpha x]. \quad \text{(A5)}$$

Here, $\alpha$ is a complex number and is any one of the four roots of

$$\alpha^4 = -\frac{K}{EI}. \quad \text{(A6)}$$

These four fundamental solutions satisfy the following normalization condition at the origin of the coordinate system

$$\begin{bmatrix} G_1(0) & G_2(0) & G_3(0) & G_4(0) \\ G_1'(0) & G_2'(0) & G_3'(0) & G_4'(0) \\ G_1''(0) & G_2''(0) & G_3''(0) & G_4''(0) \\ G_1'''(0) & G_2'''(0) & G_3'''(0) & G_4'''(0) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \text{(A7)}$$

where primes indicate differentiation with respect to $\xi$., When $K = 0$, the four fundamental solutions are reduced to

$$G_1 = 1, \quad \text{(A8)}$$

$$G_2 = x, \quad \text{(A9)}$$

$$G_3 = \frac{x^2}{2}, \quad \text{(A10)}$$

$$G_4 = \frac{x^3}{6}. \quad \text{(A11)}$$

After substituting $g_i(\xi)$ into the boundary conditions (18-21), the constants $C_{i,1}$, $C_{i,2}$, $C_{i,3}$ and $C_{i,4}$ for the general case are given as:

$$C_{1,1} = [(R_2T_3 - R_3T_2)]/Q,$$

$$C_{1,2} = [-\beta_2(R_3T_4 - R_4T_3) - (R_1T_3 - R_3T_1)]/Q,$$
$$C_{1,3} = [\beta_2(R_2T_4 - R_4T_2) + (R_1T_2 - R_2T_1)]/Q,$$
$$C_{1,4} = [-\beta_2(R_2T_3 - R_3T_2)]/Q,$$
$$C_{2,1} = -[\beta_1(R_3T_4 - R_4T_3) + (R_2T_4 - R_4T_2)]/Q,$$
$$C_{2,2} = (R_1T_4 - R_4T_1)/Q,$$
$$C_{2,3} = \beta_1(R_1T_4 - R_4T_1)/Q,$$
$$C_{2,4} = -[\beta_1(R_1T_3 - R_3T_1) + (R_1T_2 - R_2T_1)]/Q,$$
$$C_{3,1} = -T_2/Q, \quad C_{3,2} = -(\beta_2T_4 - T_1)/Q,$$
$$C_{3,3} = -\beta_1(\beta_2T_4 - T_1)/Q,$$
$$C_{3,4} = [\beta_1\beta_2T_3 + \beta_2T_2]/Q,$$
$$C_{4,1} = [\beta_1R_3 + R_2]/Q, \quad C_{4,2} = (\beta_2R_4 - R_1)/Q,$$
$$C_{4,3} = \beta_1(\beta_2R_4 - R_1)/Q,$$
$$C_{4,4} = -[\beta_1\beta_2R_3 + \beta_2R_2]/Q,$$
$$Q = \beta_1[-\beta_2(R_3T_4 - R_4T_3) - (R_1T_3 - R_3T_1)] + [-\beta_2(R_2T_4 - R_4T_2) - (R_1T_2 - R_2T_1)],$$
$$R_i = \beta_3G_i'(1) + G_i''(1),$$
$$T_i = \beta_4G_i(1) - G_i'''(1), \quad i = 1,2,3,4$$

**Appendix B: Nonlinear deflection of a beam subjected to two concentrated loads with clamped one end and nonlinear translational spring support at the other end**

The governing differential equation for the static deflection of a beam subjected to two concentrated loads with clamped one end and nonlinear translational spring support at the other end is

$$\frac{d^4W(\xi)}{d\xi^4} = -R_a\delta(\xi - a) - R_b\delta(\xi - b), \quad 0 < a < b < 1, \quad \text{(B1)}$$

where $\delta(\xi)$ is the delta function. The boundary conditions are: at $\xi = 0$:

$$\frac{dW}{d\xi} = 0, \quad W = 0,$$

$$\frac{1}{1+\beta_2}\left(\frac{\partial^3 V}{\partial \xi^3}+n\frac{\partial V}{\partial \xi}+\beta_2 V\right)=$$
$$\overline{f}_2-\overline{f}_i\left[\frac{1}{1+\beta_2}\left(\frac{\partial^3 g}{\partial \xi^3}+n\frac{\partial g}{\partial \xi}+\beta_2 g\right)\right] \quad \text{(B2)}$$

at $\xi$ = 1:

$$\frac{d^2W(\xi)}{d\xi^2}=0,\ \frac{d^3W(\xi)}{d\xi^3}-k_2W^3(\xi)=0. \quad \text{(B3)}$$

Following the procedures as revealed in Example 2, the exact static deflection of the system is

$$\begin{aligned} W(\xi) = & -\frac{(aR_a+bR_b)}{2}\xi^2+\frac{R_a+R_b}{6}\xi^3 \\ & -R_a\frac{(\xi-a)^3}{6}u(\xi-a) \\ & -R_b\frac{(\xi-b)^3}{6}u(\xi-b) \\ & +k_2W^3(1)\left(-\frac{1}{2}\xi^2+\frac{1}{6}\xi^3\right), \end{aligned} \quad \text{(B4)}$$

where $u(\xi)$ is the Heaviside function. Setting $\xi$ = 1 in the equation (B4), and solving the algebra equation, one obtains

$$\begin{aligned} W(1) = & -\frac{\left(\frac{2}{3}\right)^{1/3}}{\left(k_2^2q+\sqrt{4k_2^3+k_2^4q^2}\right)^{1/3}} \\ & +\frac{\left(k_2^2q+\sqrt{4k_2^3+k_2^4q^2}\right)^{1/3}}{(2)^{1/3}k_2} \quad (k_2\neq 0). \end{aligned} \quad \text{(B5)}$$

Here,

$$q=\frac{a^2R_a}{2}(a-3)+\frac{b^2R_b}{2}(b-3). \quad \text{(B6)}$$

When $k_2$ = 0, equation (B4) is reduced

$$\begin{aligned} W(\xi) = & -\frac{(aR_a+bR_b)}{2}\xi^2+\frac{R_a+R_b}{6}\xi^3 \\ & -R_a\frac{(\xi-a)^3}{6}u(\xi-a) \\ & -R_b\frac{(\xi-b)^3}{6}u(\xi-b). \end{aligned} \quad \text{(B7)}$$

It is the exact solution of the linear problem.