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Dynamics Analysis of Mechanical Components: a Discrete Model For Damping

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Abstract: The Cell Method is a recent numerical method that can be applied in several fields of physics and engineering. In this paper, the elasto-dynamics formulation is extended to include system internal damping, highlighting some interesting characteristics of the method. The developed formulation leads to an explicit solving system. The mass matrix is diagonal (without lumping) and in the most general case a time-dependent damping coefficient can be defined for each node. Accuracy and convergence rate have been tested with reference to the classical problem of a particle free vibration with viscous damping.
An application to mechanical components analysis has been included in order to illustrate the potentialities of the method for fatigue behaviour assessment of mechanical parts.

Keyword: Numerical methods, Cell Method, elasto-dynamics, damping

1 Introduction

All vibrating structures dissipate energy and this phenomenon is named damping. Generally, analytical models simulate damping by means of forces that are a function of the structural response. Such forces are usually able to reproduce the structural behaviour, even when the actual energy dissipation mechanism is different.

Simulation of damping is subjected to computational constraints; the structure model usually includes the correct order of magnitude of damping but the damping distribution in the real structure can be quite different. Two approaches to damping modelling are used in the Finite Element Method (FEM) [Hitchings D. (1992)].

Modal damping is included in the model only after the definition of each vibration mode equations, so that the component damping can be related to the experimental values. This approach is possible only for modal form solutions.

Proportional damping models arbitrarily assume that damping is a linear combination of the component stiffness and mass matrixes. This hypothesis has no physical significance but is convenient from the computational point of view.

Notwithstanding the importance and diffusion of FEM, efforts are still required in order to improve solution accuracy and computational speed, supplying a motivation for the development of alternative algorithms. In this framework the Cell Method (CM) has been introduced [Tonti (2001a)]. Opposite to FEM, which can be regarded as a method for the discretization of balance equations written in differential form, CM stems from the effort of expressing static and dynamic equilibrium condition directly in a discrete form. Several applications of CM have been developed by a number of authors since its introduction, i.e. in elasticity problems [Cosmi(2001), Ferretti (2005)] and electrostatics [Heshmatzadeh and Bridges (2007)]. Dynamics formulations have been developed for acoustics in fluids by Tonti (2001b) and for fluid dynamics by Straface, Troisi and Gagliardi (2006), by Marrone, Frasson and Hernàndez-Figueroa (2002) in the electromagnetic field, and by Cosmi (2005) for the elastic analysis of components. Some characteristics of CM were pointed out in these works. The CM mass matrix is diagonal and an explicit solving system is obtained without solution degradation due to lumping. Both structured and unstructured meshes can be used with accurate results. Convergence rate is two, same as II order Runge Kutta method, but CM accuracy is better.

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In this work, the dynamic balance formulation developed by Cosmi (2005) has been modified to include system internal damping, highlighting further characteristics of CM, i.e. that, in the most general case, the developed formulation allows the definition of a time-dependent damping coefficient in each node.

In the next section, the addition of viscous damping forces to the original elastodynamics formulation is shown to result in a model in which both mass and damping terms are diagonal and an explicit solving system is obtained again. The formulation is directly written in discrete form and does not require the linear combination of stiffness and mass matrixes – in this sense a non-proportional damping model is devised.

The results of several simulations are discussed in the third section. In particular, a simple case for which the analytical exact solution is known in literature has been used to test the CM solution convergence rate and accuracy in comparison with II order Runge Kutta method.

The potentialities of the method in application for mechanical components analysis are illustrated in the fourth section.

In the last section the conclusions are drawn.

2 CM elastodynamics formulation with viscous damping forces

Previous works [Tonti (2001b), Cosmi (2005)] have shown that the most important advantages in the CM solution of elastodynamics problems can be summarized as follows:

- diagonal mass matrix and explicit solving system (no locking-induced errors);
- accuracy and stability of the results with both structured and unstructured meshes;
- reduced computation time and memory requirements also in the case of meshes with a very large number of elements.

In order to extend the elastodynamic formulation to include system internal damping, it is necessary to modify the dynamic balance equation to account for viscous damping forces, which are proportional to nodal velocity.

The system configuration, namely its geometry and kinematics, is described by nodes that in the Cell Method formulation are associated to a *primal cell complex*. Displacements, strain tensor, velocity are examples of variables associated with primal cell complexes, as shown in Fig. 1(a).

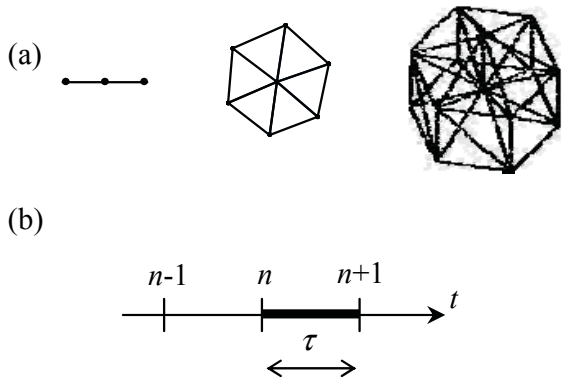


Figure 1: Examples of primal complex: (a) in 1D, 2D and 3D and (b) for the time variable.

Variables such as forces, momenta, stress tensor, which can be regarded as sources of the displacement field described by the configuration variables, are associated to a second cell complex, defined in a way that each node, or time instant, of the primal complex falls inside a cell of the *dual cell complex*. An example is shown in Fig.2(a). Each cell of this dual complex can be considered an influence region for the node inside.

When the Cell Method balance condition is written, it is not enforced on the node, but for the whole dual cell surrounding the node. The balance equation is therefore directly written in finite terms, using only global variables.

It can thus be noted that the Cell Method formulation imposes no limitations due to derivability conditions, in contrast with the requirements of differential formulations (used by FEM and other numerical methods), which enforce restrictions on the field equations, limitations that are not related to the physics of the problem.

In the CM formulation of a time-dependent problem the use of a dual complex is extended to the

time variable, as shown in Fig.1(b) and Fig.2(b).

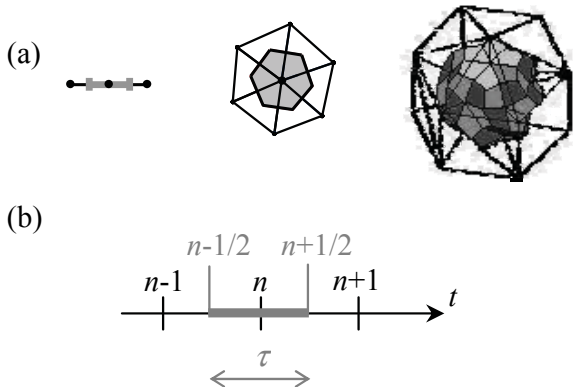


Figure 2: Examples of dual complex: (a) in 1D, 2D and 3D and (b) for the time variable.

The association rule for time-dependent variables can be summarized as follows.

Quantities whose sign is changed when time is reversed, such as displacements, are associated to elements of the primal time-complex, while variables that are invariant with respect to time-inversion, i.e. the impulse of a force, will be associated to elements of the dual time-complex. Moreover, the constant time step, τ, is assumed to be the same for the two complexes, so that time intervals in the primal complex will be associated to the corresponding central instant in the dual time complex and vice-versa.

With respect to the variables used, the following associations hold:

Displacements are associated to nodes of the primal cell complex and to instants of the primal time complex;

Velocities are associated to nodes of the primal cell complex and to intervals of the primal time complex, which are instants of the dual time complex, see Fig.2(b);

Forces are associated to the dual cell complex and to instants of the primal time complex;

Impulses are associated to the dual cell complex and to intervals of the dual time complex, which are instants of the primal time complex.

Velocity for a generic node h can be computed from the nodal displacements u_h^{n-1} and u_h^n evaluated at instants n-1 and n of the primal complex and is therefore associated to the instant n-1/2 of the corresponding dual time complex:

$$\mathbf{v}_h^{n-1/2} = \frac{1}{\tau}\left(\mathbf{u}_h^n - \mathbf{u}_h^{n-1}\right) \tag{1}$$

Assuming that the centre of mass of the dual cell is coincident with the node inside the dual cell, the linear momentum $p_h^{n-1/2}$ can be computed:

$$\mathbf{p}_h^{n-1/2} = \frac{m_h}{\tau}\left(\mathbf{u}_h^n - \mathbf{u}_h^{n-1}\right) \tag{2}$$

where m_h is the mass of the dual cell. This hypothesis is approximately verified for the nodes that don't rest on the boundary of the system, see Fig.1(a). It might introduce a larger error for the boundary nodes, but these are usually a very small percentage of the total number of nodes.

Dynamics balance imposes that the change in momentum of the dual cell is equal to the impulse. Keeping in mind the above associations, since momenta are computed at instants in the dual time complex, a difference between momenta will be associated to the corresponding instant of the primal time complex. Equilibrium can be then written for each of the N dual cells as

$$\mathbf{p}_h^{n+1/2} - \mathbf{p}_h^{n-1/2} = \tau\mathbf{T}_h^n + \tau\mathbf{F}_h^n \tag{3}$$

where, at instant n, $\mathbf{T}_h^n$ represents the surface forces acting on the sides of the influence region of node h and $\mathbf{F}_h^n$ are the resultant of the volume force acting on the influence region and the external forces applied on the influence region through its boundary.

The surface forces acting on the sides of the influence region of node h can be computed at instant n by integration of the stress components and collected in the expression

$$\{T\}^n = -[K]^n\{u\}^n \tag{4}$$

While meaning and properties of stiffness matrix $[K]^n$ are the same as in FEM, its terms depend on the choice adopted for the dual cell, which also influences convergence rate and accuracy. In this

work, barycentric dual cell complexes have been adopted. The corresponding expressions for $[K]^n$ were derived in detail in Cosmi (2005).

With the appropriate substitutions, equations (2) and (3) can be written as an explicit system that can be solved for each time-step:

$$\begin{cases} \{u\}^n = \{u\}^{n-1} + \tau\,[1/M]\,\{p\}^{n-1/2} \\ \{p\}^{n+1/2} = \{p\}^{n-1/2} - \tau[K]\,\{u\}^n + \tau\{F\}^n \end{cases} \tag{5}$$

where

$$[1/M] = \mathrm{diag}\,[1/m_1, ..., 1/m_h, ..., 1/m_N]$$

An application of this formulation for the dynamics analysis of mechanical components is discussed in the successive section 4.

In order to bring damping in the model, the dynamics balance equation (3) must be modified:

$$\mathbf{p}_h^{n+1/2} - \mathbf{p}_h^{n-1/2} = \tau\mathbf{T}_h^n + \tau\mathbf{F}_h^n + \tau\mathbf{V}_h^n \tag{6}$$

where $\mathbf{V}_h^n$ are the viscous damping forces acting on the dual cell of node h at instant n.

Since forces in eq. (6) are referred to instants of the primal time complex, while velocities are computed at instants of the dual time complex, the expression for a viscous force at instant n can be computed form the average velocity of node h at instants n+1/2 and n-1/2:

$$\begin{aligned} V_h^n &= -cv_h^n = -c\left(v_h^{n+1/2} + v_h^{n-1/2}\right) \\ &= -\frac{c}{2m_h}\left(p_h^{n+1/2} + p_h^{n-1/2}\right) \end{aligned} \tag{7}$$

where c represents the damping coefficient, numerically equal to the damping force when the velocity is equal to one.

With the appropriate substitutions, equations (2) and (6) can be re-written as:

$$\begin{cases} \{u\}^n = \{u\}^{n-1} + \tau\,[1/M]\,\{p\}^{n-1/2} \\ \{p\}^{n+1/2} = \\ \qquad [C]\left([D]\{p\}^{n-1/2} - \tau[K]\{u\}^n + \tau\{F\}^n\right) \end{cases} \tag{8}$$

where

$$[C] = \mathrm{diag}\left[\left(\frac{1}{1+\frac{\tau\cdot c_1}{2m_1}}\right), \ldots, \left(\frac{1}{1+\frac{\tau\cdot c_h}{2m_h}}\right), \ldots, \left(\frac{1}{1+\frac{\tau\cdot c_N}{2m_N}}\right)\right];$$

$$[D] = \mathrm{diag}\left[\left(1-\frac{\tau\cdot c_1}{2m_1}\right), \ldots, \left(1-\frac{\tau\cdot c_h}{2m_h}\right), \ldots, \left(1-\frac{\tau\cdot c_N}{2m_N}\right)\right].$$

Some important consequences emerge from the above. It is worth noting that, coherently with the Cell Method approach, the solving system has been directly written in discrete form without using a differential formulation. Therefore, the procedure described is not equivalent to a discretization of the operators.

The Cell Method approach to elastodynamics problems leads to a diagonal mass matrix and explicit solving system with no limitation as to the typology of the mesh, that can be both structured and unstructured [Cosmi (2005)]. This is also true when viscous damping forces are introduced in the model, as both mass and damping expressions appear as diagonal terms. The explicit system obtained is conditionally stable: Courant condition must be satisfied, i.e. the integration step must be smaller than the minimum period of time required for a disturbance to travel between two nodes of the mesh. Moreover, and quite important from the applicative point of view, it is not necessary in any way to introduce the damping matrix as a linear combination of stiffness and mass matrixes. In principle, it is therefore possible to define a unique, time-varying damping model in each single node of a component at no additional computational cost.

3 Convergence rate and accuracy

The problem of a particle free vibration with viscous damping [Thomson (1993), Timoshenko and Young, (1955)], was examined in order to test convergence rate and accuracy of Cell Method in comparison with other methods.

Let m be the mass of the body, $x(t)$ the coordinate determining the configuration of the system, c the damping coefficient and k the spring constant. The well-known balance equation

$$\ddot{x}(t) + 2\varsigma\omega_n\dot{x}(t) + \omega_n^2 x(t) = 0 \tag{9}$$

where $\omega_n^2 = k/m$ and $\varsigma\omega_n = c/2m$, then leads to the exact solution

$$\begin{aligned} &x = Ae^{-\varsigma\omega_n t}\sin\omega t, \\ &\omega^2 = \omega_n^2\left(1-\varsigma^2\right), \\ &A = \dot{x}(0)/\omega. \end{aligned} \tag{10}$$

In order to test the convergence rate and accuracy of the proposed formulation, eq. (8) is rewritten as

$$\begin{cases} x^n = x^{n-1} + \dfrac{\tau}{m}p^{n-1/2} \\ p^{n+1/2} = \left(\dfrac{1}{1+\dfrac{\tau\cdot c}{2m}}\right)\left[\left(1-\dfrac{\tau\cdot c}{2m}\right)p^{n-1/2} - \tau k x^n\right] \end{cases} \tag{11}$$

It can be recognized that the algorithm (11) can be formally reduced to the Verlet algorithm [Verlet (1967)], a central difference integrator scheme very popular for molecular dynamics modeling. This is only a formal coincidence, as the central difference integrators assume a (continuously) differentiable function, while the Cell Method algorithm has been obtained within a direct discrete framework.

Figure 3 compares the results obtained with the Cell Method and with II order Runge Kutta for different values of the damping factor c in the interval 0 – 0.5 s, assumed m=1 kg, k= 4000N/m, initial position $x(0) = 0$, initial velocity $\dot{x}(0) = 300$ m/s.

A simple approximation for the starting value of momentum, $p^{1/2}$, can be obtained by doing a single half step (the consequent error is introduced only once and does not lower the order of the method). This leads to

$$p^{1/2} = \frac{1-\dfrac{\tau\cdot c}{4m}}{1+\dfrac{\tau\cdot c}{4m}}p^0. \tag{12}$$

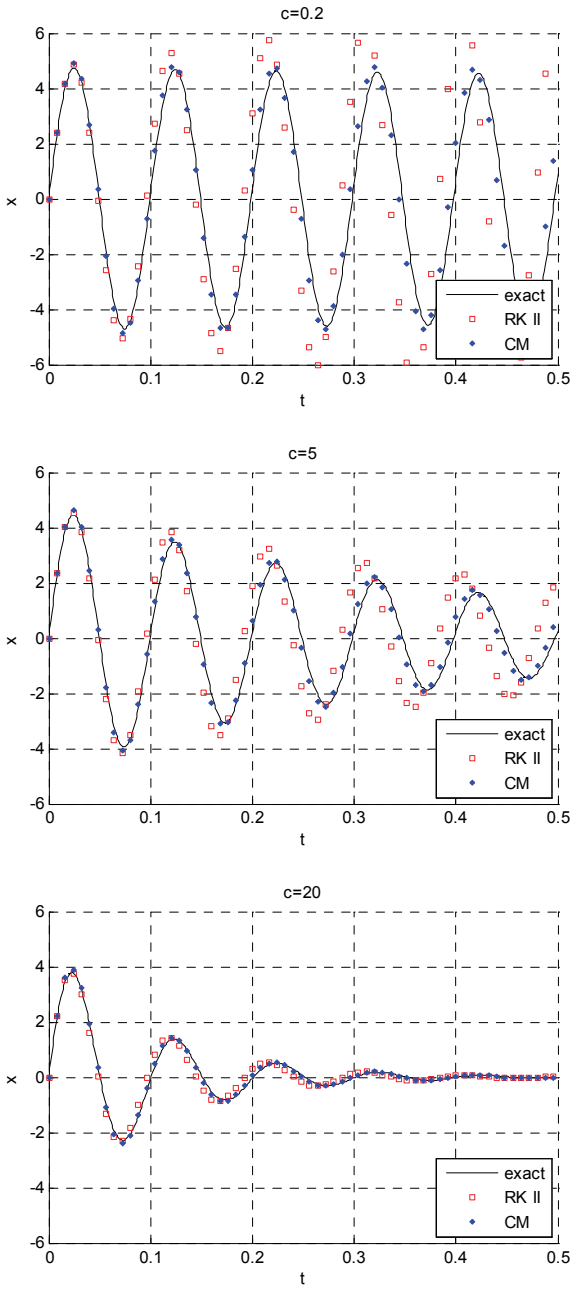


Figure 3: Time-displacement plot for different values of the damping coefficient. CM = Cell Method, RK_II = II order Runge Kutta method.

In the simulations, an integration step τ = 0.008s has been adopted. The differences among the methods and accuracy of the Cell Method can be appreciated.

For the same system, the maximum errors obtained with the Cell Method and with II order

Runge Kutta are compared in Figure 4. The Cell Method shows the same convergence rate of II order Runge Kutta method, but its accuracy is better.

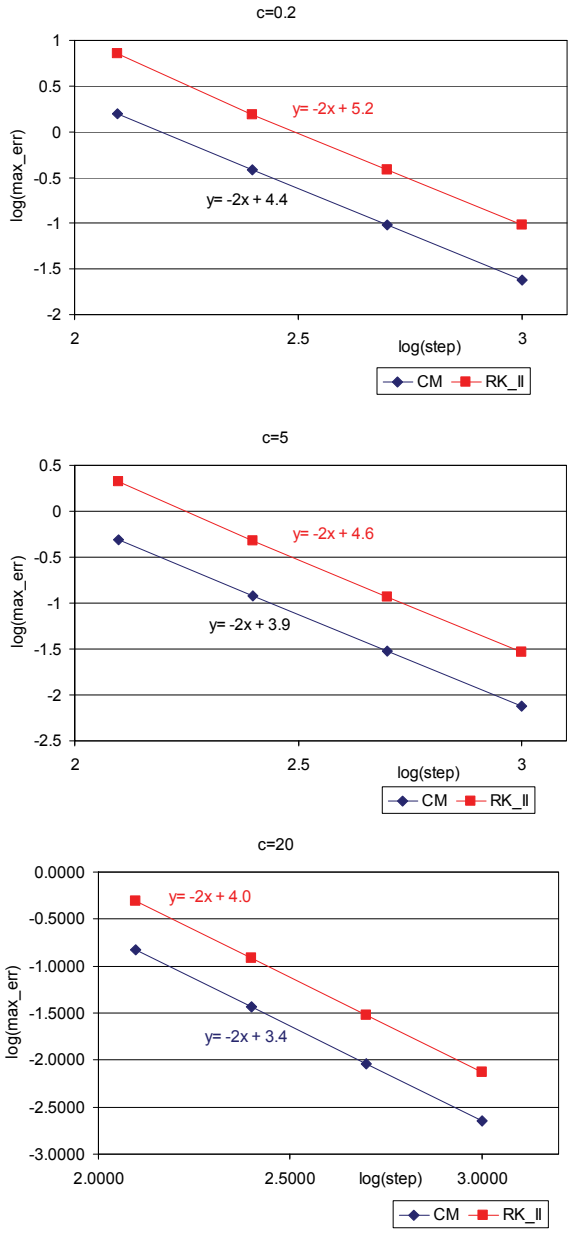


Figure 4: Convergence rate and accuracy. CM = Cell Method, RK_II = II order Runge Kutta method.

4 Application

Simulations on virtual prototypes, usually performed by dynamics analysis software (i.e. ADAMS) and Finite Element codes (i.e. ANSYS, NASTRAN) can help cutting development time and costs in the first structural evaluation phases of design.

The application of the Cell Method for the dynamics analysis of a mechanical component has been developed using as a test case a 5 mm thick L-shaped plate, as shown in Fig.5. One end of the L is fixed and the load $F_z(t)$ is applied at the other end.

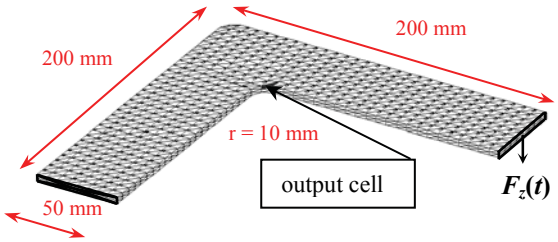


Figure 5: L-shaped plate geometry, model and output cell.

The higher stresses appear in the output cell, situated in the fillet area, also shown in Fig. 5. The parameters of the simulations are shown in Tab.1.

Table 1: Parameters of the simulations.

Material	elastic moduli $E = 210$ GPa, $\nu = 0.3$	density $\rho = 7820$ kg/m^3
Mesh	5184 primal cells (4 nodes tetrahedra)	1485 nodes
Simulation	time step $\tau = 2 \cdot 10^{-7}$ s	number of steps N = $5 \cdot 10^6$

4.1 Impulse load

In the first simulation, an impulse load F_z = 540 N, uniformly distributed on the loaded end of the plate, was applied only at the first step and then removed.

The implemented code could solve the system very quickly. Storage and post-processing of out-

put were more time consuming and will be optimized in future.

Principal components of stress in the examined cell are shown in Fig. 6 for the first 1000 steps of the simulation. It can be seen that the impulse load in input does not result in incoherent spikes in output. Therefore, it is not necessary to smooth the input time signal, as in general required by the FEM approach. Thus, a distinctive aspect of the Cell Method is possibility to simulate the application of impulse loads with increased realism.

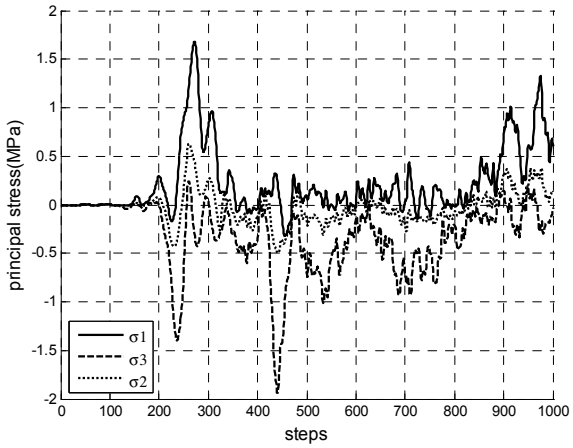


Figure 6: Impulse load – Principal components of stress in the examined cell (1000 steps).

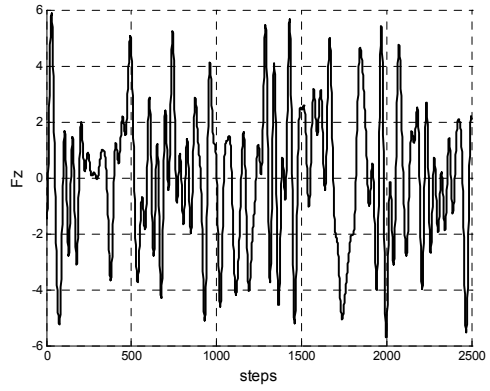
4.2 Random load

In these simulations, a random (white noise) load, characterized by a rectangular power spectral density, was used in input. The input force time-history is shown in Fig. 7 for (a) the first 2500 steps and (b) for 10^6 steps.

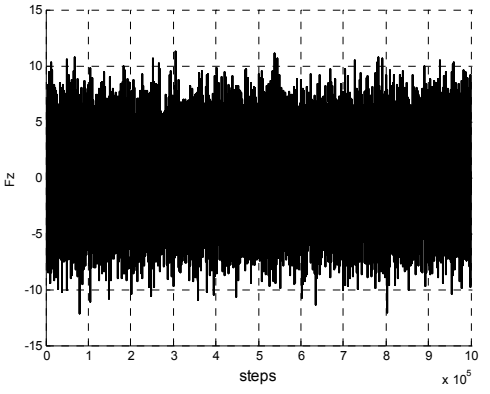
In Fig. 8, the results of the simulation are shown for (a) the first 2500 steps and (b) for 10^6 steps. No damping was included in this model.

In Fig. 9, the results of the simulation with a 0.05 damping coefficient in all nodes are shown for (a) the first 2500 steps and (b) for 10^6 steps. Again, no incoherent spikes are detected.

Such time-histories of stress components are of practical interest as, for example, they constitute the first step from which a mechanical part fatigue life can be evaluated.



(a)



(b)

Figure 7: Impulse load – Input time history (a) for the first 2500 steps and (b) for 10^6 steps.

5 Conclusions

In this work, an elastodynamics formulation that takes into account the dissipative phenomena usually named damping has been developed based on the Cell Method. Within this approach, an explicit solving system is directly obtained in discrete form and a time-dependent damping coefficient can be defined for each node. The linear combination of stiffness and mass matrixes, typical of the FEM approach, is not required. Accuracy and convergence rate of the algorithm have been discussed.

Simulations showing the application of the method for the dynamic stress analysis of a component subjected to dynamic loading have been illustrated and can constitute the first step for fatigue behaviour assessment of mechanical parts.

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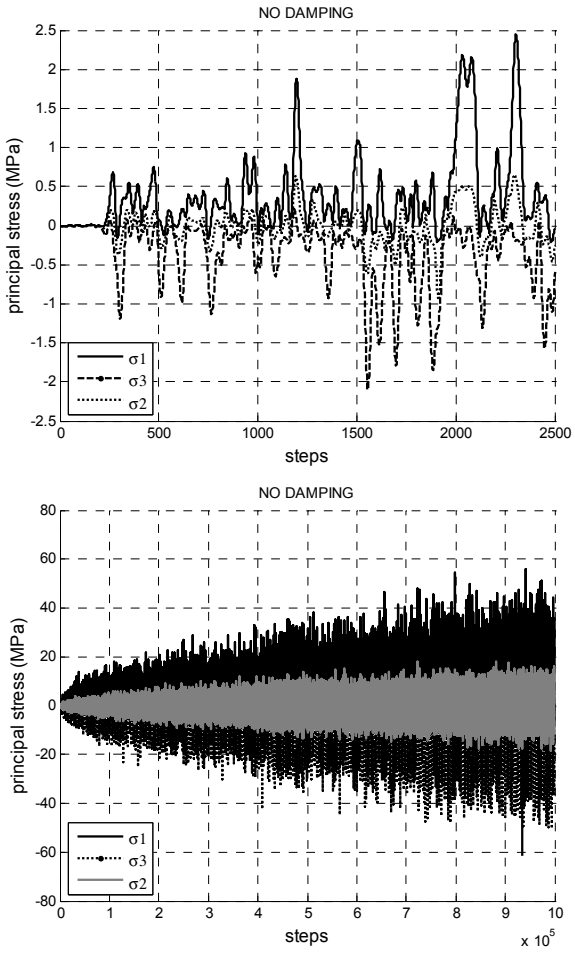

Figure 8: Random load, no damping – Principal components of stress in the examined cell (a) for the first 2500 steps and (b) for 10^6 steps.

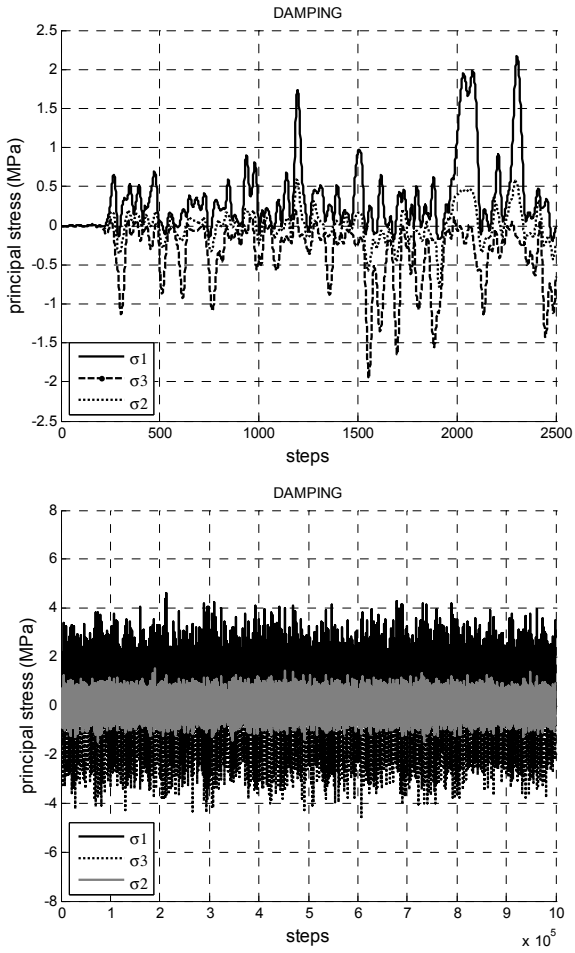

Figure 9: Random load, 0.05 damping coefficient – Principal components of stress in the examined cell (a) for the first 2500 steps and (b) for 10^6 steps.

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References

Hitchings D. (1992), *A Finite Element Dynamics Primer*, NAFEMS, Glasgow (GB).

Cosmi F. (2001): Numerical solution of Plane Elasticity Problems with the Cell Method, *CMES: Computer Modeling in Engineering & Sciences*, vol. 2, no. 3, pp. 365-372.

Cosmi F. (2005): Elastodynamics with the Cell Method, *CMES: Computer Modeling in Engineering & Sciences*, vol. 8, no. 3, pp. 191-200.

Ferretti E. (2005): A local strictly nondecreasing material law for modeling softening and size-effect: A discrete approach, *CMES: Computer Modeling in Engineering & Sciences*, vol. 9, no. 1, pp. 19-48.

Heshmatzadeh M.; Bridges G. E. (2007): A geometrical comparison between cell method and finite element method in electrostatics, *CMES: Computer Modeling in Engineering & Sciences*, vol. 18, no. 1, pp. 45-58

Marrone M.; Frasson A.M.F.; Hernàndez-Figueroa H.E. (2002): A Novel Numerical Approach for Electromagnetic Scattering: The Cell Method, *Proc. of the 2002 IEEE AP-S and URSI*, San Antonio, Texas, USA, vol. 1, pp. 160-163.

Straface S.; Troisi S.; Gagliardi V. (2006): Application of the Cell Method to the Simulation of Unsaturated Flow, *CMC: Computers, Materials and Continua*, vol. 3, no. 3, pp. 155-166.

Thomson, W.T. (1993) *Theory of Vibration with Applications*, 4^{rd} ed., Chapman & Hall., London, pp. 28-33.

Timoshenko, S.; Young, D.H. (1955) *Vibration Problems in Engineering*, 3^{rd} ed., D. Van Nostrand Co., Inc. New York, pp. 69-75.

Tonti, E. (2001a): A Direct Discrete Formulation of Field Laws: The Cell Method, *CMES: Computer Modeling in Engineering & Sciences*, vol. 2, no. 2, pp. 237-258.

Tonti, E. (2001b): A Direct Discrete Formulation for the Wave Equation, *Journal of Computational Acoustics*, vol. 9, n.4, pp. 1355-1382.

Verlet, L. (1967): Computer experiments on classical fluids. I. Thermodynamical properties of Lennard-Jones molecules. *Physical Review*, vol. 159, no. 1, pp. 98-103.