Stable PDE Solution Methods for Large Multiquadric Shape Parameters

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Abstract: We present a new method based upon the paper of Volokh and Vilney (2000) that produces highly accurate and stable solutions to very ill-conditioned multiquadric (MQ) radial basis function (RBF) asymmetric collocation methods for partial differential equations (PDEs). We demonstrate that the modified Volokh-Vilney algorithm that we name the improved truncated singular value decomposition (IT-SVD) produces highly accurate and stable numerical solutions for large values of a constant MQ shape parameter, c, that exceeds the critical value of c based upon Gaussian elimination.

Keyword: meshless radial basis functions, multiquadric, asymmetric collocation, partial differential equations, improved truncated singular value decomposition

1 Introduction

The finite element method (FEM) has been the dominant numerical method in the solution of partial differential equations (PDEs) for several decades. Nevertheless, FEM possesses some limitations in the solution of certain engineering problems: (1) The generation of a well behaved mesh may be extremely time consuming procedure for complex geometry, (2) Large deformations produce highly distorted meshes, (3) Repeated remeshing with a moving discontinuity problem such as crack growth analysis is required, (4) Very slow convergence in high gradient regions is ob-

served, etc. In order to avoid these problems, an alternative approach, called meshfree radial basis function (RBF) method, has been developed. The substantial difference between meshfree RBF methods and conventional mesh based methods is the global RBFs can extend their influence to the entire domain, without division into elements. A brief review of meshfree methods and numerous references can be found in Chen and Liu (2004). The ideal numerical method should be flexible with respect to complex geometry, computationally fast and efficient, numerically accurate and stable, easy to implement, and possess a very high rate of convergence . Meshfree RBF methods are a fairly new approach for the solution of PDEs that generally meet the above-mentioned criteria. Since the first RBF-PDE publications of Kansa (1990a,1990b), RBFs have played an increasingly important role in the solution of elliptic, parabolic and hyperbolic PDEs of engineering problems, see Fasshauer (2007).

RBFs are a family of functions that depend only on the distance $r_j = \|\mathbf{x} - \mathbf{x}_j\|$ from source point, $\mathbf{x}_j$, to a field point, $\mathbf{x}$, where $\mathbf{x}$ and $\mathbf{x}_j$ are points in $\Re^d$, and d is the dimension of the real space, $\Re$. RBFs were not well understood in the numerical community until Franke (1982) published a review paper evaluating of 29 two-dimensional interpolation methods and showed RBFs can provide very accurate approximations for multivariate scattered data interpolation. Among the investigated RBFs, the MQ proposed by Hardy (1971,1990) was ranked the best, followed by thin-plate splines.

The most popular globally supported C^∞ RBFs are multiquadrics (MQ)

$$\phi_j = (r_j^2 + c_j^2)^\beta \qquad \text{(MQ)} \qquad (1)$$

$$\phi_j = c_j^\beta (1 + (r_j/c_j)^2)^\beta \qquad \text{(MQ-prewavelet)} \qquad (2)$$

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$$\phi_j = exp(-(r_j/c_j)^2) \qquad \text{(Gaussian)} \qquad (3)$$

where c_j is the local shape parameter. Often, many authors use c = average(c_j). Very popular RBFs with limited continuity are:

$$\phi_j = r^2 \log(r), \qquad\qquad \text{(thin-plate splines)} \quad (4)$$

$$\phi_j = r^k, \ k = 1,3,5,\cdots \quad \text{(polyharmonic splines)} \tag{5}$$

Due to its spectral convergence property, the globally supported MQ is widely used in numerical analysis. The superior convergence of the MQ-RBF interpolation is supported theoretically. For example, Madych and Nelson (1990) proved that multiquadric interpolation converges at the exponential rate,

$$\text{convergence rate} \sim O(\lambda^\gamma),$$
$$\text{where } \gamma = (c/h), \text{ and } 0 < \lambda < 1, \quad (6)$$

$$h = \sup_{i,j} \|\mathbf{x}_i - \mathbf{x}_j\|; \quad \mathbf{x}_i, \mathbf{x}_j \in \Re^d \tag{7}$$

Later, Yoon (2001) proved the spectral convergence of MQ in Sobolev space. It is noteworthy that the convergence rate of MQ-RBFs is much faster than the algebraic rate $O(h^p)$ of the conventional FEM where the polynomial order is p.

For the purposes of this paper, c will be the global, constant MQ shape parameter. To improve the convergence rate, we want γ to become very large; this can be accomplished in three ways: (1) The h-scheme refines h, increasing the number, N, of data centers in a domain, Ω, (2) The c-scheme in which c increases, holding h fixed, or (3) The $c - h$ scheme in which h decreases, c increases so that γ increases.

The h-scheme greatly increases the rank of the coefficient matrix that leads to increased expense in computer storage and CPU time. In contrast, the c-scheme can be performed without any programming effort and without any extra computational cost. For purposes of efficiency, the c-scheme is superior and preferable over the h-scheme. The disadvantage of either the h-scheme, c-scheme, or $c - h$ scheme is that as $\gamma \to \infty$, the system of linear equations becomes very ill-conditioned and subject to extreme round-off errors on computers

with limited arithmetic precision. Consider a system of equations that is obtained from RBF interpolation, approximation, or collocation PDE approximations:

$$\mathbf{A}\alpha = \mathbf{b} \tag{8}$$

The absolute condition number, κ_{abs}, is defined as:

$$\kappa_{abs} = \max(\sigma)/\min(\sigma) = \|\mathbf{A}\| \cdot \|\mathbf{A}^{-1}\|, \tag{9}$$

where σ respresents the set of either the eigenvalues or singular values of the coefficient matrix, $\mathbf{A}$. Even though the absolute condition number can be very large, de Boor and Kreiss (1986) demonstrated that very accurate solutions can be obtained with very large condition numbers, because the relative condition number, κ_{rel}, defined as:

$$\kappa_{rel} = \frac{\|\mathbf{A}\alpha\|}{\|\alpha\|} \cdot \|\mathbf{A}^{-1}\|, \tag{10}$$

can be orders of magnitude smaller than the absolute condition number, producing stable and accurate results. The drawback of focusing upon the relative condition number is that of problem dependency.

A more accurate solution is obtained by increasing the shape parameter c until the matrix condition number reaches a critical value that is too large for the computing machine precision to handle, at which the round-off errors dominate and solution becomes unreliable. The trade off for this increased accuracy and ill-conditioning of the equation systems can be explained by the Schaback Uncertainty Principle (1995,1997) that states that better conditioning is associated with poorer accuracy and bad conditioning is associated with improved accuracy. However, Cheng, Golberg, Kansa, and Zammito (2004) and Huang, Lee, and Cheng (2007) demonstrated that extended arithmatic precision using Gaussian elimination, very accurate solutions of the Poisson equation can be obtained by pushing c to increasing larger limits.

Even though this paper concentrates on a constant c MQ parameter, it is worthwhile mentioning that

variable MQ shape parameters play an important role in pushing the limits of accuracy. Buhmann and Micchelli (1992a,1992b), Buhmann (1995) and Chui, Stoeckler, and Ward (1996) demonstrated that (1) A monotonic sequence of MQ shape parameters increases the rate of MQ interpolation convergence, and (2) the form of MQ in Eq (2) is a prewavelet with rotational, translational, and dilation properties. A constant value of c_j^2 is not capable of wavelet dilation. In a future paper, we intend to demonstrate this accelerated convergence with variable shape parameter with the Volokh-Vilnay (2000) method, (IT-SVD). However, in this paper, we will show that the implementation of the IT-SVD is capable of achieving superior stable results with large values of constant c.

The usual method of solving PDEs with RBFs is similar to standard mesh based methods by constructing a uniform grid that is consecutively refined, yielding progressively more ill-conditioned systems of equations. Some authors who are accustomed to finite difference and finite element methods have claimed that RBFs could not be extended to complicated, multi-dimensional PDE problems in which many thousands of data centers are required. Fasshauer (1999) and Driscoll and Heryudono (2007) found that adaptively refining the data centers in high gradient regions is more efficient that uniform h–refinement, producing better conditioned systems.

The shape parameter must be adjusted with the number of centers in order to produce equation systems that are sufficiently well conditioned to be solved with standard finite precision arithmetic; the optimal choice of the constant shape parameter is still an open question, and it is most often selected by trial and error. In order to overcome these shortcomings, many efforts have been made to find a new computational method that is capable of circumventing the ill-conditioning problems using traditional linear solver schemes.

To the best of our knowledge, main of the effort reported in the literature to reduce the ill-conditioning problems include: (1) Using variable shape parameters, see Kansa (1990a,1990b), Fedoseyev, Friedman and Kansa (2002), Wertz,

Kansa, and Ling (2006), (2) Pre-conditioning the coefficient matrix, see Ling and Kansa (2004 and 2005), Brown, Ling, Kansa, and Levelsey (2005), (3) Using domain decomposition methods in overlapping or non-overlapping schemes that decompose a very large ill-conditioned problem into many subproblems with better conditioning, see Kansa (1990b), Kansa and Hon (2000), and Ingber, Chen, and Tanski (2004), (4) Optimizing the center locations by the greedy algorithm, see Hon, Schaback, and Zhou (2003) and Ling, Opfer, and Schaback (2006), (5) Using an improved numerical solver based on affine space decomposition, see Ling and Hon (2005), (6)Removal of singularities by Pade-contour integration. Using complex MQ shape parameters as $c \rightarrow \infty$, Fornberg andWright (2004), Fornberg and Driscoll (2002), and Fornberg, Wright, and Larsson (2004), removed the singularities by Pade-contour integration and verified that the interpolation theoretical convergence rate of MQ in the limit of infinitely flat C^∞, in one-dimension, is a Lagrange polynomial due to significant cancellation. They showed that the TPS and polyharmonic splines cannot achieve such high convergence rates. (7) Authors, such as Tolstykh and Shirobokov (2005) and Sarler (2005) ,aware of the Schaback (1995) RBF trade-off principle, used localized stencils in which low order polynomials are replaced by radial basis functions obtaining well-conditioned systems yielding better convergence rates than standard low order finite difference methods, but slower rates than the more global RBF schemes. (8) Based upon the theoretical convergence rates of the MQ interpolation and spatial derivative approximations obtained by Madych (1992), Mai-Duy and Tran-Cong (2003 and 2007) developed a neural network model in which the MQ PDEs and boundary conditions are integrated without enforcing strict conservation upon the dependent variables. Since integration is a smoothing operation as opposed to differentiation, they are able to obtain rapidly converging solutions with large numbers of sample data centers, thus avoiding very ill-conditioned systems.

A main goal of this paper is to provide an efficient and easy-to-implement numerical scheme

for mitigating ill-conditioning problems arising from large values of constant shape parameters. It is claimed that in a numerical solution without round-off error, infinite accuracy can be achieved by taking c to infinity. Here we propose a stable solver for the solution of severely ill-conditioned equation systems and use it to test the above conjecture. The main focus is the stability and accuracy of RBF collocation methods with large constant shape parameter c.

The structure of the paper is as follows. First, we give a brief review of solving PDE using collocation with RBFs. Second, we introduce the improved truncated singular value decomposition (IT-SVD) as a stable solver for severely ill-conditioned equation systems. Third, we implement the proposed (IT-SVD) scheme for solving elasticity problems using the MQ RBFs. Some numerical examples are given to demonstrate the effectiveness of the IT-SVD method in handling near-singular equation systems. Finally, we discuss the results of this paper.

2 Asymmetric radial basis function collocation methods

We briefly review the RBF asymmetric collocation scheme here. The starting point for RBF-PDE numerical solutions is that a dependent variable, $U(\mathbf{x}, t)$ is well approximated by RBF interpolation, i.e., over the domain, Ω.

$$U(\mathbf{x}, t) = \sum_{j=1}^{N} \phi(\mathbf{x} - \mathbf{x}_j) \alpha_j(t) \tag{11}$$

Given a forcing function, $\mathbf{f}$, and a linear or nonlinear hyperbolic, parabolic, or elliptic differential operator, $\mathscr{L}$, over the interior, $\Omega \backslash \partial\Omega$, and a boundary forcing function, $\mathbf{g}$, and an operator, $\wp$, on $\partial\Omega$, the asymmetric collocation scheme is given by:

$$\mathscr{L}\mathbf{U} = \mathbf{f}, \quad \text{over } \Omega\backslash\partial\Omega \tag{12}$$

$$\wp\mathbf{U} = \mathbf{g}, \quad \text{on } \partial\Omega \tag{13}$$

where $\mathscr{L}$ represents either a linear or nonlinear hyperbolic, parabolic, or elliptic differential operator, f is the forcing function, $\wp$ represents a

boundary operator for either Dirichlet, Neumann, or Robin boundary conditions. The implementation of MQ-RBFs for PDEs is straightforward. Spatial derivatives are computed by simply differentiating the MQ-RBFs; time dependent derivatives are computed by differentiating the time dependent expansion coefficients. The spatial and temporal partial derivatives leads to either a linear or nonlinear set of equations:

$$\mathbf{A}\alpha = b = \begin{bmatrix} \mathscr{L}\phi(\mathbf{x}_i - \mathbf{x}_j) \\ \wp\phi(\mathbf{x}_i - \mathbf{x}_j) \end{bmatrix} \begin{pmatrix} \alpha_{\Omega\backslash\partial\Omega} \\ \alpha_{\partial\Omega} \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ \mathbf{g} \end{pmatrix} \tag{14}$$

Once the unknown expansion coefficients, α, are found, then the dependent variable $\mathbf{U}(\mathbf{x}, t)$ can be reconstructed. for all $\mathbf{x} \in \Omega$. A recent example of the success of the asymmetric MQ scheme is the solution of Maxwell's equations by Young, Chen, and Wong (2005).

3 Improved truncated singular value decomposition

Now let us briefly review the idea of treating numerical instability arising from round-off errors in the solution of ill-conditioned equation systems of RBF collocation procedure. Consider an equation system in the form of

$$\mathbf{A}\alpha = \mathbf{b} \tag{15}$$

Consider the well-known singular value decomposition algorithm for a square $N \times N$ matrix, $\mathbf{A}$, that can be decomposed as:

$$\mathbf{A} = \mathbf{V}\Sigma\mathbf{U}^T \tag{16}$$

The inverse of $\mathbf{A}$ is:

$$\mathbf{A}^{-1} = \mathbf{U}\Sigma^{-1}\mathbf{V}^T \tag{17}$$

where $\mathbf{V} = [\mathbf{V}_1, \mathbf{V}_2, \dots, \mathbf{V}_N]$, and $\mathbf{U} = [\mathbf{U}_1, \mathbf{U}_2, \dots, \mathbf{U}_N]$ are orthonormal matrices with column vectors called left and right singular vectors, respectively, and Σ is a diagonal matrix of the singular values in descending order: $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_N > 0$. The ratio of the largest

singular value to the smallest singular value gives the absolute condition number of the matrix $\mathbf{A}$,

$$\kappa_{\text{abs}}(\mathbf{A}) = \frac{\sigma_1}{\sigma_N} = \|\mathbf{A}\| \cdot \|\mathbf{A}^{-1}\| \qquad (18)$$

The condition number determines the loss in precision due to round-off errors and can be used to estimate the accuracy of results obtained from matrix inversion and linear equation solutions. The weak point of the SVD solution is the inaccurate computation of the small singular values. For an ill conditioned matrix equation the SVD solution is dominated by contributions from small singular values, and therefore it may become unbounded and oscillatory. Hence, to mitigate the ill conditioning, one often drops all terms with $\sigma_k < \mu$ for some pre-assigned value of μ, where $k < N$, and μ is the cut-off cutoff parameter. The Hensen (1997) truncated SVD (TSVD) uses the first k singular values, σ_j, $j = 1, \ldots, k$. The inverse of $\mathbf{A}$ can be computed as:

$$\mathbf{A}_{\text{trunc}}^{-1} = \mathbf{U}_{1:N,1:k} \Sigma_{1:k}^{-1} (\mathbf{V}_{1:N,1:k})^T. \qquad (19)$$

The TSVD scheme is widely used as an efficient solver for the ill-conditioned systems. However, for the RBF collocation method, solutions obtained from the TSVD method are not accurate and reliable.

Volokh and Vilnay (2000) observed that small singular values are inherent in the SVD of ill-conditioned systems and presented an algorithm to bypass the inaccurate computation of the small singular values. Their method will be summarized below.

Define matrices, $\Sigma1 = \Sigma_{1:k}$, $\mathbf{U1} = \mathbf{U}_{1:N,1:k}$, $\mathbf{U2} = \mathbf{U}_{1:N,k+1:N}$, $\mathbf{V1} = \mathbf{V}_{1:N,1:k}$, $\mathbf{V2} = \mathbf{V}_{1:N,k+1:N}$, and $\Sigma2 = \Sigma_{k+1:N}$. Their method uses the information contained in the entire SVD decomposition, but projects out the very small singular values into the null space to construct a stable scheme. They define new right and left matrices, of rank, N and k:

$$\mathbf{Unull} = \text{null}(\mathbf{U1}^T), \quad \mathbf{Vnull} = \text{null}(\mathbf{V1}^T) \qquad (20)$$

Using the small singular values, $\Sigma2$, they constructed a new matrix, $\mathbf{C}$:

$$\mathbf{C}_{k+1:N,k+1:N} = (\mathbf{VnullV2})\Sigma2(\mathbf{U2Unull})^T \qquad (21)$$

Then the complete inverse and solution vector consists of two parts:

$$\mathbf{A}^{-1} = \mathbf{A}_{\text{trunc}}^{-1} + \mathbf{Unull}\mathbf{C}^{-1}\mathbf{Vnull}, \qquad (22)$$

$$\alpha = (\mathbf{A}_{\text{trunc}}^{-1} + \mathbf{Unull}\mathbf{C}^{-1}\mathbf{Vnull})\mathbf{b}. \qquad (23)$$

The key properties of this method are that the matrices $\mathbf{C}$ and $\Sigma1$ are well conditioned in contrast to the matrix $\mathbf{A}$. The condition number of $\mathbf{C}$ depends upon the cutoff parameter μ and the well conditioning of $\mathbf{C}$. So, the performance of the proposed Volokh-Vilnay method depends upon the suitable choice of cutoff parameter, μ. In this paper, we employ a rational scheme to determine the cutoff parameter for the improved TSVD.

For a fixed machine precision, the cumulative round-off error reduces the number of accurate digits in the floating point arithmetic. The usual rule is that the exponent of the condition number indicates the number of decimal places that the computer can lose due to round-off errors. If the condition number is much greater than $\sqrt{1/\varepsilon}$, where ε is the machine precision, caution is advised for subsequent computations. For IEEE arithmetic, double precision numbers that usually used in the numerical arithmetic have about 16 decimal digits of accuracy and the machine precision is about $\varepsilon = 2.2 \times 10^{16}$, so it is advisable to keep the condition number of the matrix $\Sigma1$ less than $\sqrt{1/\varepsilon} = 6.7 \times 10^8$ by adjusting the cutoff parameter, μ. So, instead of using a constant cutoff parameter, $\mu = 10^{-8}$, as employed by Volokh and Vilney (2000) we used a floating cutoff parameter, $\mu = \sigma_1 \times 10^{-8}$, so that

$$\kappa_{\text{abs}}(\Sigma1) = \frac{\sigma_1}{\sigma_k} \leq \frac{\sigma_1}{\mu} \leq 10^8. \qquad (24)$$

This floating cutoff guarantees that the condition number of $\Sigma1$, $\kappa_{abs}(\Sigma1) \leq 10^8$, is bounded for IEEE 16 decimal digits of accuracy. In the highly ill-conditioned equation systems, very small singular values appear in matrix $\mathbf{C}$ and this matrix may also become ill-conditioned. The ill-conditioning problem of matrix $\mathbf{C}$ can be simply overcome by repeating the IT-SVD scheme on the matrix, $\mathbf{C}$.

4 Numerical implementation

The accuracy of the IT-SVD (Volokh-Vilnay method) is demonstrated through a series of numerical examples solving Navier's solid mechanics equations in $\Re^2$ space. Some recent examples of using multiquadrics in solid mechanics problems is the solution of composite laminated plates by Mai-Duy, Khennane, and Tran-Cong (2006) and the solution of the nonlinear Reissner-Mindlin plate by Wen and Hon (2007).

A constant shape parameter is assumed for simplicity in all analyses even though it is well known that the use of a variable shape parameter distribution improves the convergence rate. Madych and Nelson (1990) showed theoretically that for a constant shape parameter MQ formulation, the interpolation becomes increasingly more accurate as the shape parameter (c) increases, but this is not possible with finite precision computers to allow $c \to \infty$ because of the increasing ill-conditioning of the matrix, **A**, due to the accumulation of round-off errors. This phenomenon is studied here, through a numerical example.

The efficiency of standard Gauss elimination, singular value decomposition and the IT-SVD in numerical solutions of severely ill-conditioned problems is studied. Five numerical examples of elasticity problems are discussed here to illustrate the performance of the IT-SVD. All numerical codes are implemented with Matlab 7.2 and executed on a Core 2 Duo 2.0 GHz (4 MB Cache, 1G RAM) notebook computer running Windows XP Professional. The relative L_2 error, as defined below, is used to measure the solution accuracy.

$$L_2 = \sqrt{\frac{\sum(f_i^h - f_i^{\text{exact}})^2}{\sum(f_i^{\text{exact}})^2}} \qquad (25)$$

where f_i^h and f_i^{exact} are the approximated and exact solution values at the point, $\mathbf{x}_i$, respectively.

Example 1. Simple Tension Bar This example shows, at least numerically, that a very simple problem may suffer from ill-conditioning by increasing the shape parameter, c. The example in Figure (1) represents an elastic bar loaded axially.
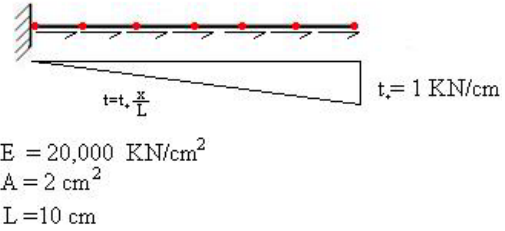

Figure 1: Regular distribution of 6 points used in tension bar problem.

Dirichlet type boundary condition imposed on the left side of ordinate $x = 0$ whereas a Dirichlet or Neumann type boundary condition is imposed on the opposite side of ordinate $x = 6$. The displacement and stress fields that are the exact solution of the mentioned problems are as follow:

$$u_x = (t_0/2EA)(Lx - x^3/3L) \qquad (26)$$

$$\sigma_{xx} = (t_0/2A)(1 + x/L)(L - x) \qquad (27)$$

where A, E, L and t_0 are the cross-section area, Young's module, the length of bar, and maximum axial traction, respectively.

Standard Gauss elimination (GE), singular value decomposition (SVD) and the IT-SVD are used for the solution of resultant equation systems.

The results show that by employing the GE or SVD scheme the errors first decreased with increasing values of the shape parameter, c, until a minimum was reached. The corresponding value of the shape parameter was termed the critical value of the shape parameter, c_{cr}, after which numerical instability appears. Beyond the critical value, the L_2 errors increased rapidly with increasing values of shape parameter due to the cumulative round-off error. This observed dependence of the L_2 error upon the shape parameter is believed to be related to the condition number of the matrix. In contrast to the GE and SVD schemes, the IT-SVD remains stable even with large values of the shape parameter, c. The IT-SVD extends the critical value of shape parameter, c_{cr}, and makes it possible to obtain good quality solution by utilizing large values of shape parameter. Table 1 shows the minimum error of each method.
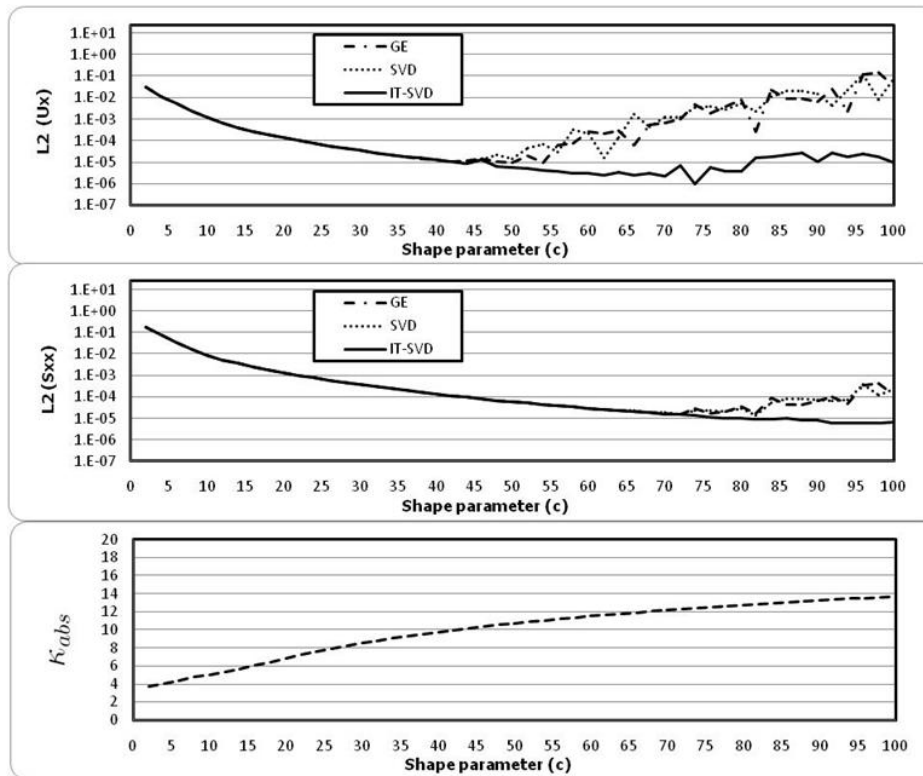
Figure 2a: The effect of c on the L_2 error of u_x, σ_{xx}, and $\kappa_{abs}(\mathbf{A})$ on the accuracy of analysis with Dirichelet boundary conditions.
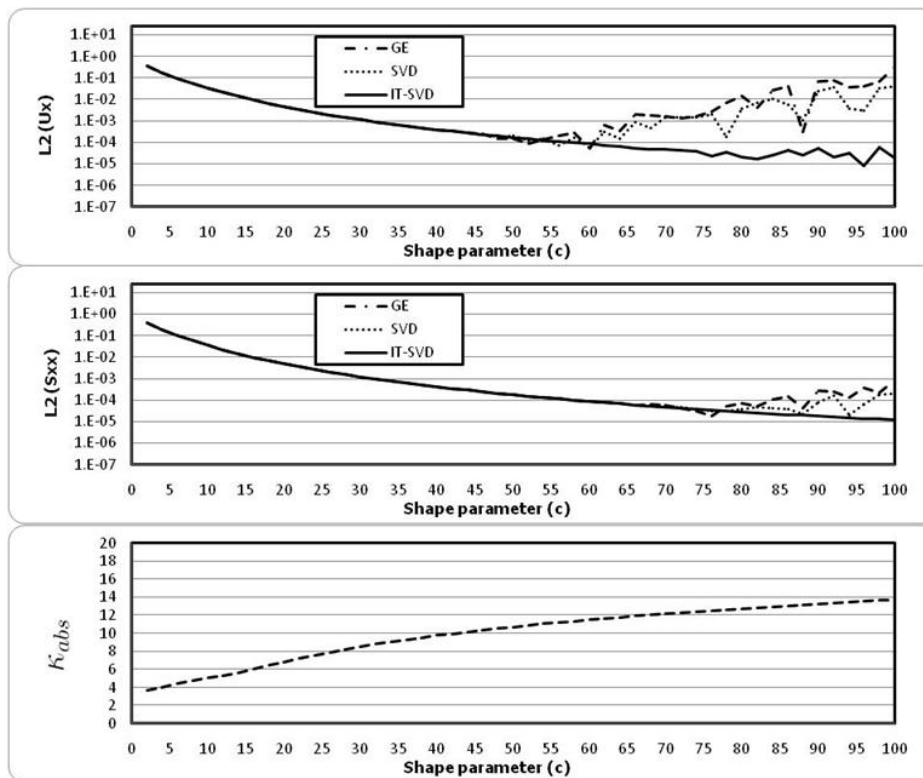

Figure 2b: The effect of c on the L_2 error of u_x, σ_{xx}, and $\kappa_{abs}(\mathbf{A})$ on the accuracy of analysis with Neumann boundary conditions.

Table 1: Minimum L_2 error in tension bar problem

Boundary Type	Neumann.	Neumann	Neumann	Dirichlet	Dirichlet	Dirichlet
SolverMethod	GE	SVD	IT-SVD	GE	SVD	IT-SVD
u_x	5.31E-5	5.57E-5	0.785E-5	0.96E-5	0.89E-5	0.14E-5
σ_{xx}	1.79E-5	2.07E-5	1.18E-5	1.56E-5	1.40E-5	0.52E-5

Table 2: Minimum error in Patch test problem (regular distribution)

Boundary Type	Neumann.	Neumann	Neumann	Dirichlet	Dirichlet	Dirichlet
SolverMethod	GE	SVD	IT-SVD	GE	SVD	IT-SVD
u_x	2.64E-4	1.72E-4	3.28E-5	9.39E-6	5.38E-6	2.22E-7
σ_{xx}	1.79E-5	2.05E-4	1.15E-5	5.41E-5	4.62E-5	3.93E-6

Table 3: Minimum error in Patch test problem (irregular distribution)

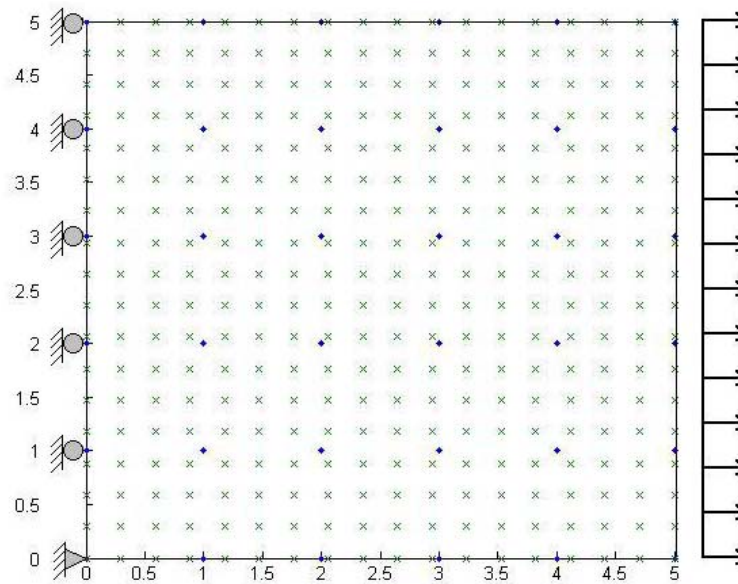| Boundary Type	Neumann.	Neumann	Neumann	Dirichlet	Dirichlet	Dirichlet
SolverMethod	GE	SVD	IT-SVD	GE	SVD	IT-SVD
u_x	4.67E-5	3.69E-5	6.60E-6	2.63E-4	2.47E-4	1.59E-5
σ_{xx}	3.2E-5	2.15E-5	9.50E-6	1.38E-4	0.92E-4	4.62E-5

Figure 3: Distribution of data centers used for example 2.

Example 2. Patch Test (Regular distribution)

A uniformly distribution of collocation points was used in the Patch test analysis. A detailed description of the elasticity patch tests can be found in Zienkiewicz andTaylor.(2000). The analytical solution for this problem is

$$u_x = x; \quad u_y = -y/4 \qquad (28)$$

$$\sigma_{xx} = 1; \quad \sigma_{xy} = 0; \quad \sigma_{yy} = 0, \qquad (29)$$

where the Young's modulus is unity and the Poisson's ratio of 0.25 are assumed in the analysis. Figure 3 shows the uniform distribution of points that are used in the analysis.

Two types of boundary condition were considered in this analysis. In the first case, the analytical displacement field is prescribed on all boundaries that represent the Dirichlet boundary condition. In the second case, Dirichlet boundary condition imposed on the boundary $\Gamma_u = [x = 0; 0 \le y \le 5]$,
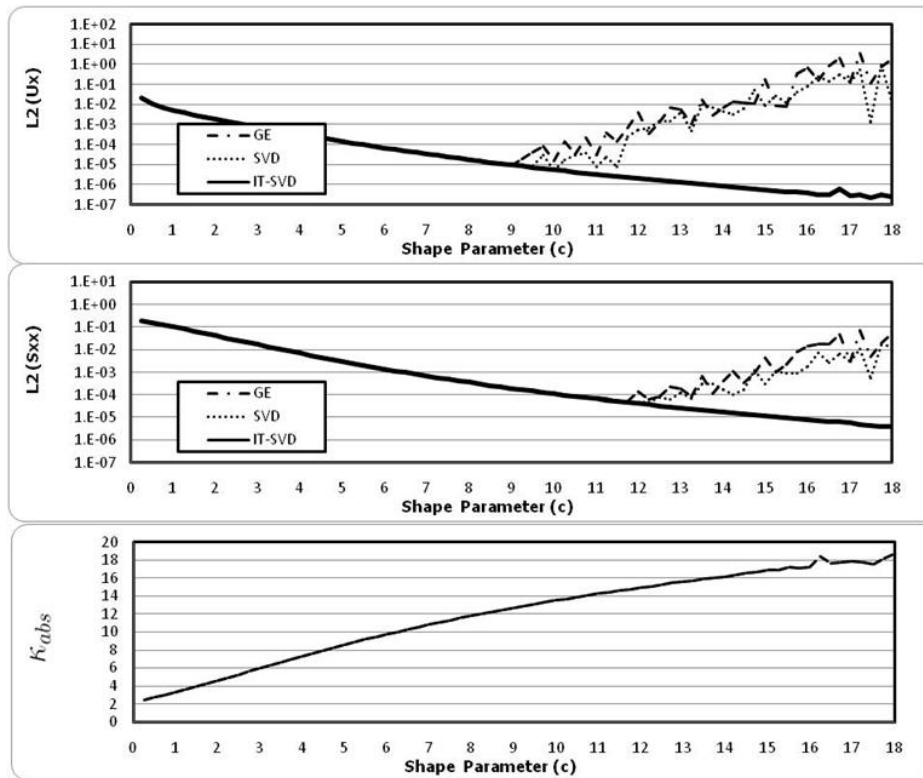
Figure 4a: The effect of c on the L_2 error of u_x, σ_{xx}, and $\kappa_{abs}(\mathbf{A})$ on the regularly distributed Patch Test with Dirichelet boundary conditions.
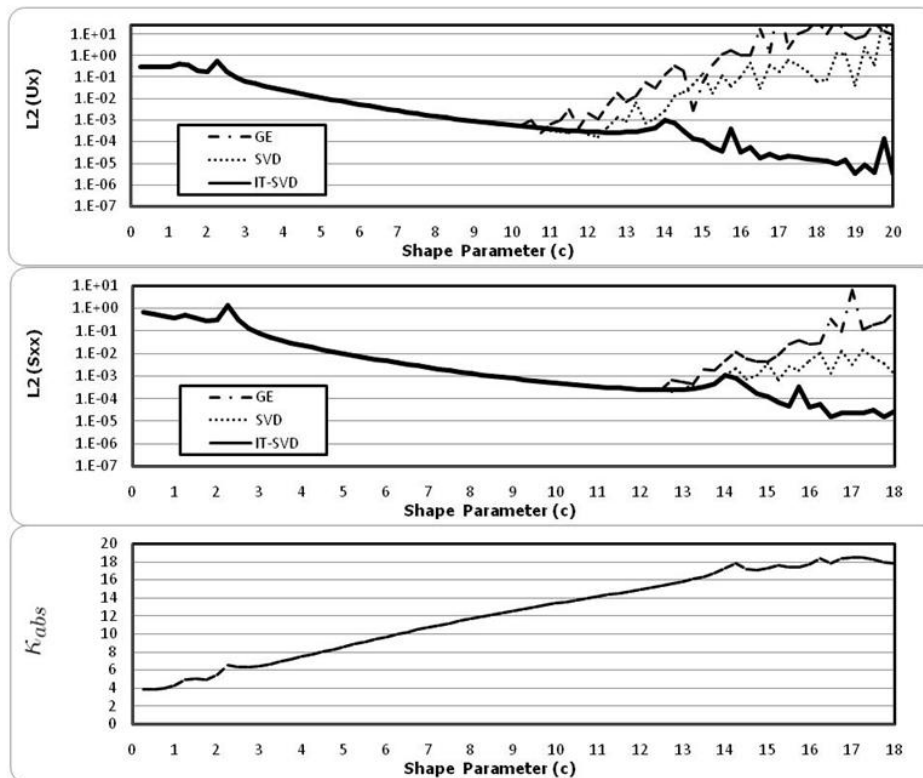

Figure 4b: The effect of c on the L_2 error of u_x, σ_{xx}, and $\kappa_{abs}(\mathbf{A})$ on the regularly distributed Patch Test with Neumann boundary conditions.
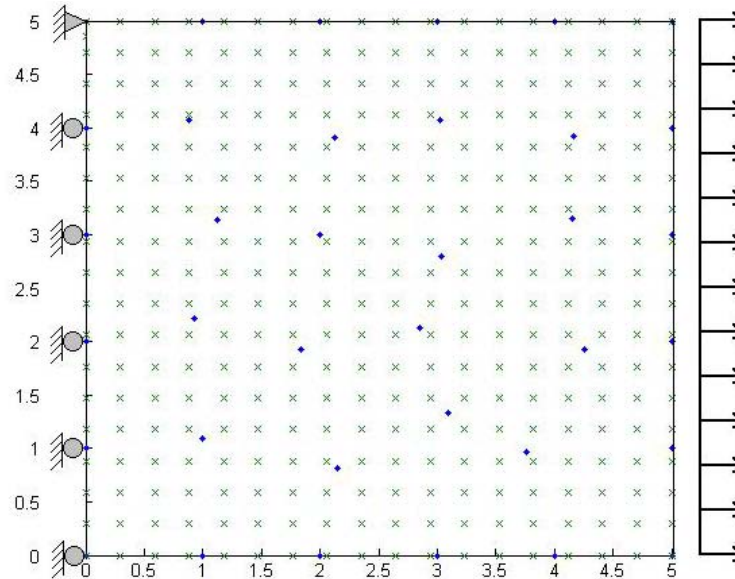
Figure 5: The irregular distribution of data centers for the second Patch Test.

and the analytical traction is applied on remaining boundaries. The later case was labeled as the Neumann boundary condition.

In the first case of the boundary condition, the L_2 error is reduced monotonically by increasing the shape parameter. But imposing second type of boundary condition, a fluctuation appeared in the L_2 error when the shape parameter varied in the range of $c < 2.5$. By increasing the shape parameter beyond $c > 2.5$, the error again reduced monotonically. So, it is possible to define a new limit for shape parameter before which the results oscillate by varying the shape parameter. This limit of shape parameter is termed the monotonic threshold shape parameter, c_t. In this example, the critical shape parameter and monotonic threshold shape parameters are $c_{cr} = 9$, $c_t = 0$ and $c_{cr} = 10$, $c_t = 2.5$ for the Neumann and Dirichlet boundary conditions, respectively.

Obtaining reliable results require tuning the shape parameter. One should select the shape parameter in the range of $c_t < c < c_{cr}$, to avoid numerical instability and to achieve monotonic convergence. The IT-SVD moderates the numerical instability and increases the critical value of the shape parameter, significantly. So, the method presented here makes it possible to fine-tune the shape parameter and improve the accuracy of so-lution. However, that is not the case in this example because there is a vast interval between two limits of $c_t = 2.5$ and $c_{cr} = 10$. We demonstrate the advantage of the IT-SVD in the following examples.

Example 3. Patch Test (Irregular distribution)
An irregular distribution of collocation points, as depicted in Figure 5 is used in the second Patch test analysis to demonstrate the effect of point arrangement on the stability and accuracy of the IT-SVD.

The analytical solution for this problem is the same as former. Figure 6 shows the effect of shape parameter on error norm of irregularly distributed patch test. Table 3 shows the minimum error of each method.

The monotonic threshold shape parameter is about $c_t = 11$ and $c_t = 1.5$ for the Neumann and Dirichlet boundary conditions, respectively while the critical value of the shape parameter is about $c_{cr} = 11.5$ for both cases of boundary conditions.

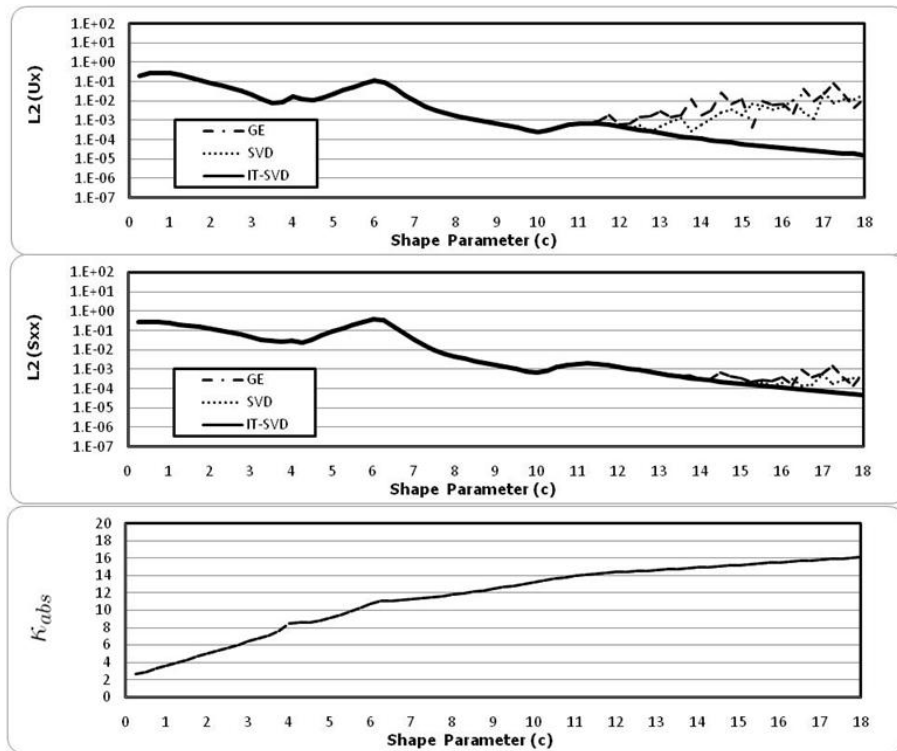Table 3 shows the minimum $L2$ error with various solver methods on the irregular distribution Patch test.

Figure 6a: The effect of c on the L_2 error of u_x, σ_{xx}, and $\kappa_{abs}(\mathbf{A})$ the irregularly distributed patch test with Dirichelet boundary conditions.
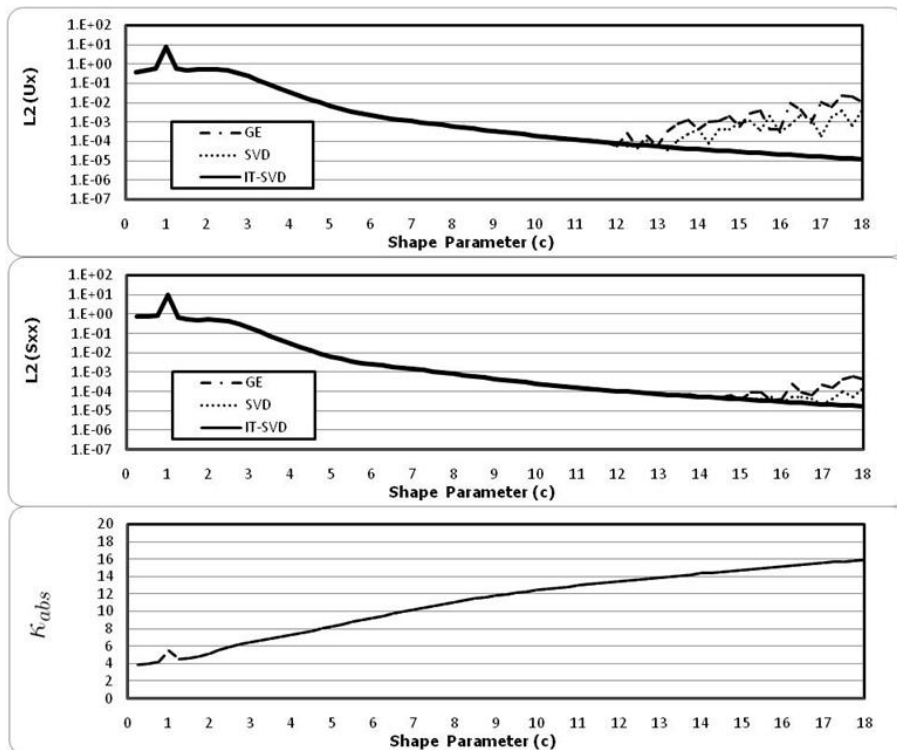

Figure 6b: The effect of c on the L_2 error of u_x, σ_{xx}, and $\kappa_{abs}(\mathbf{A})$ the irregularly distributed patch testwith Neumann boundary conditions.
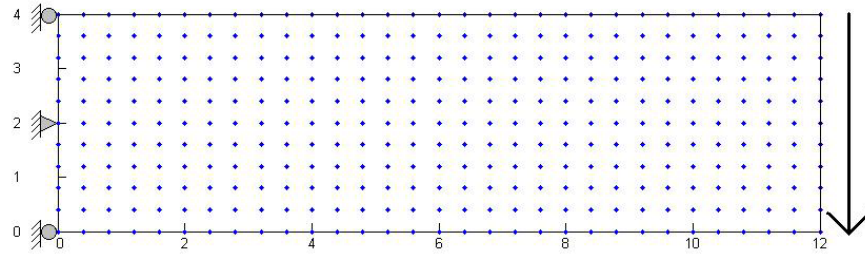
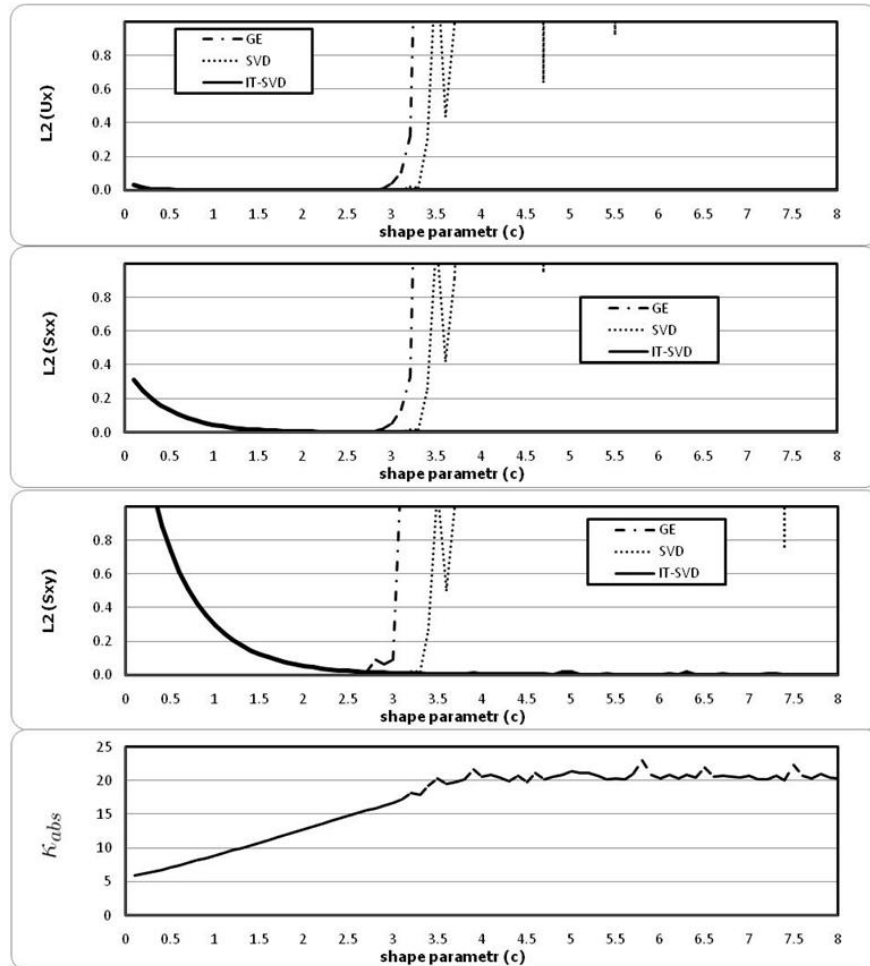Figure 7: The distribution of data centers for the cantilever beam.

Figure 8a: The L_2 errors of u_x, σ_{xx}, σ_{xy} and $\kappa_{abs}(\mathbf{A})$ as a function of c for the cantilever beam problem.with regularly distisbuted data centers and Dirichelet boundary conditions.

Example 4. Cantilever beam (Regular distribution) A cantilever beam subjected to tip shear traction as shown in Figure 7 is considered as the fourth example.

The length and height of beam are $L = 12$ and $D = 4$, respectively. The plane stress condition is assumed in the analysis with the mechanical properties of $E = 1000$ and $v = 0.33$. The exact displacement and stress field of this problem are given in Timoshenko and Goodier.(1970):

$$u_x = -(P/6EI)(y - D/2)\big[(6L - 3x)x + (2 + v)(y^2 - Dy)\big], \quad (30)$$
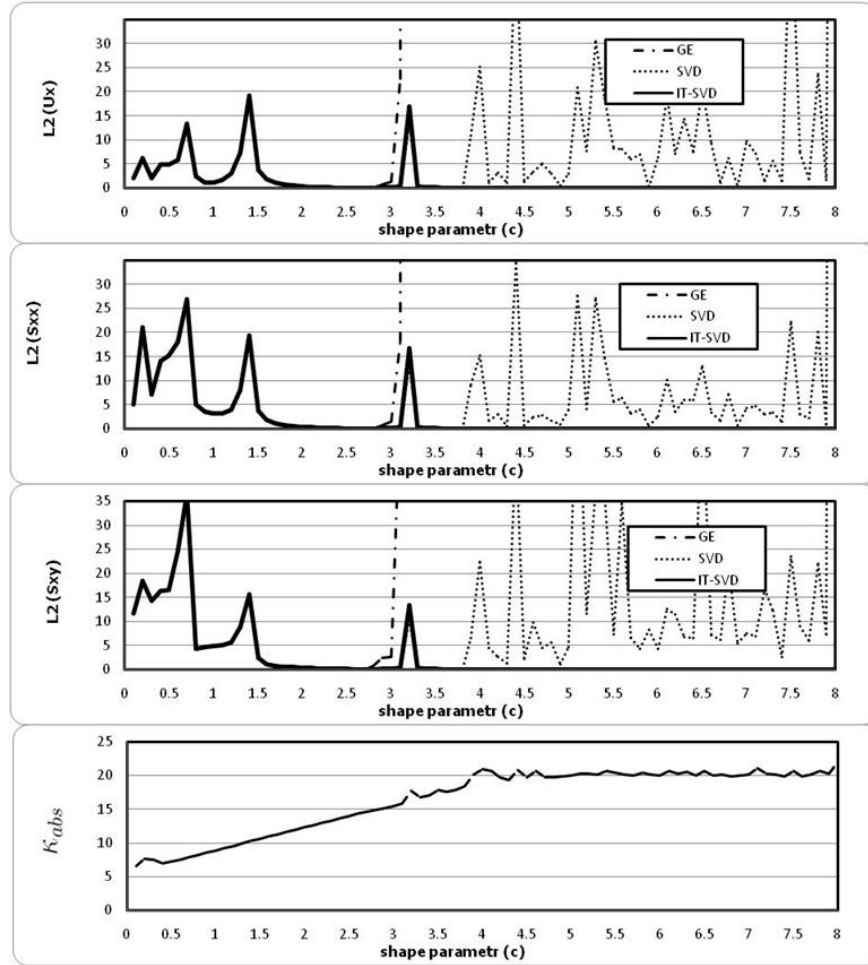
Figure 8b: The L_2 errors of u_x, σ_{xx}, σ_{xy} and $\kappa_{abs}(\mathbf{A})$ as a function of c for the cantilever beam problem.with regularly distisbuted data centers and Neumann boundary conditions.The minimum L_2 errors of each method are summarized in Table 4.
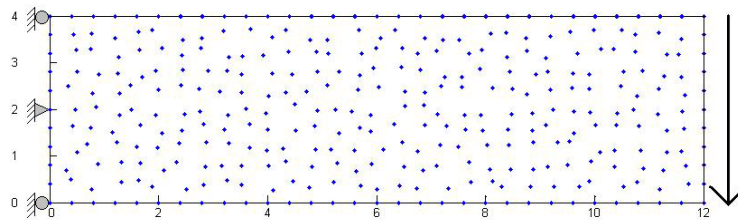

Figure 9: The irregular data center distribution for the second cantilever beam problem.

$$u_y = (P/6EI)[3v(L-x)(y-0.5D)^2$$
$$+ (4+5v)/4 \cdot D^2 x + x^2(3L-x)], \quad (31)$$

$$\sigma_{xx} = -(P/I)(L-x)(y-D/2;$$
$$\sigma_{xy} = (Py/2I)(y-D); \quad (32)$$
$$\sigma_{yy} = 0,$$

where the moment of inertia, $I = D^3/12$.

Two types of boundary condition are considered in this analysis. In the first case, the Dirichlet boundary condition is applied on all boundaries. In the second case, the analytical displacement field is prescribed on the boundary $\Gamma_u = [x = 0; 0 \leq y \leq D]$, and a Neumann type boundary condition is applied on the remaining bound-
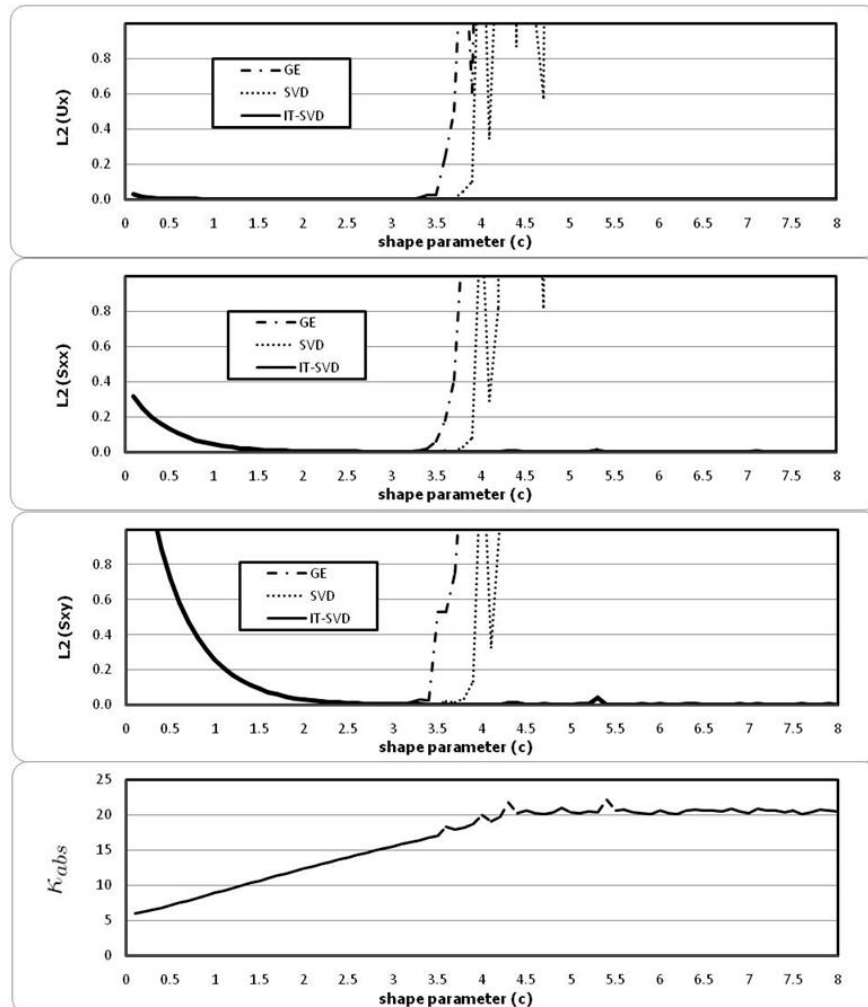
Figure 10a: The L_2 errors of u_x, σ_{xx}, σ_{xy} and $\kappa_{abs}(\mathbf{A})$ as a function of c for the cantilever beam problem.with irregularly distisbuted data centers with Dirichelet boundary conditions.

aries. The shape parameter varied in the range of $c = 0.1 \sim 8$. Figure 8 shows relative error norms of the results.

The advantages of the IT-SVD are more pronounced in a relatively large scale problem such as this example. The value of the monotonic threshold shape parameter is about c_t =3.5 for the Neumann type boundary conditions while the critical value of shape parameter is about c_t =3 for Dirichlet type boundary conditions. So, it is not possible to adjust the shape parameter in the range of $c_t < c < c_{cr}$, if $c_{cr} < c_t$. It means that numerical instability appears before the shape parameter passes the monotonic threshold shape parameter.

The efficiency of proposed method is evident in

this case. The IT-SVD reduces the round-off error and makes it possible to select an appropriate value for shape parameter. Figure 8 shows that increasing the shape parameter beyond the monotonic threshold shape parameter significantly reduced the error when the IT-SVD is used for solving resultant coefficient matrix.

Example 5. Cantilever beam (Irregular distribution) The effect of node arrangement on the stability and accuracy of proposed method is also studied numerically through the analysis of cantilever beam problem discretized by random distribution of 31×31 points. Figure 9 shows the point arrangement used in the analysis; figure 10
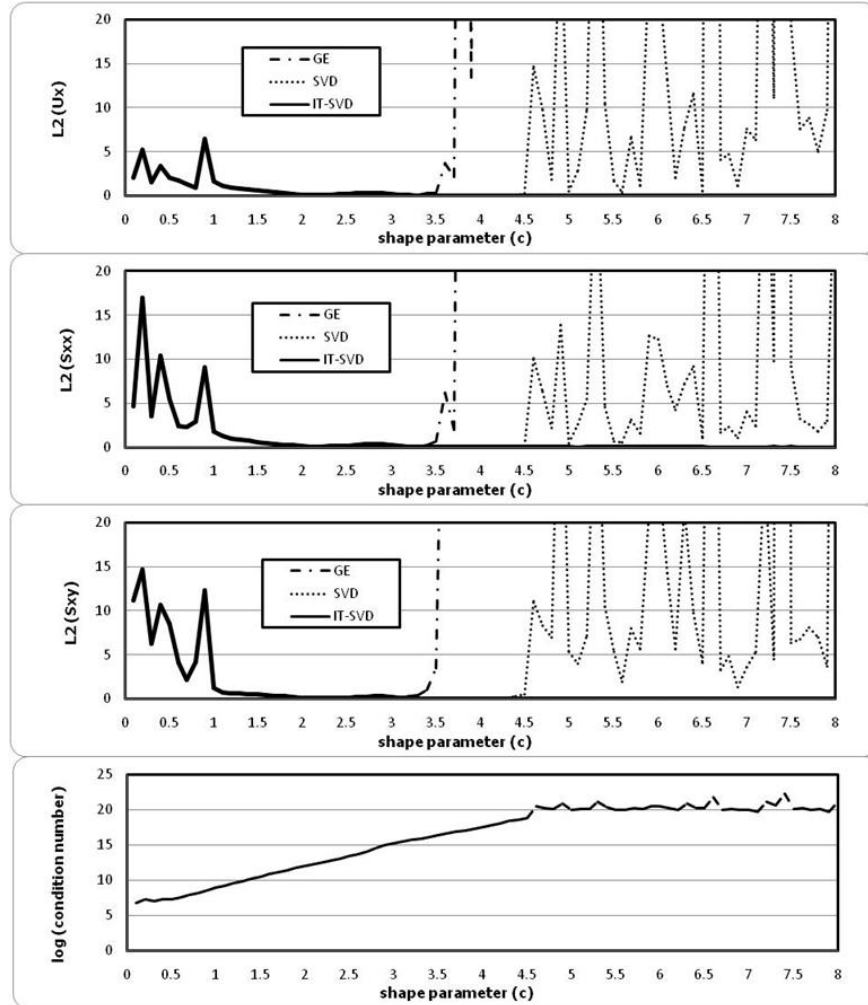
Figure 10b: The L_2 errors of u_x, σ_{xx}, σ_{xy} and $\kappa_{abs}(\mathbf{A})$ as a function of c for the cantilever beam problem.with irregularly distisbuted data centers with Neumann boundary conditions.The minimum errors of each method are summarized in Table 5.

Table 4: Minimum error in cantilever beam problem (regular distribution)

Boundary Type	Neumann	Neumann	Neumann	Dirichlet	Dirichlet	Dirichlet
Solver Method	GE	SVD	IT-SVD	GE	SVD	IT-SVD
u_x	1.48E-2	1.47E-2	5.82E-5	1.83E-4	8.38E-5	5.07E-6
u_y	1.27E-2	1.07E-2	3.35E-5	0.23E-4	1.25E-5	5.35E-7
σ_{xx}	4.34E-2	4.24E-2	8.38E-5	1.82E-3	9.13E-4	3.18E-5
σ_{xy}	3.78E-2	3.78E-2	8.82E-5	1.85E-2	1.03E-2	3.95E-4

shows the relative error norms of the results.

The threshold shape parameter is about $c_t = 0$ and $c_t=1$ for the Neumann and Dirichlet boundary conditions, respectively while the monotonic threshold critical value of the shape parameter is about $c_{cr}=3.5$ for both cases of boundary conditions.

5 Discussion

The approach presented can be considered as a numerical instability reducer scheme that makes it possible to obtain a good quality solution as-

Table 5: Minimum error in cantilever beam problem (regular distribution)

Boundary Type	Neumann	Neumann	Neumann	Dirichlet	Dirichlet	Dirichlet
Solver Method	GE	SVD	IT-SVD	GE	SVD	IT-SVD
u_x	1.48E-2	1.47E-2	5.82E-5	1.83E-4	8.38E-5	5.07E-6
u_y	1.27E-2	1.07E-2	3.35E-5	0.23E-4	1.25E-5	5.35E-7
σ_{xx}	4.34E-2	4.24E-2	8.38E-5	1.82E-3	9.13E-4	3.18E-5
σ_{xy}	3.78E-2	3.78E-2	8.82E-5	1.85E-2	1.03E-2	3.95E-4

Table 6: Minimum error in cantilever beam problem (irregular distribution)

Boundary Type	Neumann	Neumann	Neumann	Dirichlet	Dirichlet	Dirichlet
Solver Method	GE	SVD	IT-SVD	GE	SVD	IT-SVD
u_x	5.28E-2	0.12E-2	1.90E-5	3.06E-5	2.71E-5	1.85E-6
u_y	1.64E-2	6.20E-4	6.47E-6	5.76E-6	1.93E-6	1.49E-7
σ_{xx}	7.62E-2	0.62E-2	4.22E-5	7.62E-4	3.52E-4	4.42E-5
σ_{xy}	7.36E-2	0.47E-2	9.08E-5	3.32E-3	1.97E-3	2.29E-4

Table 7: Maximum accuracy improvement by using SVD or IT-SVD instead of the GE solver.

Boundary condition		Dirchelet	Neumann	Neumann	Neumann
Solver method	variable	SVD	IT-SVD	SVD	IT-SVD
Tension Bar	u_x	1.1	7	0.9	7
Tension Bar	σ_{xx}	1.1	3	0.9	1.5
Patch test (regular)	u_x	1.8	42	1.5	80
Patch test (regular)	σ_{xx}	1.2	15	1.1	20
Patch test (irregular)	u_x	1.1	17	1.3	7
Patch test (irregular)	σ_{xx}	1	3	1.5	7
Cantilever beam (regular)	u_x	2.2	36	1	255
Cantilever beam (regular)	σ_{xx}	2	57	1	518
Cantilever beam (irregular)	u_x	1.1	16	44	2780
Cantilever beam (irregular)	σ_{xx}	2.2	17	12	1805

sociated with high values of the shape parameter, c. Table 6 shows the efficiency of proposed method on improving accuracy of the solution. The proposed method may be used for enhancing the numerical solution of boundary vale problems with combinations of Dirchelet and/ or Neumann boundary conditions. The efficiency of method is more pronounced in large scale problems where numerical instability occurs even by considering low value of shape parameter in the analysis when h is refined.

It would help the readers to better understand the benefits of the IT-SVD method. For instance, it is useful to mention that the monotonic threshold shape parameter, c_t, is problem dependent, and it is significantly affected by node arrangement or boundary conditions. On the other hand, the critical shape parameter mainly depends on solver scheme. These facts can be simply concluded by comparing the 2nd and 3rd examples or the 4th and 5th examples. So, to reach a reliable solution in problems with different node arrangement and boundary conditions that may have different c_t, one should use large values of shape parameter to certainly pass the criteria, $c > c_t$. In practical problems, this is only possible when we use a stable solver e.g. the IT-SVD to circumvent ill-conditioning problems.

Also, we could emphasize the simplicity and efficiency of the IT-SVD scheme. In spite of pervious works in the literature e.g. those presented by Fornberg et.al., the IT-SVD is simple, easy to im-

plement, and efficient in accuracy and CPU time. In addition, one can utilize the IT-SVD in his/her codes by including a supplementary subroutine without any modifications on the other parts of the codes.

Numerical studies show that the value of the monotonic threshold shape parameter is significantly affected by number of node, arrangement of nodes and the type of imposed boundary condition. However, it seems that the value of critical value of the shape parameter is mainly affected by condition number of coefficient matrix. In a future paper, we will examine a hybrid combination of methods to permit the impact of high precision arithmetic on improving the IT-SVD based on the Volokh-Vilnay method, using preconditioning, variable shape parameters, and domain decomposition.

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