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Numerical Computation of Electromagnetic Fields by the Time-Domain Boundary Element Method and the Complex Variable Method

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Abstract: This work presents an alternative procedure to compute time-domain electromagnetic fields. The Boundary Element Method is here adopted to numerically analyze wave propagation problems, computing just a so-called primary field (either the electric or the magnetic field can be selected as primary field; the complementary field is here named secondary field). The secondary field is obtained following Maxwell's equations, i.e., considering space derivatives of the primary field (computed by the Complex Variable Method) and time integration procedures. This methodology is more efficient and flexible since fewer systems of equations must be solved at each time-step. At the end of the paper, numerical applications illustrate the accuracy and potentialities of the proposed technique.

Keyword: Time-Domain Boundary Element Method; Complex Variable Method; Maxwell's Equations; Wave Propagation Problems; Electromagnetic Fields.

1 Introduction

In the present work, the Time-Domain Boundary Element Method (TD-BEM) and the Complex Variable Method (CVM) are combined to numerically compute electromagnetic fields.

In the last years, numerical computation of electromagnetic fields has been subject of intensive research [Reitich and Tamma (2004), Bleszynski, Bleszynski and Jaroszewicz (2004), Milazzo, Benedetti and Orlando (2006)], specially taking into account wave propagation applications. In the present work, the TD-BEM is employed to numerically analyze scalar wave propagation problems in homogeneous media (it is important to note that each Cartesian component of a electric/magnetic vectorial field may be treated independently, as a scalar wave propagation problem, when homogeneous medium is considered). As it is well known, the TD-BEM has several advantages, as for instance the capability to deal with infinite media in a very elegant manner, describing perfectly radiation conditions; moreover, time-domain procedures are quite general, allowing proper analysis of high or low frequency problems.

The objective of the present work is to apply the TD-BEM to compute just a primary electromagnetic field. This primary field can be either the electric or the magnetic field (the selection is considered taking into account the characteristics of the problem to model). Once the so-called primary field is evaluated (analyzing wave propagation problems), the secondary field (magnetic or electric field – the one which complements the primary field) is computed directly, considering Maxwell's equations.

The above-described procedure aims to be more efficient, since fewer systems of equations (just the systems related to the primary field) need to be solved at each time step of the analysis. Moreover, since just one electromagnetic field (primary field) is modeled, more flexible and easier to implement simulations take place.

Following Maxwell's equations, space derivatives of the primary field are necessary in order to compute the secondary field. In a TD-BEM context, however, internal point space derivatives are not accurately computed when simple numerical procedures are considered [Mansur (1983)] and

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the adoption of analytical derivative expressions within the TD-BEM kernels generates high computational cost procedures, which are mathematically elaborated and difficult to implement [Carrer and Mansur (1994)]. In the present work, the Complex Variable Method is adopted to numerically compute space derivatives in a TD-BEM context (as one will observe, this numerical technique is robust and accurate). According to Maxwell's equations, time-integration procedures are also necessary in order to compute secondary fields. This is a very simple numerical task and the classical trapezoid rule (two point Newton-Cotes quadrature rule) is here adopted.

The present work is focused on two-dimensional applications; however, the procedures here presented are general and may easily be extended to three-dimensional problems. The scalar two-dimensional time-domain BEM formulation was first introduced by Mansur (1983) and, latter on, Mansur and Carrer (1993) re-presented the kernels of the problem employing the concept of finite part integrals, as introduced by Hadamard (1952). Accurate space derivative calculations for the scalar wave propagation problem were introduced by Carrer and Mansur (1994). In that work, conventional boundary integral equation procedures were adopted and the derivatives of the finite part integrals were presented. Further on, Carrer and Mansur (1999) extended these procedures to elastodynamic analysis.

An interesting alternative approach (the CVM) to compute space derivatives may be developed based on the work of Lyness and Moler (1967). This approach is very attractive, specially in a BEM context, since it provides a straightforward way to compute derivatives: the kernels that appear in the BEM integral equations are written as function of complex source point coordinates and their derivatives are computed using second-order expressions that arise from complex Taylor's expansions. In this methodology, only imaginary parts of complex functions contribute to the final result and accuracy is controlled taking into account imaginary increments (which may be considered very small). The motivation to employ such a methodology (combination of BEM and CVM procedures) in electromagnetic wave propagation analyses is justified by the reliable results obtained by Soares Jr *et al.* (2002, 2005), concerning the numerical evaluation of internal stress and velocity fields in elastodynamics, as well as the good results obtained by Gao *et al.* [Gao, Liu and Chen (2001, 2002), Gao and He (2005)] concerning elastostatic nonlinear analysis and heat conduction problems.

In the present paper, first the Maxwell's equations and the wave propagation governing equations are briefly presented. In the sequence, the TD-BEM and the CVM are discussed, describing the methodology developed to numerically compute primary and secondary fields. At the end of the paper, numerical applications are presented, verifying the accuracy and potentialities of the proposed technique.

2 Governing equations

The Maxwell's equations in differential form can be written as follows

$$\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t \tag{1a}$$

$$\nabla \times \mathbf{H} = \partial \mathbf{D}/\partial t + \mathbf{J} \tag{1b}$$

$$\nabla \cdot \mathbf{D} = \rho \tag{1c}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{1d}$$

where $\mathbf{E}$ and $\mathbf{H}$ are the electric and magnetic field intensity, respectively; $\mathbf{D}$ and $\mathbf{B}$ are the electric and magnetic flux density, respectively; and $\mathbf{J}$ and ρ are the electric current and electric charge density, respectively. The constitutive relations between the field quantities are specified as

$$\mathbf{D} = \varepsilon \mathbf{E} \tag{2a}$$

$$\mathbf{B} = \mu \mathbf{H} \tag{2b}$$

$$\mathbf{J} = \sigma \mathbf{E} \tag{2c}$$

where the parameters ε, μ and σ denote, respectively, the permittivity, permeability and conductivity of the medium.

Combining equations (1) and (2) for homogeneous media, vectorial wave equations describing the electric and magnetic fields are obtained, as shown by equations (3).

$$\nabla^2\mathbf{E} - \varepsilon\mu\partial^2\mathbf{E}/\partial t^2 = \mu\partial\mathbf{J}/\partial t + (1/\varepsilon)\nabla\rho \quad (3a)$$

$$\nabla^2\mathbf{H} - \varepsilon\mu\partial^2\mathbf{H}/\partial t^2 = -\nabla\times\mathbf{J} \quad (3b)$$

In each space-coordinate direction, considering proper boundary and initial conditions, equations (3) can be treated independently, describing scalar wave problems, as indicated by equation (4).

$$\nabla^2 p - (1/c^2)\partial^2 p/\partial t^2 = \gamma \quad (4)$$

In equation (4), $c = (\varepsilon\mu)^{-1/2}$ stands for the medium wave propagation velocity and p and γ are generic field and source terms, respectively.

The numerical solution of the scalar wave equation (4) (computation of the primary field p), in two-dimensional applications, is briefly discussed next, taking into account the Time-Domain Boundary Element Method.

3 The Boundary Element Method solution

The basic integral equation for the scalar wave propagation problem (equation (4)), can be written as (initial condition terms are disregarded, for simplicity)

$$\begin{aligned}4\pi c(\xi)p(\xi,t) &= \int_\Omega\int_0^{t+} p*(\mathbf{X},t;\xi,\tau)\gamma(\mathbf{X},\tau)d\tau d\Omega \\ &+ \int_\Gamma\int_0^{t+} p*(\mathbf{X},t;\xi,\tau)q(\mathbf{X},\tau)d\tau d\Gamma \\ &- \int_\Gamma \frac{\partial r(\mathbf{X};\xi)}{\partial\mathbf{n}} \fint_0^{t+} p_r*(\mathbf{X},t;\xi,\tau)p(\mathbf{X},\tau)d\tau d\Gamma\end{aligned} \quad (5)$$

in which $p(\mathbf{X},t)$, as previously discussed, is a generic function representing the components of the electric or magnetic field (primary field) and $q(\mathbf{X},t)$ is defined by $q = \partial p/\partial\mathbf{n}$ (it denotes the flux along the boundary with outward normal vector $\mathbf{n}$). In equation (5), the geometric coefficient $c(\xi)$ is equal to 1, if $\xi\in\Omega$; or equal to $\alpha/(2\pi)$, if $\xi\in\Gamma$, where α is the boundary internal angle (Ω and Γ stand for the domain and the boundary of the problem, respectively). The fundamental functions presented in equation (5) are given by

$$\begin{aligned}p*(\mathbf{X},t;\xi,\tau) &= P*(\mathbf{X},t;\xi,\tau)\cdot H[t_R] \\ &= 2c(c^2(t-\tau)^2 - r^2)^{-1/2}\cdot H[t_R]\end{aligned} \quad (6a)$$

$$\begin{aligned}p_r*(\mathbf{X},t;\xi,\tau) &= P_r*(\mathbf{X},t;\xi,\tau)\cdot H[t_R] \\ &= 2cr(c^2(t-\tau)^2 - r^2)^{-3/2}\cdot H[t_R]\end{aligned} \quad (6b)$$

where H stands for the Heaviside function, $t_R = (t-\tau) - r/c$ defines a retarded time, r stands for the distance between the field point ($\mathbf{X}$) and the fundamental source point (ξ) and c is the wave velocity . The symbol $\fint$ on the last term on the right-hand-side of equation (5) denotes the finite part of an integral. Following Hadarmard [Hadamard (1952)], this operation can be written as

$$\begin{aligned}&\fint_0^{t+} p_r*(\mathbf{X},t;\xi,\tau)p(\mathbf{X},\tau)d\tau \\ &= \lim_{\tau\to t-r/c}\Big(\int_0^{\tau} P_r*(\mathbf{X},t;\xi,\tau)p(\mathbf{X},\tau)d\tau \\ &\quad - \frac{1}{c}P*(\mathbf{X},t;\xi,\tau)p(\mathbf{X},\tau)\Big)\end{aligned} \quad (7)$$

In order to numerically implement the TD-BEM, approximations in time and along the boundary are introduced as indicated by equations (8)

$$p(\mathbf{X},t) = \sum_{j=1}^{J}\sum_{m=1}^{M}\phi_p^m(t)\eta_p^j(\mathbf{X})p_j^m \quad (8a)$$

$$q(\mathbf{X},t) = \sum_{j=1}^{J}\sum_{m=1}^{M}\phi_q^m(t)\eta_q^j(\mathbf{X})q_j^m \quad (8b)$$

where $\eta_p^j(\mathbf{X})$ and $\eta_q^j(\mathbf{X})$ are space interpolation functions corresponding to a boundary node $\mathbf{X}^j$ and $\phi_p^m(t)$ and $\phi_q^m(t)$ are time interpolation functions corresponding to a discrete time t^m. In the present work, linear interpolation functions are adopted for η_p, η_q and ϕ_p and piecewise constant functions are adopted for ϕ_q. The coefficients p_j^m and q_j^m, presented in equations (8), are defined by (nodal values): $p_j^m = p(\mathbf{X}^j,t^m)$ and $q_j^m = q(\mathbf{X}^j,t^m)$.

Adopting matrix notation and taking into account the approximations given in equations (8), equation (5) can be written at each boundary node and generic time-step t^n as

$$\begin{aligned}&\left(\mathbf{C}+\mathbf{H}^1\right)\mathbf{P}^n = \\ &\mathbf{G}^1\mathbf{Q}^n + \sum_{m=1}^{n-1}\left(\mathbf{G}^{n-m+1}\mathbf{Q}^m - \mathbf{H}^{n-m+1}\mathbf{P}^m\right) + \mathbf{S}^n\end{aligned} \quad (9)$$

where $\mathbf{H}^n$ and $\mathbf{G}^n$ are influence matrices computed at time-step t^n and vector $\mathbf{S}^n$ stands for source terms (first integral term on the right-hand-side of equation (5)). After introducing boundary conditions in equation (9), the following expression is obtained

$$\mathbf{A}\mathbf{x}^n = \mathbf{B}\mathbf{y}^n + \mathbf{R}^n + \mathbf{S}^n \tag{10}$$

where the entries of $\mathbf{x}^n$ are unknown nodal variables, while the entries of vector $\mathbf{y}^n$ are the corresponding known boundary values (boundary conditions). $\mathbf{R}^n$ is the vector related to the time-convolution process of the TD-BEM; it represents the complete history up to t^{n-1}. Further details on the time-domain boundary element formulation can be found in Mansur (1983) and Dominguez (1993).

The electromagnetic wave propagation problem described by equations (3) can be analyzed taking into account equations (4)-(10) and the components (for each component, a different system of equations must be solved) of the electric or magnetic fields (i.e., E_k or H_k) may be evaluated along the boundary of the problem. Once the boundary node values are obtained, one may directly (i.e., without solving any system of equations) evaluate internal point values by adopting $c(\xi) = 1$ in equation (5).

In the present work, instead of analysing both electric and magnetic problems (equations (3a) and (3b)) taking into account equations (4)-(10), just the electric or the magnetic field (as discussed before, the selected field will be named here "primary" and the other one "secondary") is computed by this procedure (TD-BEM). The secondary field is calculated by taking into account equations (1) and (2), i.e., by directly employing Maxwell's equations. Thus, a more efficient procedure is achieved since fewer systems of equations must be solved.

In order to evaluate the secondary field, space derivatives of the primary field are necessary (see equations (1)) and the Complex Variable Method is here adopted to evaluate these derivatives.

4 The Complex Variable Method

The CVM was originally proposed by Lyness and Moler (1967) and was successfully applied by Soares Jr *et al.* (2002) to numerically compute internal stress and velocity fields in elastodynamic problems, taking into account the TD-BEM.

The CVM can be described as follows: given a real function $f(x)$, if the variable x is replaced by a complex variable $x + i\Delta x$, and Δx is sufficiently small, $f(x+i\Delta x)$ can be expanded in the neighborhood of $f(x)$ taking into account Taylor's series, as indicated below

$$\begin{aligned} f(x+i\Delta x) =& f(x) + \frac{i\Delta x}{1!}\frac{df(x)}{dx} - \frac{\Delta x^2}{2!}\frac{d^2f(x)}{dx^2} \\ &- \frac{i\Delta x^3}{3!}\frac{d^3f(x)}{dx^3} + \frac{\Delta x^4}{4!}\frac{d^4f(x)}{dx^4} + \ldots \end{aligned} \tag{11}$$

Hence, the first derivative in expression (11) can be written as

$$df(x)/dx = \mathrm{Im}\left[f(x+i\Delta x)\right]/\Delta x + O(\Delta x^2) \tag{12}$$

where Im stands for imaginary part. As one may observe, the expression for the first derivative (equation (12)) is second order accurate with respect to the step-size.

In order to evaluate space derivatives of the primary field considering equation (12), the fundamental solutions that appear in equations (5)-(6) must be written as function of complex coordinates (as indicated by equation (13)) and the imaginary part of the computed integrals considered.

$$\begin{aligned} &4\pi\partial p(\xi,t)/\partial\xi_k \\ &= \mathrm{Im}\left[\int_\Omega\int_0^{t+} p*(\mathbf{X},t;\xi+i\Delta\xi_k,\tau)\gamma(\mathbf{X},\tau)d\tau d\Omega\right]/\Delta\xi_k \\ &+ \mathrm{Im}\left[\int_\Gamma\int_0^{t+} p*(\mathbf{X},t;\xi+i\Delta\xi_k,\tau)q(\mathbf{X},\tau)d\tau d\Gamma\right]/\Delta\xi_k \\ &- \mathrm{Im}\left[\int_\Gamma \frac{\partial r(\mathbf{X};\xi+i\Delta\xi_k)}{\partial\mathbf{n}}\right. \\ &\left.\times \fint_0^{t+} p_r*(\mathbf{X},t;\xi+i\Delta\xi_k,\tau)p(\mathbf{X},\tau)d\tau d\Gamma\right]/\Delta\xi_k \end{aligned} \tag{13}$$

The following algorithm describes the methodology adopted to compute internal secondary fields, once primary fields are already evaluated (section 2) along the boundary of the model:

1. A CVM step-size $\Delta\xi_k$ for each Cartesian coordinate is selected;

2. The Cartesian coordinate ξ_k of the source point ξ is written as $\xi_k + i\Delta\xi_k$;

3. The space derivatives of the primary field are computed as described by equation (13): i.e., taking into account a complex version of equation (5) for internal points and adopting the approximations indicated by equation (12);

4. The space derivatives of the primary field are combined regarding equations (1)-(2) and the time derivative of the secondary field is obtained;

5. The time derivative of the secondary field is time integrated adopting numerical procedures (the trapezoid rule, for instance) and the secondary field is finally computed.

It is important to observe that, in order to apply the present methodology, the analyticity of the functions involved must be preserved after their arguments replacement. Considering intricate time-space fundamental solutions, as it is the case, this can be accomplished by proper numerical implementation, i.e., considering the causality phenomenon, wave front velocities, suitable branch-cuts etc.

Though dealing with a numerical derivative concept, this approach is very attractive. All the complicated operations that arise from an analytical treatment of equation (5), in order to evaluate its space derivatives (see Carrer and Mansur (1994), for instance) may be avoided. Much simpler procedures result considering equation (12) and the precision of this numerical derivative methodology is easily controlled by the step-size of the input parameter $\Delta\xi_k$ (which may be very small).

By combining the TD-BEM and the CVM, as described above, both electric and magnetic field components may be computed at once, avoiding several systems of equations to be considered. In the next section some numerical applications are presented, illustrating the accuracy of the proposed methodology.

5 Numerical applications

Two numerical applications are described in the present work. For both applications the electric field intensity is considered the primary field and the magnetic flux density is computed as the secondary field. In the first application, natural boundary conditions are prescribed along the entire model for the primary field wave propagation analysis; in the second application, essential boundary conditions are considered. As one will observe, both applications are intimately related to the electromagnetic fields that arise when infinitely long wires, carrying linear time-dependent currents, are considered.

5.1 Cylindrical surface with circular cross-section

In the present application the electromagnetic fields surrounding a cylindrical surface with circular cross section are analysed. The present application highlights the ability of the TD-BEM to deal with infinite media in a very elegant manner, describing perfectly radiation conditions.

A sketch of the model in focus is depicted in Fig.1. The cross section radius is $R = 0.10m$ and the proprieties of the infinite medium are: $\mu = 1.2566 \cdot 10^{-6} H/m$ (magnetic permeability) and $\varepsilon = 8.8544 \cdot 10^{-12} F/m$ (electric permittivity). In the present application the electric field is considered the primary field and the following natural boundary condition is prescribed along the whole cylindrical surface

$$\partial E_z(R,t)/\partial \mathbf{n} = -\mu/(2\pi)\cdot(ct/R)\cdot(c^2t^2-R^2)^{-1/2} \cdot H[ct-R] \quad (14)$$

Considering the above-described boundary condition, the analytical expressions for the electric and magnetic fields that arise in the infinity medium are given by

$$\mathbf{E}(\rho,t) = \frac{\mu}{2\pi} \cdot \ln\left[\frac{ct+(c^2t^2-\rho^2)^{1/2}}{\rho}\right] \cdot H[ct-\rho]\cdot \mathbf{z} \quad (15a)$$

$$\mathbf{H}(\rho,t) = \frac{1}{2\pi c} \cdot \left(\frac{ct}{\rho} - \frac{\rho}{ct+(c^2t^2-\rho^2)^{1/2}}\right) \cdot H[ct-\rho]\cdot \theta \quad (15b)$$

where ρ is the distance between the field point and the centre of the cylindrical surface cross section and $\mathbf{z}$ and θ are unit vectors (cylindrical coordinates).

The boundary element discretization adopted to numerically analyze the model is depicted in Fig.1: Six linear boundary elements are employed. In the present analysis the double symmetry of the problem is taken into account. An interesting feature of the boundary element formulation is that symmetric bodies under symmetric loads can be analyzed without discretization of the symmetry axes. This can be accomplished by an automatic condensation process, which integrates over reflected elements and performs the assemblage of the final matrices in reduced size [Telles (1983)]. The time-step adopted in the TD-BEM analysis is $\Delta t = 2.0 \cdot 10^{-11} s$ and the step-size adopted in the CVM analysis is $\Delta\xi_k = 10^{-10} m$.

Fig.2 shows the modulus of the electric field intensity (primary field) and of the magnetic flux density (secondary field) obtained at point A ($x = 0.2m$ and $y = 0.0m$ – see Fig.1) considering the proposed methodology. Analytical time histories are also depicted in Fig.2, highlighting the excellent accuracy of the numerical results. As it can be observed, the present methodology is a very efficient tool to simulate transmitting boundaries and to compute electric and magnetic fields (both together) along infinite media.

In fact, the electromagnetic fields considered in the present application describe the fields that arise when an infinitely long wire carrying a current $I(t) = t$ is located along the adopted z-axis

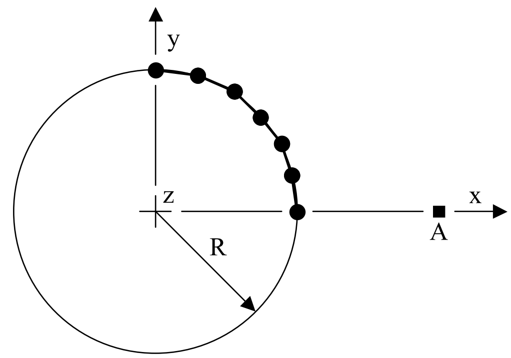
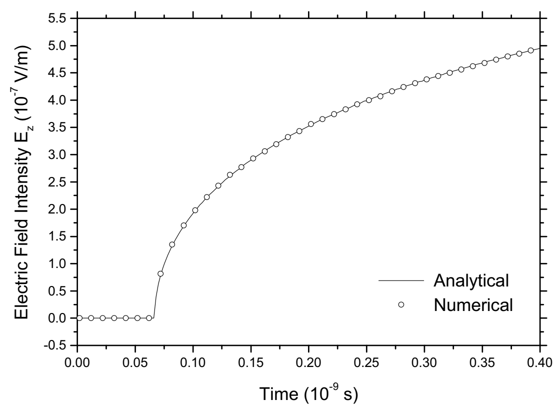


Figure 1: Infinite domain surrounding a cylindrical surface with circular cross section: sketch of the model and boundary element mesh.

(a)



(b)

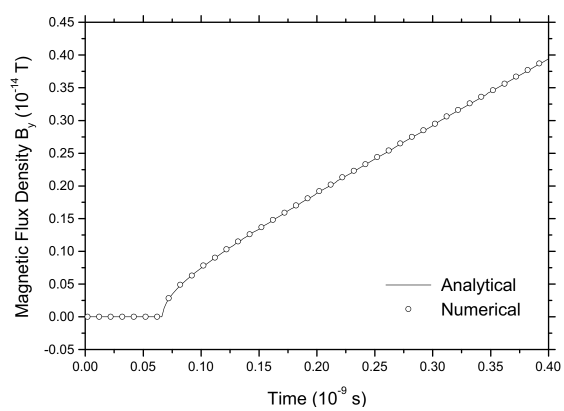


Figure 2: Electromagnetic time-history fields at point A: (a) electric field intensity (primary field); (b) magnetic flux density (secondary field).

[Machado (2006)]. Thus, in the present application, the selected BEM mesh may also be interpreted as a perfect silent boundary for any internal field computation, as in a BEM-FEM coupling context, for instance.

5.2 Cylindrical surface with rectangular cross-section

In the present application the analytical expressions (15), due to an infinitely long wire, parallel to a given cylindrical surface, are further explored. Two wires are here considered (both carrying$I(t) = t$) and they are located as indicated by Fig.3. A boundary element mesh composed by 6 linear elements (double symmetry is once again considered) is adopted. The geometry of the model is defined by $a = 0.05m$ (see Fig.3) and the physical proprieties of the infinite medium are as described in sub-section 5.1. The time-step adopted in the TD-BEM analysis is $\Delta t = 5.0 \cdot 10^{-11}s$ and the step-size adopted in the CVM analysis is $\Delta\xi_k = 10^{-10}m$.

Fig.4 shows the electric field intensity (primary field) obtained at point A ($x = 0.15m$; $y = 0.10m$), at point B ($x = 0.15m$; $y = 0.0m$) and at point C ($x = 0.0m$; $y = 0.10m$). In Fig.5, the magnetic flux density (secondary field) is depicted. As one can observe, Figs.4 and 5 highlight once again the good accuracy of the proposed methodology.

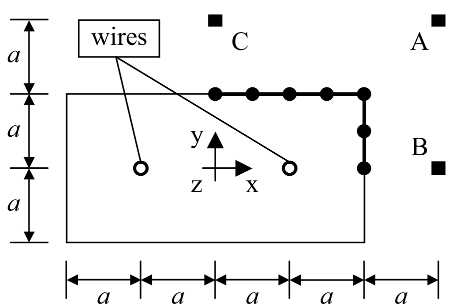


Figure 3: Infinite domain surrounding a cylindrical surface with rectangular cross section: sketch of the model and boundary element mesh.

6 Conclusions

A time-domain methodology to numerically compute electromagnetic fields was presented. The

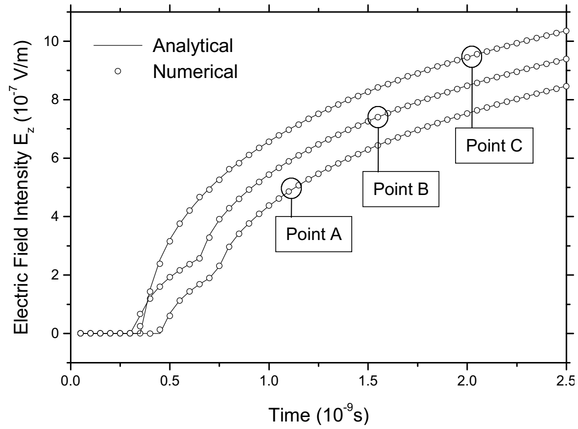
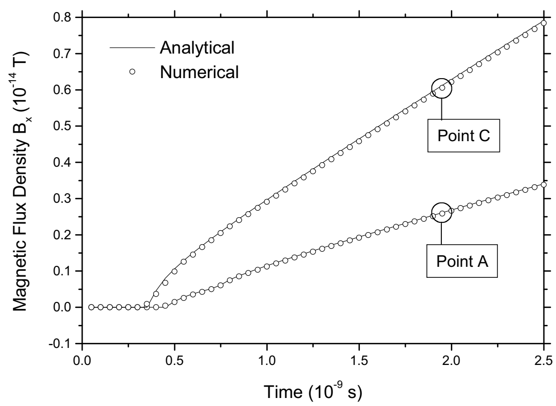


Figure 4: Electric field intensity (primary field) at points A, B and C.

(a)



(b)

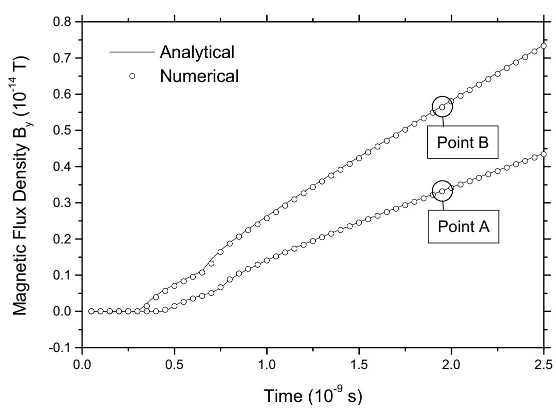


Figure 5: Magnetic Flux density (secondary field) at points A, B and C: (a) B_x; (b) B_y.

Boundary Element Method and the Complex Variable Method were combined, allowing the components of electric and magnetic fields to be computed at once, as primary and secondary fields. The proposed methodology is efficient, since it avoids the computational effort related to the calculation of different fields as new wave propagation problems; and it is simple to implement, since it does not introduce any new elaborated mathematical expression into pre-existing codes. The applications considered at the end of the paper illustrated the excellent accuracy of the proposed methodology.

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