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The Stochastic α Method: A Numerical Method for Simulation of Noisy Second Order Dynamical Systems

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Abstract: The article describes a numerical method for time domain integration of noisy dynamical systems originating from engineering applications. The models are second order stochastic differential equations (SDE). The stochastic process forcing the dynamics is treated mainly as multiplicative noise involving a Wiener Process in the Itô sense. The developed numerical integration method is a drift implicit strong order 2.0 method. The method has user-selectable numerical dissipation properties that can be useful in dealing with both multiplicative noise and stiffness in a computationally efficient way. A generalized analysis of the method including the multiplicative noise is presented. Strong order convergence, user-selectable numerical dissipation and stability properties are established in the analysis of the method. The concept of stochastic contractivity has been developed in this context. The integration method is illustrated with numerical examples of noisy mechanical systems. The method addresses the need for higher strong order convergent stochastic schemes for efficient simulation and design analysis of stiff and highly oscillatory engineering systems with multiplicative noise.

Keyword: Second order Stochastic Differential Equations, Multiplicative Noise, Multiple Itô Integrals, Numerical Dissipation

1 Introduction

Mechanical and electrical systems often are modeled as second order differential equations. However unmodeled dynamics and structural behavior contribute to noise for such system. For certain kind of devices taking the noise into account for numerical simulation is essential in order to get correct estimates of design behavior with respect to manufacturing variations, thus affecting yield. This estimate is also crucial for control design and for obtaining desired operating performance levels. Micro-mechanical systems subject to manufacturing process variation and operating condition noises is one such example. The control of an aerospace vehicle subjected to noise such as wind gusts is another example. Long slender interconnects and circuits in sub-100 nm VLSI CMOS design are subjected to process variations, introducing noise that must accounted for, so that a desirable level of yield can be maintained for the design [Winkler (2004)].

Modeling of the above engineering applications yields second order Stochastic Differential Equations (SDE). As a general model, we shall consider equations of the form

$$\ddot{x} = f(x,\dot{x},t) + B(x,\dot{x},t)\xi_t \tag{1}$$

where $x \in \mathbb{R}^n$, $f : \mathbb{R}^n \times \mathscr{T} \to \mathbb{R}^n$, $B : \mathbb{R}^n \times \mathscr{T} \to \mathbb{R}^{n\times m}$, $t \in \mathscr{T}$; ξ_t is m-dimensional Gaussian white noise and $\mathscr{T}$ is the time interval $[t_0, t_f]$. An Itô interpretation of (1) may be written as

$$dx = v\,dt \tag{2a}$$
$$dv = f(x,v,t)dt + B(x,v,t)\,dW_t \tag{2b}$$

where, in the sense of generalized stochastic processes, m-dimensional Wiener Process, W_t, has been introduced corresponding to the white noise. B is the diffusion coefficient affecting only the first order changes in x. For mechanical systems, this translates to noisy changes in velocities only.

Such second order engineering systems are often stiff and oscillatory [Petzold, Jay, and Yen

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(1997)] in the drift term f. Numerical methods for integration thus need to handle oscillation through some numerical damping of small amplitude high frequency responses. Also the numerical scheme needs to be implicit for being able to deal with stiffness. For deterministic dynamical systems, the generalized [Chung and Hulbert (1993)] and other α and similarly numerically dissipative methods [Hoff and Pahl (1988),Wood, Bossak, and Zienkiewicz (1981), Yen, Petzold, and Raha (1998),Liu (2007), Cho and Kim (2002), Briseghella, Majorana, and Pavan (2003), Cornwell and Malkus (1992)] have been developed to meet the above requirements and achieve computational efficiency, A-stability and larger time step sizes fixed over almost entire interval of integration. Preserving the efficiency of deterministic α-methods, the stochastic version of the generalized α-method is developed in this paper.

1.1 Background

Special structures of the equations (2) under consideration have been used to construct lower order methods in the form of two step schemes in [Lépingle and Ribémont (1992)]. Multistep schemes like stochastic BDF-2 may be found in [Buckwar, Horváth-Bokor, and Winkler (2006)]. Since the numerical method constructed in the present paper is a one-step scheme, a brief review confined only to one step methods is touched upon in this section. On numerical methods for second order SDE systems, treatment of Runge-Kutta, Heun, leap-frog and other schemes may be found in [Burrage, Burrage, and Tian (2004), Burrage, Lenane, and Lythe (2007)] along with analysis on measure-exact methods in the context of stationary densities. Since straightforward extension of numerical schemes for deterministic counterpart of the same stochastic equations may not produce the desired consistency and stability [Kloeden (2002)] the derivation should be done systematically. Systematically derived strong order 1.0 Milstein schemes [Milstein and Tretyakov (2004)] and strong order 1.5 strong schemes [Kloeden and Platen (1999)] have been traditionally and successfully used for numerical integration of first order SDE systems. Runge-Kutta Methods [Carletti (2006)] have been extended for Stratonovich and Itô interpretation of second order SDEs. A stochastic version of the Newmark Beta method [Roy and Dash (2002)] may be mentioned as a one step scheme providing controlled damping. The present method belongs to this last class.

2 Preliminaries

The design goal of the numerical method is to introduce user-controllable numerical damping in order to damp highly oscillatory small amplitude responses including those from small multiplicative noise and to focus the computational effort on overall responses in relatively lower frequencies which usually carry more information for the engineering design. An accompanying strategy toward computational efficiency is to take larger time steps and keep the time step size fixed over large sub-intervals of simulation, if not the entire time interval of simulation. This time step efficiency is important in the engineering context since path-wise simulations must be computed economically enough to obtain the stochastic process governing the behavior of the response with reasonable computing resources. A strongly convergent scheme is necessary for most engineering system since each component state's response is important for failsafe working of the design. Stiffness and hence implicit formulation, is particularly important when dealing with models from flexible multibody systems, structural engineering applications and multi-physics models.

The Itô-Taylor expansion is used to construct the strong order 2.0 approximation and numerical damping is affected through implicit introduction of the drift-related terms. In line with the observation in [Milstein and Tretyakov (2004)], the diffusion term in the present method is not designed to be implicit. The scheme is parameterized for user-control of numerical damping.

Throughout the paper the following standard assumptions are made as in [Kloeden and Platen (1999)]. The noise in equations (2b) is assumed to be standard m-dimensional Wiener Process with pairwise independent components. Let there be a common probability space $(\Omega, \mathscr{A}, P)$ with index $t \in \mathscr{T} \subset \mathbb{R}$ on which stochastic processes $x(t)$ and $v(t)$ are collections of random variables. The Wiener process $W = \{W_t, t \geq t_0\}$ is associated with an increasing family of σ-algebras $\{\mathscr{A}_t, t \geq t_0\}$. Each component of W_t is $\mathscr{A}_t$-measurable with $\mathbf{E}(W(t_0)) = 0$ with probability 1, $\mathbf{E}(W(t)|\mathscr{A}_{t_0}) = 0$, $\mathbf{E}((W^i_t - W^i_s)(W^j_t - W^j_s)|\mathscr{A}_s) = \delta_{i,j}(t-s)$ for $t_0 \leq s \leq t$ and $\Delta W = W(t_{n+1}) - W(t_n)$, the increments are independent at all points in the partition of the time interval $\mathscr{T}$: $t_0 \leq t_1 \leq t_2 \ldots \leq t_r \leq t_{r+1} \leq \ldots \leq t_N = t_f$.

It is assumed that in the equations (2b) f and B are component-wise jointly $\mathscr{L}^2$-measurable in $(t, \begin{pmatrix} x \\ v \end{pmatrix}) \in \mathscr{T} \times \mathbb{R}^{2n}$, where $\mathscr{L}$ is the σ-algebra of Lebsgue subsets of $\mathbb{R}$. Also, f and B are Lipschitz-continuous for all time over the simulation interval and for all values of x and v. They satisfy the linear growth bound condition, i.e., $\|.\|_2^2 \leq K^2 \left(1 + \left\| \begin{pmatrix} x \\ v \end{pmatrix} \right\|_2^2\right)$, $K > 0$ is constant. The initial value condition that $\begin{pmatrix} x \\ v \end{pmatrix}_{t_0}$ is $\mathscr{A}_{t_0}$-measurable with $\mathbf{E}\left(\left\| \begin{pmatrix} x \\ v \end{pmatrix}_{t_0} \right\|_2^2\right) < \infty$ is also assumed.

3 Numerical Scheme

Starting with the Itô-Taylor expansions, the numerical scheme leading to the stochastic version of the generalized-α method is developed in this section. Define

$$
\begin{aligned}
Y_t &:= \begin{pmatrix} x \\ v \end{pmatrix}_t \\
a(x_t, v_t, t) &:= \begin{pmatrix} v \\ f(x_t, v_t, t) \end{pmatrix} \\
\sigma^{:,j}(x_t, v_t, t) &:= \begin{pmatrix} 0 \\ B(x_t, v_t, t) \end{pmatrix} (:, j) \quad (j^{th} \text{ column})
\end{aligned}
$$

The stochastic process $\begin{pmatrix} x \\ v \end{pmatrix}_t$ in the equations (2) can be written as

$$
\begin{pmatrix} x \\ v \end{pmatrix}_t = \begin{pmatrix} x \\ v \end{pmatrix}_s + \int_s^t \begin{pmatrix} v \\ f(x_u, v_u, u) \end{pmatrix} du + \sum_{j=1}^m \int_s^t \begin{pmatrix} 0 \\ B \end{pmatrix}^{(:,j)} dW^j_u. \tag{3}
$$

The equations in (3) can be rewritten in the context of Itô-Taylor expansion as

$$
Y_t = Y_s + \int_s^t L^0 Y_u \, du + \sum_{j=1}^m \int_s^t L^j Y_u dW^j_u \tag{4a}
$$

where

$$
L^0 := \frac{\partial}{\partial t} + \sum_{k=1}^{2n} a^k \frac{\partial}{\partial y^k} + \frac{1}{2} \sum_{k,l=1}^{2n} \sum_{j=1}^m \sigma^{k,j} \sigma^{l,j} \frac{\partial^2}{\partial y^k \partial y^l} \tag{4b}
$$

$$
L^j := \sum_{k=1}^{2n} \sigma^{k,j} \frac{\partial}{\partial y^k}, \quad j \in \{1, \ldots, m\} \tag{4c}
$$

with superscripts indicating appropriate entries in the matrix or vector. Expanding equations (3) in Itô-Taylor fashion, we get

$$
\begin{aligned}
Y_t =& Y_s + a_s \int_s^t du + L^0 a_s \int_s^t \int_s^\tau du \, d\tau \\
&+ \sum_{k=1}^m \sigma^{:,j}_s \int_s^t dW^j_u \\
&+ \sum_{k=1}^m \sum_{l=1}^m L^l \sigma^{:,k}{}_s \int_s^t (W^l_\tau - W^l_t) \, dW^k_\tau \\
&+ \sum_{k=1}^m L^0 \sigma^{:,k}{}_s \int_s^t \int_s^\tau du \, dW^k_\tau \\
&+ \sum_{k=1}^m L^k a_s \int_s^t (W^k_\tau - W^k_s) \, d\tau \\
&+ \sum_{k=1}^m \sum_{l=1}^m \sum_{i=1}^m L^i L^l \sigma^{:,k}{}_s \\
&\times \int_s^t \left(\int_s^u (W^i_\tau - W^i_s) \, dW^l_\tau \right) dW^k_u \\
&+ \sum_{k=1}^m \sum_{l=1}^m \sum_{i=1}^m \sum_{j=1}^m L^j L^i L^l \sigma^{:,k}{}_s \\
&\times \int_s^t \int_s^{\tau_1} \left(\int_s^u (W^j_\tau - W^j_s) \, dW^i_\tau \right) dW^l_u \, dW^k_{\tau_1} \\
&+ \sum_{k=1}^m \sum_{l=1}^m L^l L^k a_s \int_s^t \left(\int_s^u (W^l_\tau - W^l_s) \, dW^k_\tau \right) du \\
&+ \sum_{k=1}^m \sum_{l=1}^m L^l L^0 \sigma^{:,k}{}_s \int_s^t \int_s^u \int_s^\tau dW^l_\eta \, d\tau \, dW^k_u \\
&+ \sum_{k=1}^m \sum_{l=1}^m L^0 L^l \sigma^{:,k}{}_s \int_s^t \int_s^\tau \int_s^u d\eta \, dW^l_u \, dW^k_\tau
\end{aligned}
$$

$$+R \tag{5}$$

where the terms of the form $L^{(.)}(.)_s$ denote the operator applied to the vector function component-wise and evaluated at time instant s and R is the remainder with mean square expectation of $O(h^{2.5})$. The above expansion is relatively cheaper to expand owing to the special structure of (2). After applying the operators, the expansion (5) is obtained, component-wise for vectors x and v, as

$$x_t = x_s + v_s h + f_s \frac{h^2}{2} + B_s \underbrace{\int_s^t (W_\tau - W_s) d\tau}_{I_{.,0}}$$
$$+ \sum_{k=1}^{m}\sum_{l=1}^{m}\underbrace{\sum_{i=1}^{n} B^{i,l}\frac{\partial B^{:,k}}{\partial v^i}_s}_{C^{1B}_{l,k}} \underbrace{\int_s^t \left(\int_s^t (W^l_\tau - W^l_s) dW^k_\tau\right) du}_{I_{l,k,0}}$$
$$+R_x \tag{6a}$$

$$v_t = v_s + f_s h + \frac{\partial f}{\partial t}\frac{h^2}{2} + \sum_{i=1}^{n}\left(v^i\frac{\partial f}{\partial x^i} + f^i\frac{\partial f}{\partial v^i}\right)\frac{h^2}{2}$$
$$+ \frac{1}{2}\sum_{k=1}^{n}\sum_{l=1}^{n}\sum_{j=1}^{m} B^{k,j}B^{l,j}\frac{\partial^2 f}{\partial v^k \partial v^l}\frac{h^2}{2} + B_s\int_s^t dW_\tau$$
$$+ \sum_{k=1}^{m}\sum_{l=1}^{m}\sum_{i=1}^{n} B^{i,l}\frac{\partial B^{:,k}}{\partial v^i}_s \underbrace{\int_s^u \left(W^l_\tau - W^l_s\right) dW^k_\tau}_{I_{l,k}}$$
$$+ \sum_{k=1}^{m}\left(\underbrace{\frac{\partial B^{:,k}}{\partial t} + \sum_{i=1}^{n}\left(v^i\frac{\partial B^{:,k}}{\partial x^i} + f^i\frac{\partial B^{:,k}}{\partial v^i}\right)}_{C^{2B}_k}\right)$$
$$\times \underbrace{\int_s^t\int_s^\tau du dW^k_\tau}_{I_{0,k}}$$
$$+ \underbrace{\frac{1}{2}\sum_{q=1}^{n}\sum_{l=1}^{n}\sum_{j=1}^{m} B^{q,j}B^{l,j}\frac{\partial^2 B^{:,k}}{\partial v^q \partial v^l}}_{C^{3B}_k}\int_s^t\int_s^\tau du dW^k_\tau$$
$$+ \sum_{k=1}^{m}\underbrace{\sum_{i=1}^{n} B^{i,k}\frac{\partial f}{\partial v^i}_s}_{C^{1f}_k}\underbrace{\int_s^t \left(W^k_\tau - W^k_\tau\right) d\tau}_{I_{k,0}} + R_v \tag{6b}$$

The above expansions are strong order 2.0-accurate in x and 1.5-accurate in v respectively. The notations under the under-brace have been introduced in equations (6) for referring to appropriate coefficient and multiple Itô integral terms.

3.1 The Stochastic-α Method

Based on the expansions in (6) and following the deterministic generalized-α method of [Chung and Hulbert (1993)], the drift implicit Stochastic-α method can be constructed as

$$x_{n+1} = x_n + v_n h + ((1-2\beta)\phi_n + 2\beta\phi_{n+1})\frac{h^2}{2}$$
$$+ B_n I_{.,0} + \sum_{k=1}^{m}\sum_{l=1}^{m} C^{1B}_{l,k\,n} I_{l,k,0} \tag{7a}$$
$$v_{n+1} = v_n + ((1-\gamma)\phi_n + \gamma\phi_{n+1})h$$
$$+ B_n(W_{n+1} - W_n) + \sum_{k=1}^{m}\sum_{l=1}^{m} C^{1B}_{l,k\,n} I_{l,k} \tag{7b}$$

$$\alpha_m\phi_{n+1} = (\alpha_m - 1)\phi_n + (1-\alpha_f)f_n + \alpha_f f_{n+1}. \tag{7c}$$

For the deterministic generalized α method, the algorithmic parameters β, γ, α_f, α_m are real-valued functions of a real valued user selectable parameter ρ chosen in the interval $[0,1]$ and are given as

$$\beta = \frac{1}{(1+\rho)^2} \tag{8a}$$
$$\gamma = \frac{3-\rho}{2(1+\rho)} \tag{8b}$$
$$\alpha_m = \frac{2\rho-1}{1+\rho} \tag{8c}$$
$$\alpha_f = \frac{\rho}{1+\rho}. \tag{8d}$$

For the Stochastic-α method the same parameters are needed to be assigned in terms of ρ and time step size h. This is done in the context of designing the algorithm for stability later in this paper.

4 Stability of the Stochastic-α Method

The generalized α method is practically A-stable [Chung and Hulbert (1993)], i.e, for a given problem ρ and h can be selected so that the problem lies in the induced stability region of the method.

Similar properties for the Stochastic-α method is analyzed. Being a one-step method, the stability properties are investigated by constructing an amplification matrix A on a linear test equation on the normalized time interval $[0,1]$ with multiplicative noise:

$$dq = u\,dt \tag{9a}$$

$$du = -\omega^2\, q\, dt + \sum_{l=1}^{m} (\chi_l q + \eta_l u)\; dW_t^l. \tag{9b}$$

where $q,u \in \mathbb{R}$, $\omega \in \mathbb{C}$, W is am m-dimensional Wiener Process. The amplification matrix A for the test equation (9) is defined such that $\Xi_{n+1} := \begin{pmatrix} q_{n+1} & u_{n+1}h & \phi_{n+1}h^2 \end{pmatrix}^T = A \begin{pmatrix} q_n & u_n h & \phi_n h^2 \end{pmatrix}^T =: A\Xi_n$ and $\Omega := \omega h$. Then, A can be split into deterministic part A_d and the stochastic part A_s:

$$A = A_d + A_s. \tag{10}$$

The deterministic part corresponds to the amplification matrix of the generalized-α method and the stochastic part contains the Itô integrals. Thus the parts of the amplification matrix can be written as

$$A_d := \begin{pmatrix} \frac{\alpha_f \beta\Omega^2 + \alpha_m - 1}{D_\omega} & \frac{\alpha_m - 1}{D_\omega} & \frac{\alpha_m + 2\beta - 1}{2D_\omega} \\ \frac{\gamma\Omega^2}{D_\omega} & \frac{A_{d22}}{D_\omega} & \frac{A_{d23}}{2D_\omega} \\ \frac{\Omega^2}{D_\omega} & -\frac{(\alpha_f - 1)\Omega^2}{D_\omega} & \frac{A_{d33}}{2D_\omega} \end{pmatrix}$$

$$A_s := \begin{pmatrix} \frac{(\alpha_m - 1)(X_x + Z_x)}{D_\omega} & \frac{(\alpha_m - 1)(X_v + Z_v)}{hD_\omega} & 0 \\ \frac{A_{s21}}{D_\omega} & \frac{A_{s22}}{hD_\omega} & 0 \\ -\frac{(\alpha_f - 1)\Omega^2 (X_x + Z_x)}{D_\omega} & -\frac{(\alpha_f - 1)\Omega^2 (X_v + Z_v)}{hD_\omega} & 0 \end{pmatrix}$$

where

$$\begin{aligned} D_\omega &:= (\alpha_f - 1)\,\beta\Omega^2 + \alpha_m - 1 \\ A_{d22} &:= (\alpha_f - 1)\,\beta\Omega^2 + (1 - \alpha_f)\,\gamma\Omega^2 + \alpha_m - 1 \\ A_{d23} &:= (\alpha_f - 1)\,(2\beta - \gamma)\,\Omega^2 + 2\,(\alpha_m + \gamma - 1) \\ A_{d33} &:= (\alpha_f - 1)\,(2\beta - 1)\,\Omega^2 + 2\alpha_m \\ A_{s21} &:= (S_x + W_x)h\left((\alpha_f - 1)\beta\Omega^2 + \alpha_m - 1\right) \\ &\quad - (\alpha_f - 1)\gamma\Omega^2 (X_x + Z_x) \\ A_{s22} &:= h\left((\alpha_f - 1)\beta\Omega^2 + \alpha_m - 1\right)(S_v + W_v) \\ &\quad - (\alpha_f - 1)\gamma\Omega^2 (X_v + Z_v) \end{aligned}$$

in which Z represents the summation over Itô integrals of the form $I_{(l,0)}$, X represents the summation over Itô integrals of the form $I_{(l,k,0)}$, S represents summation over the integrals $I_{(l,k)}$, and W represents the summation over the integrals of the form $\int_{t_n}^{t_{n+1}} dW^l$; with $l,k = 1,2,\ldots,m$. The subscript v to any of the Itô integrals S and X denotes summation over the expressions of the form $\eta_l \eta_k I_\varphi$ and the subscript x denotes that over $\eta_l \chi_k I_\varphi$; φ being a multi-index (as defined in [Kloeden and Platen (1999)]) either of the form $(l,k,0)$ or of the form (l,k) as appropriate. For Itô integrals W and Z, the subscript v denotes summation over expressions of the form $\eta_l I_\varphi$ and for subscript x, it denotes summation over $\chi_l I_\varphi$; φ being a multi-index of the form $(l,0)$ or (l). In all the above, l and k take non-zero integer values $1,2,\ldots,m$. The time step size is $h = t_{n+1} - t_n$. Letting $|\omega| \to \infty$, the amplification matrix A for the very stiff case of the test equation (9) becomes

$$A_{stiff} := \begin{pmatrix} -\rho & 0 & 0 \\ \frac{A_{stiff21}}{\beta} & \frac{A_{stiff22}}{\beta h} & 1 - \frac{\gamma}{2\beta} \\ -\frac{\rho + X_x + Z_x + 1}{\beta} & -\frac{h + X_v + Z_v}{\beta h} & 1 - \frac{1}{2\beta} \end{pmatrix} \tag{11}$$

where

$$\begin{aligned} A_{stiff21} &:= \beta h(S_x + W_x) - \gamma(\rho + X_x + Z_x + 1) \\ A_{stiff22} &:= \beta h(S_v + W_v + 1) - \gamma(h + X_v + Z_v) \end{aligned}$$

For $|\omega| \to 0$ in the test equation (9b) with only multiplicative noise affecting the changes in velocity, the amplification matrix A reduces to

$$A_0 = \begin{pmatrix} X_x + Z_x + 1 & \frac{h + X_v + Z_v}{h} & \frac{3\beta}{\rho - 2} + \beta + \frac{1}{2} \\ A_{021} & A_{022} & \frac{3\gamma}{\rho - 2} + \gamma + 1 \\ 0 & 0 & 2 + \frac{3}{\rho - 2} \end{pmatrix}$$

where

$$\begin{aligned} A_{021} &:= \frac{(\rho + 1)}{(\rho - 2)}(S_x + W_x)h\left(\frac{\rho - 2}{\rho + 1}\right) \\ A_{022} &:= S_v + W_v + 1 \end{aligned}$$

For the deterministic (no noise) case of the test equation (9) (i.e., $\eta_l, \chi_l \to 0$ for $l = 1,2,\ldots,m$) the amplification matrix A_d, after putting the values of the parameters in equation (8), becomes

$$A_d = \begin{pmatrix} 1-\frac{\Omega^2(\rho+1)}{D_r} & 1-\frac{\Omega^2}{D_r} & \frac{\rho-\rho^3}{-2D_r} \\ \frac{\Omega^2(\rho-3)(\rho+1)^2}{2D_r} & \frac{A_{d22}}{2D_r} & -\frac{A_{d23}}{4D_r} \\ -\frac{\Omega^2(\rho+1)^3}{D_r} & -\frac{\Omega^2(\rho+1)^2}{D_r} & -\frac{A_{d33}}{2D_r} \end{pmatrix} \tag{12}$$

where

$$D_r := -\rho^3+3\rho+\Omega^2+2$$
$$A_{d22} := -2\rho^3+6\rho+\Omega^2((\rho-2)\rho-1)+4$$
$$A_{d23} := (\rho-1)\left(2(\rho+1)^2-\Omega^2(\rho-1)\right)$$
$$A_{d33} := (\rho(\rho+2)-1)\Omega^2+2\rho^2(2\rho+3)-2.$$

The eigenvalues $\lambda(A_d)$ of the deterministic part of the amplification matrix as in (12) are as follows. Let

$$P^3 = \frac{\left(27(\rho-1)\Omega^4+2(\rho+1)^4\right)(\rho+1)^5}{\left(-\rho^3+3\rho+\Omega^2+2\right)^3} + \frac{9(\rho+1)^2(3(\rho-3)\rho+8)\Omega^2}{\left(-\rho^3+3\rho+\omega^2+2\right)^3} + 3\sqrt{3} \cdot \sqrt{\frac{\Omega^2(\rho+1)^{10}\left(27\Omega^2(\rho-1)^2+4(\rho+1)^2\right)}{\left(-\rho^3+3\rho+\Omega^2+2\right)^4}}$$

$$Q = \frac{5-\rho\left(3\Omega^2+\rho(4\rho+3)-6\right)}{3\left(-\rho^3+3\rho+\Omega^2+2\right)}$$

$$L = -\frac{(\rho+1)^4\left(3(3\rho-4)\Omega^2+(\rho+1)^2\right)}{\left(-\rho^3+3\rho+\Omega^2+2\right)^2}$$

and the eigenvalues are evaluated as

$$\lambda_1 = \frac{\sqrt[3]{2}L}{3P} - \frac{P}{3\sqrt[3]{2}} + Q \tag{13a}$$

$$\lambda_2 = -\frac{2\sqrt[3]{-2}L+P\left((-2)^{\frac{2}{3}}P-6Q\right)}{6P} \tag{13b}$$

$$\lambda_3 = -\frac{\left(1-i\sqrt{3}\right)L}{3(2^{2/3})P} + \frac{\left(1+i\sqrt{3}\right)P}{6\sqrt[3]{2}} + Q. \tag{13c}$$

For $\Omega \in \mathbb{C}$ and $\rho \in [0,1]$, the eigenvalues in (13) have $\max_\lambda(|\lambda(A_d)|) \le 1$ everywhere on the complex plane except at $\Omega = \pm\sqrt{(\rho^3-3\rho-2)}$ at which points the eigenvalues become non-analytic. This means the overall stability region for the deterministic generalized α method induced by a particular choice of ρ and h, is the entire complex plane except for a point each in the intervals $[-2,-\sqrt{2}]$ and $[\sqrt{2},2]$ on the real axis. Thus, for a particular choice of $\rho \in [0,1]$, the induced stability region excludes the points $\pm\Im\left(\sqrt{(\rho^3-3\rho-2)}\right)$ on the real axis. However, for a given ω, the parameter ρ and time step size h can be chosen so that the problem is always in the induced stability region, i.e., does not fall on one of the non-analytic points on the real axis. This gives the method A-stability for all practical purposes.

4.1 Choice of β and γ

The values of parameters β and γ as in (15) in terms of the user selectable dissipation parameter ρ is based on the expected value of the maximum absolute value of the eigenvalues of A_{stiff} as defined in (11). In order to dissipate the highly oscillatory smaller amplitude responses in the extremely stiff modes, the design goal is to choose the parameters in such a way that the all the eigenvalues of A_{stiff} are real and their absolute values are equal to ρ, thus giving a dissipation proportionate to the user selectable dissipation parameter [Chung and Hulbert (1993)]. The eigenvalues of A_{stiff} are given as

$$\lambda_1(A_{stiff}) = -\rho \tag{14a}$$

$$\vartheta_1 := \frac{4\beta h-2\gamma h+2\beta S_v h+2\beta W_v h-h-2\gamma X_v-2\gamma Z_v}{4\beta h}$$

$$\vartheta_2 := (h(2\gamma-2\beta(S_v+W_v+2)+1)+2\gamma(X_v+Z_v))^2$$

$$\vartheta_3 := -8\beta h^2(-2\gamma-S_v-W_v+2\beta(S_v+W_v+1)+1) + 16\beta h(\gamma-1)(X_v+Z_v)$$

$$\lambda_{2,3}(A_{stiff}) = \vartheta_1 \pm \frac{\sqrt{\vartheta_2+\vartheta_3}}{4\beta h} \tag{14b}$$

In order the preserve the properties similar to the deterministic generalized-α method, it is required that $\mathbf{P}(|\vartheta_1+\rho|>0) \le O(h^2)$ and $\mathbf{P}(|\vartheta_2+\vartheta_3|>0) \le O(h^2)$ for a sufficiently small time step h. Applying Markov inequality this condition may be modified as $\mathbf{E}(\lambda_2(A_{stiff})) = \mathbf{E}(\lambda_3(A_{stiff})) = -\rho$ in (14b) so that $\mathbf{E}(\vartheta_1) = -\rho$ and $\mathbf{E}(\vartheta_2+$

$\vartheta_3) = 0$. The resulting equations are solved for β and γ. Taking the solutions least sensitive to change in the time step size h we obtain

$$\begin{aligned}
\vartheta_4 &:= \\
&1152 - h^4\hat{\eta}^8 - 6h^3\hat{\eta}^6 - 6h^2(4(\rho-2)\rho+5)\hat{\eta}^4 \\
&- 96h(\rho-2)\rho\hat{\eta}^2 \\
\vartheta &:= \sqrt{2(\rho+1)^2\vartheta_4} \\
\beta_1 &:= \left(\left(h\hat{\eta}^2+4\right)\rho+1\right)\hat{\eta}^2 - 96\rho + 2\vartheta - 96 \\
\beta &= -\frac{\hat{\eta}^2 h\left(h\hat{\eta}^2+4\right)(\rho+1)}{-4h(\rho+1)\beta_1} \qquad (15a) \\
\gamma_1 &:= h^2(2\rho-1)\hat{\eta}^4 \\
\gamma_2 &:= h^2\left(2\rho^2+1\right)\hat{\eta}^4 + 2h\left(4\rho^2+2\rho+1\right)\hat{\eta}^2 \\
\gamma &= \frac{\gamma_1 + 6h\rho\hat{\eta}^2 - 24\rho^2 + \vartheta + 24}{2\left(\gamma_2 + 24(\rho+1)^2\right)} \qquad (15b)
\end{aligned}$$

$$\alpha_m = \frac{2\rho-1}{1+\rho} \quad \text{and} \quad \alpha_f = \frac{\rho}{1+\rho} \qquad (15c)$$

where $\hat{\eta}^2 = \sum_{l=1}^m \eta_l^2$. Expanding the equations (15a) and (15b) in series,

$$\begin{aligned}
\beta \approx &\left(1 - \frac{\hat{\eta}^2 h}{16} + O\left(h^2\right)\right) \\
&+ \left(-2 + \frac{\hat{\eta}^2 h}{4} + O\left(h^2\right)\right)\rho \\
&+ \left(3 - \frac{17\hat{\eta}^2 h}{24} + O\left(h^2\right)\right)\rho^2 \\
&+ \left(\frac{5\hat{\eta}^2 h}{3} - 4 + O\left(h^2\right)\right)\rho^3 \\
&+ \left(5 - \frac{27\hat{\eta}^2 h}{8} + O\left(h^2\right)\right)\rho^4 + \dots \\
\approx &\frac{1}{(\rho+1)^2} \quad \text{for very small } h \text{ or very small } \hat{\eta}^2
\end{aligned} \qquad (16a)$$

$$\begin{aligned}
\gamma \approx &\left(1 - \frac{\hat{\eta}^2 h}{16} + O\left(h^2\right)\right) \\
&+ \left(-2 + \frac{\hat{\eta}^2 h}{4} + O\left(h^2\right)\right)\rho \\
&+ \left(3 - \frac{17\hat{\eta}^2 h}{24} + O\left(h^2\right)\right)\rho^2 \\
&+ \left(\frac{5\hat{\eta}^2 h}{3} - 4 + O\left(h^2\right)\right)\rho^3 \\
&+ \left(5 - \frac{27\hat{\eta}^2 h}{8} + O\left(h^2\right)\right)\rho^4 + \dots \\
\approx &\frac{3-\rho}{2(\rho+1)} \text{ for very small } h \text{ or very small } \hat{\eta}^2.
\end{aligned} \qquad (16b)$$

Thus, for large noise multiplicative in velocities, the parameters β and γ are time step size dependent and must be computed using (15). However, for small noise and extremely small time step sizes, the same β and γ as in the deterministic generalized α method can be used. The α_f and α_m remain same as the the deterministic generalized α method since these parameters are concerned with only the drift terms and also, the noise does not affect the column corresponding to ϕ in A_s. While solving for β and γ, the following estimates (using expectations of multiple Itô integrals given in [Kloeden and Platen (1999)]) are used for finding the expectation of various second moment terms involving multiple Itô integrals.

$$\begin{aligned}
\mathbf{E}(I_{(j_1)}I_{(j_2,0)}) &= \frac{h^2}{2}\delta_{j_1,j_2} && \text{in WZ} \quad (17a) \\
\mathbf{E}(I_{(j_1)}I_{(j_2,j_3)}) &= 0 && \text{in WS} \quad (17b) \\
\mathbf{E}(I_{(j_1,j_2)}I_{(j_3,0)}) &= 0 && \text{in SZ} \quad (17c) \\
\mathbf{E}(I_{(j_1,j_2)}I_{(j_3,j_4,0)}) &= \frac{h^3}{6}\delta_{j_1,j_3}\delta_{j_2,j_4} && \text{in SX} \quad (17d) \\
\mathbf{E}(I_{(j_1,0)}I_{(j_2,j_3,0)}) &= 0 && \text{in ZX} \quad (17e) \\
\mathbf{E}(I_{(j_1)}I_{(j_2,j_3,0)}) &= 0 && \text{in WX} \quad (17f) \\
\mathbf{E}(I_{(j_1,0)}I_{(j_2,0)}) &= \frac{h^3}{3}\delta_{j_1,j_2} && \text{in } Z^2 \quad (17g) \\
\mathbf{E}(I_{(j_1,j_2,0)}I_{(j_3,j_4,0)}) &= \frac{h^4}{12}\delta_{j_1,j_3}\delta_{j_2,j_4} && \text{in } X^2 \quad (17h) \\
\mathbf{E}(I_{(j_1)}I_{(j_2)}) &= h\delta_{j_1,j_2} && \text{in } W^2 \quad (17i) \\
\mathbf{E}(I_{(j_1,j_2)}I_{(j_3,j_4)}) &= \frac{h^2}{2}\delta_{j_1,j_3}\delta_{j_2,j_4} && \text{in } S^2. \quad (17j)
\end{aligned}$$

where the components j_l of the multi-indices take non-zero integer values $1,2,\dots,m$. Also, $\mathbf{E}(I_{(j_l)}) = \mathbf{E}(I_{(j_l,j_k)}) = \mathbf{E}(I_{(j_l,0)}) = \mathbf{E}(I_{(j_l,j_k,0)}) = 0$.

4.2 Stochastic Perturbation to the Deterministic Amplification Matrix

The stability property of the Stochastic-α method, everywhere in $\mathbb{C}$ except at the two points on the real axis where the deterministic eigenvalues become singular for a given ρ, depends the largest magnitude of the eigenvalues of the amplification matrix being less than or equal to 1. The path-wise stability for the Stochastic-α method can be viewed as a perturbation in the eigenvalues of the deterministic part A_d of the amplification matrix A, i.e., as a noisy perturbation to the generalized-α method. Hence an estimate of the suitable norms of the perturbation A_s is needed.

4.2.1 An Estimate for the Norm of A_s

Because of the structure of A_s, viz., a null column and similar entries containing the multiple Itô integrals in the two other columns, the 2-Norm is expected to be $O(\sqrt{h})$. We shall use the Frobenius Norm to upper bound the estimate of $\mathbf{E}(||A_s||_2)$.

LEMMA 4.1 *For small time step size h,* $\mathbf{E}(||A_s||_F) \leq \sqrt{\frac{7}{3}}\sqrt{\hat{\eta}^2}\sqrt{h} + \frac{\sqrt{\hat{\eta}^2}\left(13\hat{\eta}^2 + \frac{8\Re(\omega)}{(\rho-2)(\rho+1)^2}\right)h^{3/2}}{8\sqrt{21}} + O(h^{5/2})$ *when both $\hat{\eta}$ and $\hat{\chi}$ are not zero. If $\hat{\eta}^2 \to 0$,* $\mathbf{E}(||A_s||_F) \leq \frac{\sqrt{\hat{\chi}^2}h^{3/2}}{\sqrt{3}} + \frac{\sqrt{\hat{\chi}^2}\Re(\omega)h^{5/2}}{\sqrt{3}(\rho-2)(\rho+1)^2} + O\left(h^{7/2}\right)$ *and* $\mathbf{E}(||A_s||_F) = 0$*, when both $\hat{\eta}^2 \to 0$ and $\hat{\chi}^2 \to 0$.*

Proof. Squaring the absolute value of every entry of A_s and adding, we take the expectation of the square of the Frobenius norm of A_s:

$$\begin{aligned}\mathbf{E}(||A_s||_F^2) \leq{}& (1-\alpha_m)^2 E_\chi \\ &+ (\alpha_f - 1)^2(2\gamma^2+1)|\Omega|^2 E_\chi \\ &+ 2\left(\hat{\chi}^2\hat{\eta}^2\frac{h^4}{2} + \hat{\chi}^2 h^3 + \hat{\eta}^4\frac{h^2}{2} + \hat{\eta}^2 h\right)\end{aligned} \tag{18}$$

where

$$E_\chi := \left(\hat{\chi}^2\frac{h^3}{3} + \hat{\chi}^2\hat{\eta}^2\frac{h^4}{12} + \hat{\eta}^2\frac{h}{3} + \hat{\eta}^4\frac{h^2}{12}\right) \times \left|\frac{1}{(\alpha_f-1)\beta\Omega^2 + \alpha_m - 1}\right|^2$$

using the expectations in (17) and well-known inequalities for random variables, such as, $\mathbf{E}(|z(X+Z)+(S+W)h|^2) \leq 2(\mathbf{E}(|z(X+Z)|^2) + \mathbf{E}(|hS+hW|^2))$, where X,Z,S,W are random variables and $z \in \mathbb{C}$ is a deterministic scalar. Substituting the values of the parameters as in (15) for small h, followed by taking the square root and expanding the right hand side in a series in h about 0, establishes the first estimate. For the second estimate, appropriate limits are taken before the series expansion in h. The second estimate is thus obtained as $\frac{\sqrt{\hat{\chi}^2(\rho+1)^2(\rho(3\rho-2)+11)}h^{3/2}}{\sqrt{6}} + O\left(h^{7/2}\right)$ as $\Re(\omega) \to \infty$. The last estimate in the Lemma follows from making $\hat{\eta}$ and $\hat{\chi}$ zero in the expression for (18). □

It may be noted that as $\Re(\omega) \to \infty$, the first estimate becomes

$$\begin{aligned}&\mathbf{E}(||A_s||_F) \leq \\ &\sqrt{\frac{1}{6}(\rho+1)^2(\rho(3\rho-2)+11)\hat{\eta}^2 + 2\hat{\eta}^2}\sqrt{h} \\ &+ \frac{\hat{\eta}^4(\rho(\rho(\rho(3\rho+4)+10)+20)+35)h^{3/2}}{8\sqrt{6}\sqrt{\hat{\eta}^2\left((\rho(3\rho-2)+11)(\rho+1)^2+12\right)}} \\ &+ O\left(h^{5/2}\right).\end{aligned}$$

When significant noise multiplicative in velocities is present, $\hat{\chi}^2$, the norm of the coefficients of noise multiplicative in positions, does not play a significant role in the the first estimate for $\mathbf{E}(||A_s||_F)$ and appears in the coefficients of $h^{\frac{5}{2}}$ and higher order terms in h. Noise multiplicative in positions alone produces lesser perturbation to A_d since the estimate for $\mathbf{E}(||A_s||_F)$ then becomes $O(h^{\frac{3}{2}})$.

4.2.2 An Estimate of the Norm of the Strictly Upper Triangular Matrix of Schur Decomposition of A_d

The sensitivity of the eigenvalues of A_d due to the additive perturbation A_s depends on the norm of the strictly upper triangular matrix from the Schur Decomposition of A_d [Golub and Van Loan (1996)]. Hence, this estimate. Let $Q^H(D+N)Q$ be the Schur Decomposition of A_d, where Q is a unitary matrix and D, a diagonal matrix and N is a strictly upper triangular matrix, such that $|N|^p = 0$ with an integer p, with $1 \leq p \leq n$. For the test equation (9), p can take the values $1,2$ or 3. A result on smooth decompositions, useful for estimating the norm of N, is stated.

LEMMA 4.2 (DIECI AND EIROLA) *[Dieci and Eirola (1999)] A_d has Schur Decomposition, Q,D,N that are smooth wherever A_d is smooth.*

Let the stability region for the deterministic generalized α method be modified to exclude a small neighborhood around each of the points of singularity on the real axis and let the modified stability region be denoted by $\tilde{\mathbb{S}}$. A_d as a function of Ω is smooth over $\tilde{\mathbb{S}}$. The spectral radius of A_d increases smoothly with ρ everywhere in $\tilde{\mathbb{S}}$. The strictly upper triangular N from the Schur Decomposition is smooth over $\tilde{\mathbb{S}}$ for the deterministic generalized α method, by Lemma (4.2). Also, $||N||_2 \leq ||A_d||_2 + ||-D||_2$, since norms remain unchanged under unitary transformation. As D is the diagonal matrix with eigenvalues of A_d, $||D||_2 = \max|\lambda(A_d)| \leq 1$ on $\tilde{\mathbb{S}}$. Thus we estimate $||N||_2 \leq ||A_d||_2 + 1$. Since, A_d is smooth everywhere on $\tilde{\mathbb{S}}$, $||A_d||$ is also smooth. Then, the following estimate is used.

$$||A_d||_2^2 \leq ||A_d||_F^2 \leq \frac{\varpi}{D_{\rho\Omega}} \tag{19}$$

where

$$D_{\rho\Omega} := \left(\rho^3 - 3\rho + \Im(\Omega)^2 - \Re(\Omega)^2 - 2\right)^2 + 4\Im(\Omega)^2\Re(\Omega)^2$$

and

$$\begin{aligned}
\varpi :=\ & \left(\frac{5\Im(\Omega)^4}{4} + \frac{5\Re(\Omega)^2\Im(\Omega)^2}{2} + \frac{5\Re(\Omega)^4}{4} + \frac{15}{2}\right)\rho^6 \\
&+ \left(\frac{11\Im(\Omega)^4}{2} + 11\Re(\Omega)^2\Im(\Omega)^2 - \frac{3\Im(\Omega)^2}{4}\right)\rho^5 \\
&+ \left(\frac{11\Re(\Omega)^4}{2} + \frac{3\Re(\Omega)^2}{4} + \frac{25}{2}\right)\rho^5 \\
&+ \left(\frac{229\Im(\Omega)^4}{16} + \frac{229\Re(\Omega)^2\Im(\Omega)^2}{8} - \frac{45\Im(\Omega)^2}{4}\right)\rho^4 \\
&+ \left(\frac{229\Re(\Omega)^4}{16} + \frac{45\Re(\Omega)^2}{4} - \frac{39}{4}\right)\rho^4 \\
&+ \left(\frac{99\Im(\Omega)^4}{4} + \frac{99\Re(\Omega)^2\Im(\Omega)^2}{2} - \frac{17\Im(\Omega)^2}{2}\right)\rho^3 \\
&+ \left(\frac{99\Re(\Omega)^4}{4} + \frac{17\Re(\Omega)^2}{2} - 17\right)\rho^3 \\
&+ \left(\frac{249\Im(\Omega)^4}{8} + \frac{249\Re(\Omega)^2\Im(\Omega)^2}{4} + \frac{29\Im(\Omega)^2}{2}\right)\rho^2 \\
&+ \left(\frac{249\Re(\Omega)^4}{8} - \frac{29\Re(\Omega)^2}{2} + 21\right)\rho^2 \\
&+ \left(\frac{69\Im(\Omega)^4}{4} + \frac{69\Re(\Omega)^2\Im(\Omega)^2}{2} + \frac{53\Im(\Omega)^2}{4}\right)\rho \\
&+ \left(\frac{69\Re(\Omega)^4}{4} - \frac{53\Re(\Omega)^2}{4} + \frac{73}{2}\right)\rho \\
&+ \frac{77\Im(\Omega)^4}{16} + \frac{77\Re(\Omega)^4}{16} + \frac{3\Im(\Omega)^2}{4} \\
&+ \frac{77\Im(\Omega)^2\Re(\Omega)^2}{8} - \frac{3\Re(\Omega)^2}{4} + \frac{53}{4}.
\end{aligned}$$

$||A_d||_F^2$ thus smoothly increases with $\rho \in [0,1]$ everywhere on $\tilde{\mathbb{S}}$ for a given step size h and circular frequency ω. Then, $\max||A_d||_F^2 = ||A_d||_F^2|_{\rho=1} \leq \frac{99\Im(\Omega)^4+2\left(99\Re(\Omega)^2+4\right)\Im(\Omega)^2+99\Re(\Omega)^4-8\Re(\Omega)^2+64}{\Im(\Omega)^4+2\left(\Re(\Omega)^2-4\right)\Im(\Omega)^2+\left(\Re(\Omega)^2+4\right)^2}$ which has a minima at $\pm\frac{1}{\sqrt{5}}$ on the imaginary axis and a point of inflexion at the origin. Elsewhere on the $\tilde{\mathbb{S}}$ the norm grows smoothly and becomes 99 as $|\Omega| \to \infty$. The points of singularity of A_d have been excluded along with a neighborhood from $\tilde{\mathbb{S}}$ such that at the boundary of the excluded region $||A_d||_F$ is bounded by its limiting value for $|\Omega| \to \infty$. Thus,

$$||N||_2 \leq \sqrt{99} + 1$$

and

$$||N||_2^2 + ||N||_2 + 1 \leq 109.95$$

However, excluding the poles with a neighborhood on the boundary of which $||N||_2$ is bounded as above would exclude almost all of the real axis, i.e., between $(1.3784, \infty)$ and $(-\infty, -1.3784)$ from $\tilde{\mathbb{S}}$. This still ensures I-Stability. Since the eigenvalues are no longer analytic on on entire $\mathbb{C}^-$, A-Stability is lost. In order to include most of the $\mathbb{C}^-$, the bounds above may be relaxed to exclude only a reasonable neighborhood around the points of singularity on the real axis of the stability plane. The definition of the neighborhood depends on the engineering problem being solved.

For multibody systems relaxing the upper bound to

$$||N||_2^2+||N||_2+1 \leq 140$$

works well and results in defining

$$\tilde{\mathbb{S}} := \mathbb{C}\setminus\{(-\left|\sqrt{\rho^3-3\rho-2}\right|-4.06157, \left(0.576424-\left|\sqrt{\rho^3-3\rho-2}\right|\right)) \bigcup(\left(\left|\sqrt{\rho^3-3\rho-2}\right|-0.576424\right), \left(4.06157+\left|\sqrt{\rho^3-3\rho-2}\right|\right))\} \quad (20)$$

i.e., including all of $\mathbb{C}$ in $\tilde{\mathbb{S}}$ *except* $(-|\sqrt{(\rho^3-3\rho-2)}|-4.06157, -|\sqrt{(\rho^3-3\rho-2)}|+0.576424)$ and $(|\sqrt{(\rho^3-3\rho-2)}|-0.576424, |\sqrt{(\rho^3-3\rho-2)}|+4.06157)$. For the rest of this paper this relaxed definition of $\tilde{\mathbb{S}}$, (20) is used to denote the stability region of the Stochastic-α method.

4.2.3 Noise as Perturbation to the Deterministic Amplification Matrix

Using a result similar to Bauer-Fike (Theorem 7.2.3, [Golub and Van Loan (1996)]) the expected perturbation due to the Weiner Process, to the eigenvalues of the deterministic part A_d of the amplification matrix A, can be estimated. For A_d, the above result may be stated as follows.

LEMMA 4.3 *Let $Q^H A_d Q = D+N$ be a Schur Decomposition of $A_d \in \mathbb{C}^{n\times n}$. If $\lambda_s \in \lambda(A_d+A_s)$ and p is the smallest positive integer (at most 3 for A_d in the test problem (9)) such that $|N|^p = 0$, then,*

$$\min_{\lambda\in\lambda(A_d)} \quad |\lambda-\lambda_s| \leq \max(\theta, \theta^{\frac{1}{p}}) \quad (21)$$

where $\theta := ||A_s||_2 \sum_{k=0}^{p-1} ||N||_2^k \leq ||A_s||_F \sum_{k=0}^{p-1} ||N||_2^k$

Proof. Using the Theorem 7.2.3, in [Golub and Van Loan (1996)], and applying A_s as perturbation to A_d, the result is obtained. □

Using the estimates for $\mathbf{E}(||A_s||_F)$, we obtain

$$\mathbf{E}(\theta) = \mathbf{E}(||A_s||_2)\sum_{k=0}^{p-1}||N||_2^k \leq \mathbf{E}(||A_s||_F)\sum_{k=0}^{p-1}||N||_2^k$$

and

$$\mathbf{E}(\theta^{\frac{1}{p}}) \leq (\mathbf{E}(\theta))^{\frac{1}{p}}$$

applying Lyapunov inequality and observing that θ is positive and real. Again, using Lemma (4.1) followed by taking only the first significant terms in h, and observing that N is deterministic,

$$\mathbf{E}(\theta) \approx \begin{cases} \left(\sqrt{\frac{7}{3}}\sqrt{\hat{\eta}^2}\sqrt{h}\right)\sum_{k=0}^{p-1}||N||_2^k, & \text{for } \hat{\eta}^2, \hat{\chi}^2 > 0 \\ \left(\frac{\sqrt{\hat{\chi}^2}h^{3/2}}{\sqrt{3}}\right)\sum_{k=0}^{p-1}||N||_2^k, & \text{for } \hat{\eta}^2 \to 0 \text{ only} \\ 0, & \text{for } \hat{\eta}^2, \hat{\chi}^2 \to 0 \end{cases}$$

Next, we estimate the expected perturbation to the spectral radius of the deterministic amplification matrix.

LEMMA 4.4 *With the Schur Decomposition of and the perturbation A_s to A_d and the estimates as defined in Lemma (4.3), the expected perturbation to the eigenvalues of A_d is, almost surely,*

$$\min_{\lambda\in\lambda(A_d)} \quad |\lambda-\mathbf{E}(\lambda_\mathbf{s})| \leq \max(\mathbf{E}(\theta), \mathbf{E}(\theta^{\frac{1}{p}})) \quad (22)$$

where p is an integer such that $1 \leq p \leq 3$.

Proof. In Lemma(4.3) taking expectations on both sides, and using Jensen's inequality, the result is obtained. The degree of nilpotency of $|N|$ in the equation (9) is at most 3. □

At $\rho = 1$, the eigenvalues of (A_s+A_d) can be evaluated explicitly and using Jensen's inequality, it may be shown that the expected magnitude of the spectral radius is in between 1 and $1+\sqrt{\hat{\eta}^2}\sqrt{h}$. This is a sharper and more useful estimate since at $\rho = 1$ the perturbation based stability estimate may be quite pessimistic, thus giving a much lower probability of stability than that observed while solving SDEs from engineering applications. Similarly, as $|\omega| \to 0$, the expected spectral radius is in between 1 and $1+\sqrt{\hat{\eta}^2}\sqrt{h}$, thus again giving a sharper and realistic estimate for stability at low frequencies. For noise multiplicative only in positions, the expected magnitude of the spectral radius of the stochastic amplification matrix $A_d + A_s$ is in between 1 and $1+\sqrt{\hat{\chi}^2}h^{3/2}$. This and the preceding analysis shows that noise multiplicative in velocities is likely to have worse destabilizing effect on the Stochastic-α method than that due to the positions. Also, the destabilizing effect is no worse than the analytical destabilization, due to multiplicative noise, of the SDE in its analytical form.

4.3 Contractivity

For a choice of small enough $h > 0$ and $\rho \in [0,1]$, the Stochastic-α method exhibits contractivity in probability and mean. Contractivity for a numerical scheme for stochastic differential equations can be defined as follows.

DEFINITION 4.1 (STOCHASTIC CONTRACTIVITY) *A numerical method of the form $\Xi_{r+1} = A\Xi_r$ is said to have stochastic contractivity property*

(a) in Probability (i.e, with probability 1*)*
if $\mathbf{P}(\|\Xi_{r+1}\| \leq \|\Xi_r\|) = 1$
and
(b) in the p^{th} Mean
if $\mathbf{E}(\|\Xi_{r+1}\|^p) \leq \mathbf{E}(\|\Xi_r\|^p)$

where $A \in \mathbb{C}^{\dim(\Xi)\times\dim(\Xi)}$ is the amplification matrix of the numerical method and Ξ_r is the numerical solution for the test equation (9) generated at the r^{th} time-step in a partitioning $t_0 \leq t_1 \leq t_r \leq t_{r+1} \cdots \leq t_N = t_f$ of the time interval $\mathcal{T}$.

Following the definition a conditional contractivity property for the Stochastic-α method is established in the following Lemma.

LEMMA 4.5 (CONTRACTIVITY) *For the choice of a sufficiently small time step size h and an appropriate choice of $\rho \in [0,1]$ the Stochastic-α method has contractivity property with probability* 1 *and contractivity property in the mean.*

Proof. For the test equation (9), the stochastic-α method generates iterations of the form $\Xi_{r+1} = A\Xi_r$ at the r-th time step. The amplification matrix can be written as: $A = A_d + A_s^{(r)}$ (for r-th time step). There exists a suitable norm $\|A\| \leq \max_{\lambda\in\lambda(A)} |\lambda| + \varepsilon$ with the smallest $\varepsilon \geq 0$. Then if Ξ_r is expressed as $QX_A a$, some rotational transformation on a linear combination of the eigenvectors of A, then using the above norm on $A\Xi_r$, we obtain in the asymptotic growth rate:

$$\|\Xi_{r+1}\| \leq (\max_{\lambda\in\lambda(A)})|\lambda|)\|\Xi_r\|. \tag{23}$$

Using Lemma (4.4) and (4.3), choosing $0 < h << 1$ and then applying Markov inequality, we get

$$\begin{aligned}
&\mathbf{P}(\|\Xi_{r+1}\| \leq \|\Xi_r\|) = \mathbf{P}\left(\max_{\lambda(A)} |\lambda| \leq 1\right)\\
&= 1 - \mathbf{P}\left(\max_{\lambda(A),\lambda_(A_d)} \ \min_{\lambda(A_d)} |\lambda(A_d) - \lambda(A)| \geq \varepsilon\right)\\
&\quad(\text{since } \max_{\lambda\in\lambda(A_d)} |\lambda| \leq 1)\\
&\geq 1 - \max\left(\frac{1}{\varepsilon}\left(\sqrt{\frac{7}{3}\hat{\eta}^2 h}\right)\sum_{k=0}^{p-1} ||N||_2^k\,,\right.\\
&\left.\frac{1}{\varepsilon}\left(\sqrt{\frac{7}{3}\hat{\eta}^2 h}\sum_{k=0}^{p-1} ||N||_2^k\right)^{1/p}\right)
\end{aligned}$$

where $0 < \varepsilon << 1$, p is an integer in $[1, \dim(\Xi)]$ such that $|N|^p = 0$; and N is a strictly upper triangular matrix obtained from Schur Decomposition of $A_d := Q^H(D+N)Q$. For a sufficiently small choice of h and a suitable choice of the tolerance $0 < \varepsilon < 1 - \max_{\lambda\in\lambda(A_d)} |\lambda|$ for a given ρ and $\Omega \in \tilde{\mathbb{S}}$, $\mathbf{P}(\|\Xi_{r+1}\| \leq \|\Xi_r\|) \approx 1$. Taking into account equivalence of norms, contractivity in probability is thus shown.

From the results in Lemma (4.4) and (4.4), one may write

$$\mathbf{E}\left(\max_{\lambda\in\lambda(A_d)} |\lambda|\right) \leq 1$$

if

$$\begin{aligned}
&\mathbf{E}\left(\max_{\lambda(A),\lambda(A_d)} \ \min_{\lambda(A_d)} |\lambda(A_d) - \lambda(A)|\right)\\
&\leq 1 - \max_{\lambda\in\lambda(A_d)} |\lambda|
\end{aligned}$$

for a given $\rho \in [0,1]$, $h > 0$ and $\Omega \in \tilde{\mathbb{S}}$. Then, $h > 0$ and $\rho \in [0,1]$ can be chosen such that

$$\max\left(\left(\sqrt{\frac{7}{3}\hat{\eta}^2 h}\right)\sum_{k=0}^{p-1}||N||_2^k, \left(\sqrt{\frac{7}{3}\hat{\eta}^2 h}\sum_{k=0}^{p-1}||N||_2^k\right)^{1/p}\right)$$
$$\leq 1 - \max_{\lambda\in\lambda(A_d)} |\lambda|.$$

Taking expectation on both sides of equation (23), we obtain, almost surely,

$$\mathbf{E}(\|\Xi_{r+1}\|) \quad \leq \quad \mathbf{E}(\max_{\lambda\in\lambda(A)} |\lambda|)\mathbf{E}(\|\Xi_r\|), \tag{24}$$

using the fact that $A_s^{(r)}$ (i.e, A_s at the rth time-step) are independent across the time-steps due to the independence of increments (across time-steps) of the m-dimensional Weiner process affecting the test equation (9). With h and ρ as chosen above along with the consideration for equivalence of norms, stochastic contractivity in mean is established. □

4.4 Numerical Asymptotic Stability

Let ω be chosen such that the test equation (9) is *stochastically asymptotically stable* [Burrage, Burrage, and Mitsui (2000)] i.e, $\forall\, \varepsilon > 0$, $\forall\, t_0$,

$$\lim_{Y_0\to 0} \quad \mathbf{P}\left(\sup_{t\geq t_0} ||Y(t;t_0,Y_0)|| \geq \varepsilon\right) = 0$$

and

$$\lim_{Y_0\to 0} \quad \mathbf{P}\left(\lim_{t\to\infty} ||Y(t;t_0,Y_0)|| = 0\right) = 1$$

where $||.||$ is a suitable norm.

Definition 4.2 (Numerical Asymptotic Stability) *[Burrage, Burrage, and Mitsui (2000)] A numerical method with time step-size $h > 0$, applied to the test equation (9) with $\Omega \in \tilde{\mathbb{S}}$, and $\eta, \chi \in \mathbb{R}^m$ such that it is stochastically asymptotically stable, is defined to be* numerically asymptotically stable *if almost surely, $\lim_{N\to\infty} ||\Xi_N|| = 0$, where $\{\Xi_r\}$ is the sequence of numerical solutions to the test equation (9) generated at the end of each time step for N time steps.*

In this section the stability properties of the Stochastic-α method are analyzed in the sense of the definition (4.2). In course of the analysis, it'd be shown that due to stability requirements, noise multiplicative in positions imposes more restriction on the choice of a larger step-size for the Stochastic-α method.

Theorem 4.1 (Numerical Asymptotic Stability) *For a choice of sufficiently small step size $h > 0$, and $0 \leq \rho < 1$ Stochastic-α method is numerically asymptotically stable when applied to a stochastically asymptotically stable second order stochastic differential equation.*

Proof. At the end of r time steps, the solution generated with the linear test equation (9) is Ξ_r, and the amplification matrix is $A^{(r)} = A_d + A_s^{(r)}$. Then,

$$\Xi_r = A_d\Xi_{r-1} + A_s^{(r)}\Xi_{r-1} \tag{25a}$$

$$\text{i.e.,}\Xi_r = A_d^r\Xi_0 + \sum_{k=1}^{r-1} A_s^{(k)}A_d^{r-k-1}\Xi_k \tag{25b}$$

Taking appropriate norms,

$$\|\Xi_r\| \leq \|A_d^r\|\|\Xi_0\| + \sum_{k=1}^{r-1} \|A_s^{(k)}\|\|A_d^{r-k-1}\|\|\Xi_k\| \tag{25c}$$

For $r = N$ and $N \to \infty$, $\|A_d^N\| \to 0$ due to the $\max_{\lambda\in\lambda(A)} |\lambda| \leq 1$ property of the deterministic generalized-α method. Also, consistent with the assumptions in section (2), it is assumed that $\mathbf{P}(\Xi(0) = \Xi_0) = 1$ and $\|\Xi_0\| < \delta_0$ where $\delta_0 > 0$ is finite. Due to Lemma (4.5), $\mathbf{P}(\|\Xi_r\| \leq \delta_0) \approx 1$ for $r > 0$.. In equation (25a) with $r = N$, and $N \to \infty$ and due to Lemma (4.5) and independence of Weiner process increments, we have

$$\mathbf{P}(\lim_{N\to\infty} \|\Xi_N\| = 0) = \mathbf{P}(\|\Xi_{k-1}\| \leq \delta_0)\mathbf{P}(\|A_s^{(k)}\| < \varepsilon),$$

(where $1 >> \varepsilon > 0$, $1 \leq k \leq N-1$)

$$\geq 1 - \frac{\sqrt{7\hat{\eta}^2 h}}{\sqrt{3}\varepsilon}$$

(using Markov inequality and Lemmas (4.1), and (4.5))

$$\approx 1$$

(for a sufficiently small h and ε)

This proves numerical asymptotic stability (in probability). □

4.5 Stability in Mean and Mean Square

Stability associated with expected values of various moments is defined as follows.

DEFINITION 4.3 (STABILITY IN p-TH MEAN) *[Burrage, Burrage, and Mitsui (2000)] A numerical method is said to be stable in the p-th mean if* $\lim_{N\to\infty}\mathbf{E}(\|\Xi_N\|^p)=0$, *where* Ξ_N *is the solution generated at the N-th time step for a test equation of the form (9).*

For the Stochastic-α method, the stability in mean, i.e., for $p=1$ is shown by the following theorem.

THEOREM 4.2 (MEAN STABILITY) *For a choice of sufficiently small step size h, and* $\rho\in[0,1]$ *Stochastic-α method is numerically stable in the mean when applied to a stochastically asymptotically stable second order stochastic differential equation with multiplicative noise as in equation (9b).*

Proof. Taking expectation on both sides in equation (25c), as $r=N\to\infty$, and using Lemma(4.1) and independence of the increment of Weiner Process across time steps, we obtain, almost surely

$$
\begin{aligned}
&\lim_{N\to\infty}\mathbf{E}(\|\Xi_N\|)\\
&\le \lim_{N\to\infty}\sum_{k=1}^{N-1}\mathbf{E}(\|A_s^{(k)}\|)\|A_d^{N-k-1}\|\mathbf{E}(\|\Xi_k\|)\\
&\le \lim_{N\to\infty}\sum_{k=1}^{N-1}\sqrt{\frac{7\hat{\eta}^2h}{3}}\|A_d^{N-k-1}\|\delta_k
\end{aligned}
$$

(using Lemma(4.1) and that

$$\mathbf{E}(\|\Xi_k\|):=\delta_k<\delta_0$$

by Lemma (4.5))

$$\to 0 \quad \text{for sufficiently small } h$$

noting that $\mathbf{E}(\|\Xi_k\|)\to 0$ for $k=N-j,\ j<<N$, as $N\to\infty$. This proves the stability in mean. □

COMMENT. For engineering application purposes, obviously, the stability in probability (i.e, numerical asymptotic stability) is more restrictive in step sizes than that in the mean. In fact, for large noise multiplicative in velocities and tight tolerances in Newton Iteration, the stability in probability can be too inefficient and render the method inapplicable. Same observation applies to contractivity in probability being more restrictive on step size than contractivity in mean. Hence for step size selection, stability and contractivity in mean are more practical consideration and works well with mechanical multibody systems applications, as is illustrated in the last section.

THEOREM 4.3 (MEAN SQUARE STABILITY) *For a sufficiently small time step size h and a choice of* $\rho\in[0,1]$, *the Stochastic-α method is mean-square stable.*

Proof. Consider equation (23). Squaring both sides and taking expectation (similar to the recurrence (24)), we obtain, almost surely,

$$\mathbf{E}\left(\|\Xi_{r+1}\|^2\right) \quad\le\quad \mathbf{E}(\max_{\lambda\in\lambda(A)}|\lambda|^2)\mathbf{E}\left(\|\Xi_r\|^2\right) \tag{26}$$

using the contractivity property (that is, $\mathbf{E}(\max_{\lambda\in\lambda(A)}|\lambda|)\le 1$ in Lemma (4.5). For a suitable choice of ρ, we can make, almost surely, $\mathbf{E}(\max_{\lambda\in\lambda(A)}|\lambda|)<1$ strictly. Also, as before it is assumed that $\|\Xi_0\|=\delta_0<\infty$ almost surely. As $r=N\to\infty$, the recurrence (26) contracts $\mathbf{E}\left(\|\Xi_N\|^2\right)$ to zero, provided h has been chosen to satisfy

$$
\begin{aligned}
\max&\left(\left(\sqrt{\frac{7}{3}\hat{\eta}^2h}\right)\sum_{k=0}^{p-1}||N||_2^k,\right.\\
&\qquad\left.\left(\sqrt{\frac{7}{3}\hat{\eta}^2h}\right)^{1/p}\left(\sum_{k=0}^{p-1}||N||_2^k\right)^{1/p}\right)\\
&<\frac{1-\max_{\lambda\in\lambda(A_d)}\ |\lambda|^2}{2\max_{\lambda\in\lambda(A_d)}\ |\lambda|},\quad 1\le p\le\dim(\Xi)
\end{aligned}
$$

This establishes the mean square stability. □

Thus, as seen in the preceding proofs, the mean and the mean-square stability are only as restrictive on time step sizes as contractivity in mean.

5 Strong Order Consistency

The convergence of the method for ϕ, velocity v and position x can be derived by comparing the method with the Itô-Taylor expansion. Expanding f_{n+1} using Itô-Taylor expansion gives the following expression

$$f_{n+1} = f_n + \int_{t_n}^{t_{n+1}} L^0 f_s ds + \sum_{j=1}^{m} \int_{t_n}^{t_{n+1}} L^j f_s dW_s^j$$

Further expanding the integral terms, we obtain

$$\begin{aligned} f_{n+1} = & \qquad (27) \\ & f_n + hL^0 f_n + \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} L^0 L^0 f_u du ds \\ & + \int_{t_n}^{t_{n+1}} \sum_{k=1}^{m} \int_{t_n}^{s} L^k L^0 f_u dW_u^k ds + \sum_{k=1}^{m} L^k f_n (\Delta W_n^k) \\ & + \sum_{k=1}^{m} \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} L^0 L^l f_u du dW_s^k \\ & + \sum_{l=1}^{m} \sum_{k=1}^{m} \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} L^l L^k f_u dW_u^k dW_s^l \qquad (28) \end{aligned}$$

where $\Delta W_n^k = \Delta W_{n+1}^k - \Delta W_n^k$. Applying this expansion to (7c) and rearranging yields

$$\begin{aligned} \phi_{n+1} - f_{n+1} = & \left(\frac{\alpha_m}{1-\alpha_m}\right)(\phi_n - f_n) \\ & + \left(\frac{\alpha_f - \alpha_m}{1-\alpha_m}\right)\left(hL^0 f_n + \sum_{j=1}^{m} L^j f_n \Delta W_n^j\right) \qquad (29) \end{aligned}$$

where $\sqrt{\mathbf{E}\left(\left(\sum_{j=1}^{m} L^j f_n \Delta W_n^j\right)^2\right)}$ is $O(h^{0.5})$. Consistent with the assumptions for initial conditions, it is assumed that $\phi_0 = f_0$ almost surely. The root mean square error is estimated as: $\sqrt{\mathbf{E}(|\phi_{n+1} - f_{n+1}|^2)} = O(h^{0.5})$. The recurrence (29) indicates that the accumulated (global) root mean square error for ϕ_N (after N time steps) remains $O(h^{0.5})$. This establishes strong order 0.5 for the consistency of root mean square error in ϕ (the acceleration term).

The root mean square error in v, $\sqrt{\mathbf{E}(|v(t_{n+1}) - v_{n+1}|^2)}$, is estimated from the recurrence obtained by expanding f_{n+1} in (7b) using (28):

$$\begin{aligned} v_{n+1} = & v_n + B_n \Delta W_n + h f_n + \sum_{k=1}^{m} \sum_{l=1}^{m} C_{l,k}^{1B} I_{l,k} \\ & + \frac{h^2 \gamma \alpha_f}{\alpha_m} L^0 f_n + \frac{h \gamma \alpha_f}{\alpha_m} \sum_{j=1}^{m} L^j f_n \Delta W_n^j + R_v \end{aligned} \qquad (30)$$

Using the above recurrence (30), one gets

$$\begin{aligned} v(t_{n+1}) - v_{n+1} = & h^2 L^0 f_n \left(1 - \frac{\gamma \alpha_f}{\alpha_m}\right) \\ & + \frac{h \gamma \alpha_f}{\alpha_m} L^1 f_n \sum_{k=1}^{m} \Delta W_n^k \\ & - L^1 f_n \sum_{k=1}^{m} \left(\int_{t_n}^{t_{n+1}} (W_\tau^k - W_{t_n}^k) d\tau\right) \\ & + R_v \qquad (31) \end{aligned}$$

The root mean square estimate $\sqrt{\mathbf{E}\left(\left(\sum_{k=1}^{m} \left(\int_{t_n}^{t_{n+1}} (W_\tau^k - W_{t_n}^k) d\tau\right)\right)^2\right)}$ is $O(h^{1.5})$. Applying this in a time step recurrence for error, the strong order consistency in global root mean square error in v is obtained as 1.5.

Expanding the update of position x in (7a) using (28) we get

$$\begin{aligned} x_{n+1} = & x_n + v_n h + B_n \Delta W_n + \frac{h^2}{2} f_n \\ & + \frac{h^2}{2} \left(\frac{2\beta \alpha_f}{\alpha_m}\right) \sum_{j=1}^{m} L^j f_n \Delta W_n^j + \frac{h^3}{2} \frac{2\beta \alpha_f}{\alpha_m} L^0 f_n \\ & + \sum_{k=1}^{m} \sum_{l=1}^{m} C_{l,k}^{1B} I_{l,k,0} + R_x \end{aligned}$$

Comparing the recurrence above against an Itô-Taylor expansion, we can estimate the root mean square error in x. The resulting error expression

$$\begin{aligned} x(t_{n+1}) - x_{n+1} = & \frac{h^2}{2} \left(\frac{2\beta \alpha_f}{\alpha_m}\right) \sum_{j=1}^{m} L^j f_n \Delta W_n^j \\ & + \frac{h^3}{2} \frac{2\beta \alpha_f}{\alpha_m} L^0 f_n + R_x \end{aligned}$$

can be used to estimate the global consistent root mean square error in x. Since $\sqrt{\mathbf{E}(h^4 (W_{n+1} - W_n)^2)}$ is of $O(h^{2.5})$, the root mean square error in x is consistent with strong order 2.0.

THEOREM 5.1 (CONVERGENCE WITH STRONG ORDER) *The Stochastic-α method is convergent with strong order* 2.0.

Proof. The discussion preceding the theorem establishes that the method is strong order 2.0 consistent in root mean square error. The stability property as established in theorems (4.1), (4.2) and (26) together with the root mean square error consistency proves the convergence. □

6 Numerical Dissipation Property

The Stochastic-α method introduces user controllable numerical dissipation. From the contractivity property, Lemma (4.5), it is seen that, under an appropriate norm, this dissipation is proportional to the square of the spectral radius of the method, in both mean and probability. Since the expected spectral radius of the method at each time step depends on the user selectable parameter ρ, the dissipation can be introduced accordingly. Also, the expected spectral radius is influenced by the high frequency oscillations, thus allowing the user to introduce greater dissipation for highly oscillatory but small amplitude responses. The parameters β and γ are chosen (section (4.1)) such that the expected dissipation at very high frequencies tends to be $\rho^2+O(\sqrt{h})$ The numerical dissipation property is estimated from equation(26) in theorem (4.3). In the following equation

$$\mathbf{E}\left(\|\Xi_{r+1}\|^2\right) \quad \leq \quad \mathbf{E}(\max_{\lambda\in\lambda(A)}|\lambda|^2)\mathbf{E}\left(\|\Xi_r\|^2\right)$$

with a suitable norm expressing the energy of the engineering system, the expected energy dissipation across each time step is $\mathbf{E}(\max_{\lambda\in\lambda(A)}|\lambda|^2)$. Considering the result in theorem (4.2) along with Jensen's inequality yields

$$\begin{aligned}\mathbf{E}(\max_{\lambda\in\lambda(A)}|\lambda|^2) &\leq \left(\max_{\lambda\in\lambda(A_d)}|\lambda|+\sqrt{\frac{7}{3}\hat{\eta}^2h}\right)^2\\ &\leq \max_{\lambda\in\lambda(A_d)}|\lambda|^2+O(\sqrt{h})\end{aligned}$$

As $\Re(\omega)\to\infty$ in the test equation (9), the expected dissipation, in the limit is $\rho^2+O(\sqrt{h})$. Similarly, for low frequencies, the expected dissipation tends to disappear: $1+O(\sqrt{h})$.

7 Numerical Examples

The Stochastic-α method because of its dissipative property, is supposed to work well for engineering problems where stiffness arising from spatial discretization and/or multiplicative noise is a computational efficiency issue, in the sense of nonlinear method convergence and smallness of time step sizes. For the brevity of presentation, four simple stiff and nonlinear problems are used as examples. Two of the examples where the analytical behavior is well known with respect to computational efficiency of the α methods, computational work efficiency data are also presented.

7.1 Elastic Beam Excited with 3-Channel White Noise

This example is taken from the IVP Test Set at `http://pitagora.dm.uniba.it/$\sim$testset/problems/beam.php` but excited with a three channel white noise. The detailed description of the deterministic problem including that of the spatial discretization may be found in `http://pitagora.dm.uniba.it/$\sim$testset/report/beam.pdf` and in [Hairer and Wanner (1996)]. The problem has been chosen to demonstrate the numerical damping of responses in the highly oscillatory modes due to the spatial discretization of the beam. The initial conditions were chosen to be $z=(0.00.0...0.0), \dot{z}=(0.00.0...0.0)$ at time $t=0.0$ almost surely. The simulation was done for $n=10$ elements for spatial discretization with 3 channels of noise, multiplicative in position, velocity and an additive channel.The lth row of the noise diffusion coefficient matrix

$$B^{l,:}_{3\ channel} := \quad \begin{pmatrix}0.04\dot{z}_l & 0.04z_l & -0.4\end{pmatrix} \tag{32}$$

where $l=1,2,..10$ are the corresponding element nodes. The results are presented in figures 1-4.

7.1.1 Euler Bernoulli Beam Model of Connecting Rod in a Slider Crank

A variant of the slider-crank problem described in `http://pitagora.dm.uniba.it/~testset/problems/crank.php` is considered. The connecting rod of the mechanism is an Euler-Bernoulli beam discretized with two-node Lagrangian elements in the longitudinal displacements and with spectral (sinusoidal, first two

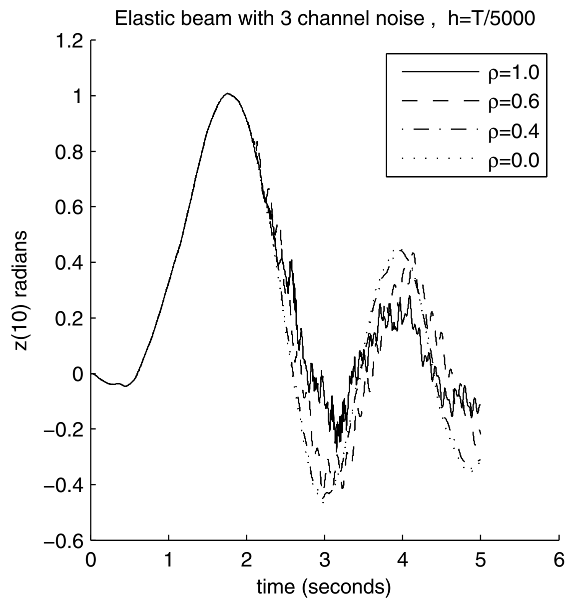

Figure 1: Beam Problem: Response at Beam Middle for Same sample Path with Various Dissipations

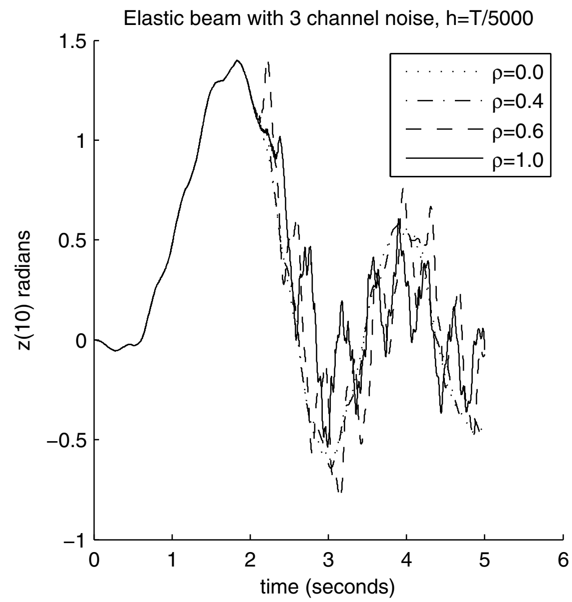

Figure 2: Beam Problem: Response at Beam End for Same sample Path with Various Dissipations

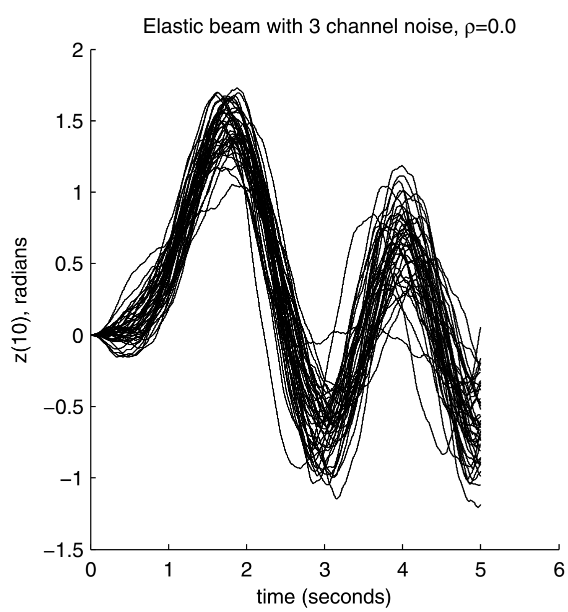

Figure 3: Beam Problem: Response at Beam End for Different Sample Paths with Full Dissipation

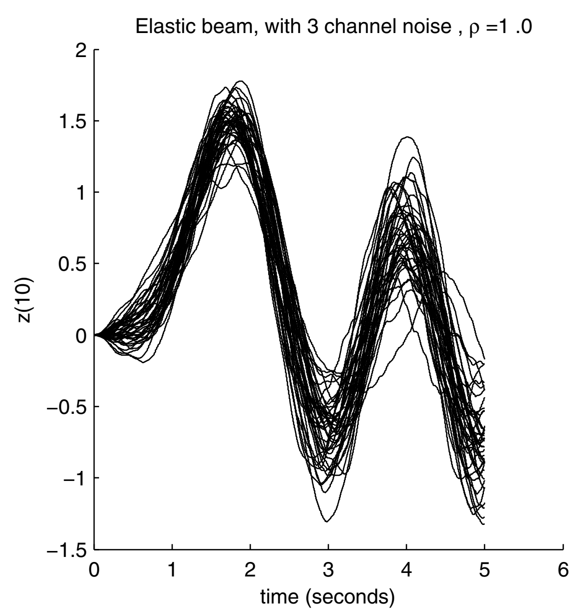

Figure 4: Beam Problem: Response at Beam End for Different Sample Paths with No Dissipation

modes) finite elements in the lateral displacements. The original partial differential-algebraic model is differentiated and simulated as a stochastic differential equation with the above mentioned spatial finite element discretization. The lateral deflection at the midpoint of the beam subjected to a single channel additive noise with coefficient 0.5 is simulated with the Stochastic-α method. Stochastic Newmark Beta Method, a special case of the present method with $\gamma = 0.5$ and $\alpha_m = \alpha_f = 1$, is used for comparison purposes. Figure 5 for $\rho = 0.8$ shows the improved dissipation of more erroneous higher frequency modes of the finite element discretization.

7.2 Double pendulum

This example demonstrates the working of the method for nonlinearity arising from rigid multibody systems. A double pendulum system shown in Fig. 6 is described by the following system of equations in its deterministic form.

$$\begin{aligned}
\ddot{\theta}_1 L_1 D &= -g(2m_1+m_2)sin\theta_1 - m_2 g sin(\theta_1 - 2\theta_2) \\
&\quad -2sin(\theta_1-\theta_2)m_2(\dot{\theta}_2^2 L_2 \\
&\quad +dot\theta_1^2 L_1 cos(\theta_1-\theta_2)) \qquad (33a)\\
\ddot{\theta}_2 L_2 D &= 2sin(\theta_1-\theta_2)(\dot{\theta}_1^2 L_1(m_1+m_2) \\
&\quad +g(m_1+m_2)cos\theta_1 \\
&\quad +\dot{\theta}_2^2 L_2 m_2 cos(\theta_1-\theta_2)) \qquad (33b)\\
&\quad \text{where,} \\
D &:= (2m_1+m_2-m_2cos(2\theta_1-2\theta_2))
\end{aligned}$$

The simulation is done with $L_1 = 1.0, L_2 = 1.0, m_1 = 1.0, m_2 = 2.0$ and initial conditions $\theta_1 = \frac{\pi}{2}, \theta_2 = \frac{\pi}{2}, \dot{\theta}_1 = 0.0, \dot{\theta}_2 = 0.0$ at $t = 0$, almost surely. A single channel Wiener Process, which forces the deterministic equations, with the diffusion coefficients matrix

$$B_{multiplicative} := \begin{pmatrix} 0.2\dot{\theta}_1 \\ 0.1\dot{\theta}_2 \end{pmatrix}$$

is used multiplicative noise in the simulation examples.

Sample paths are shown in the figures 7 - 12.

7.3 Noisy bushing problem

This well-used example illustrates a stiff noisy system similar to those obtained from Finite Element Method (FEM) discretization of elastic systems. The Stochastic-α method is illustrated by simulating the sample paths of a three-variable bushing example. In 2-dimensional cartesian coordinates, the deterministic version of problem is given as

$$\ddot{x} - \frac{f_x}{\varepsilon^2} = 0 \qquad (34a)$$

$$\ddot{y} - \frac{f_y}{\varepsilon^2} - 1 = 0 \qquad (34b)$$

$$\ddot{\theta} + \frac{\theta}{10\varepsilon^2} - \frac{1}{2}\left(cos\theta(1+\frac{f_y}{\varepsilon^2}) - sin\theta\frac{f_x}{\varepsilon^2}\right) = 0 \qquad (34c)$$

where $f_x = \frac{1}{2} - x + \frac{1}{2}cos\theta$, $f_y = -y + \frac{1}{2}sin\theta$, and $\varepsilon = 10^{-5}$. A Wiener Process with diffusion coefficient B is then used to excite the bushing system. The initial conditions are, almost surely, $x = 1.0, y = 0.0, \theta = 0.0$ and $\dot{x} = 0.0, \dot{y} = 0.0, \dot{\theta} = 0.0$ at the time $t = 0.0$. The time step size is fixed at $h = \frac{t_f}{200}$. The simulation was done with both additive and multiplicative noise. The deterministic model is forced by a Wiener Process with different coefficient matrices B in three separate simulations:

$$B_{multiplicative} := \begin{pmatrix} 0.012\dot{x} \\ 0.005\dot{y} \\ -.017\dot{\theta} \end{pmatrix} \qquad (35a)$$

$$B_{2channel} := \begin{pmatrix} 0.022\dot{x} & 0.42 \\ 0.045\dot{y} & 0.25 \\ -0.017\dot{\theta} & 0.73 \end{pmatrix} \qquad (35b)$$

Some sample paths are shown in figures 13, 14, 15 and 16. The numerical dissipation in the same sample path through different choices of ρ is demonstrated in figures 17 and 18. The average of 40 sample paths is compared against the case with no noise in figures 19 - 22.

The nonlinear bushing system was also simulated with a 2 channel noise. The results are shown in figures 23, 24 and 25.

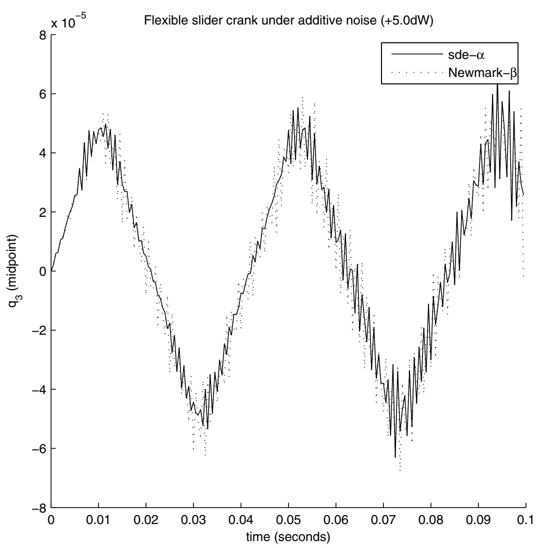

Figure 5: FEM Example: Euler-Bernoulli Beam Model of Connecting Rod of a Noisy Slider Crank Mechanism

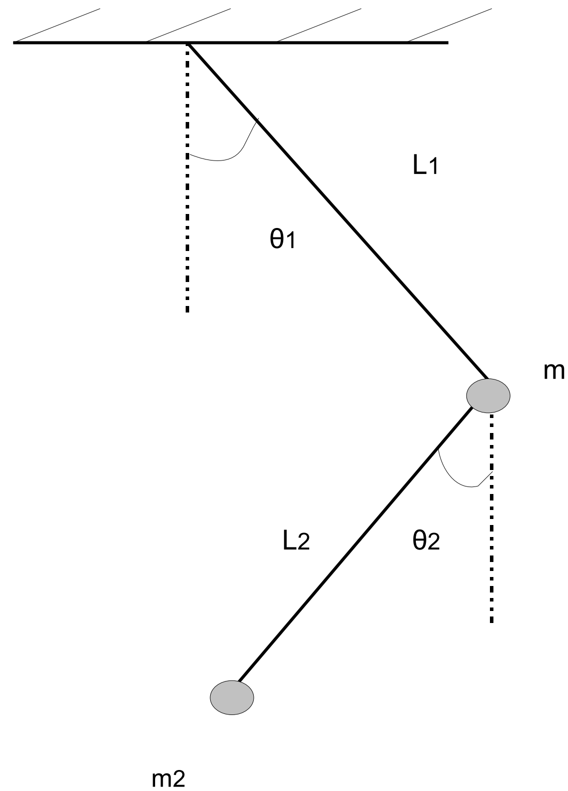

Figure 6: double pendulum

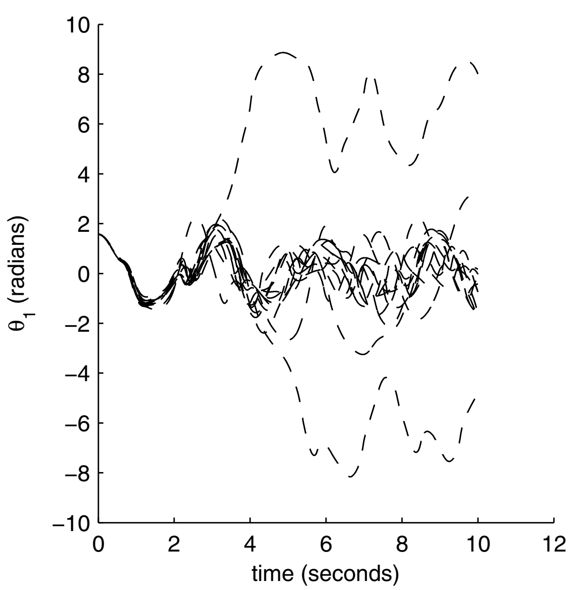

Figure 7: double pendulum θ_1 vs. time with multiplicative noise and $\rho = 1.0$

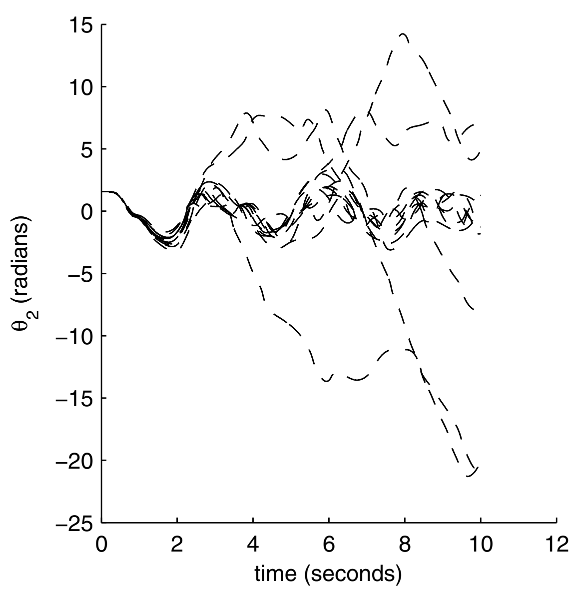

Figure 8: double pendulum θ_2 vs. time with multiplicative noise and $\rho = 1.0$

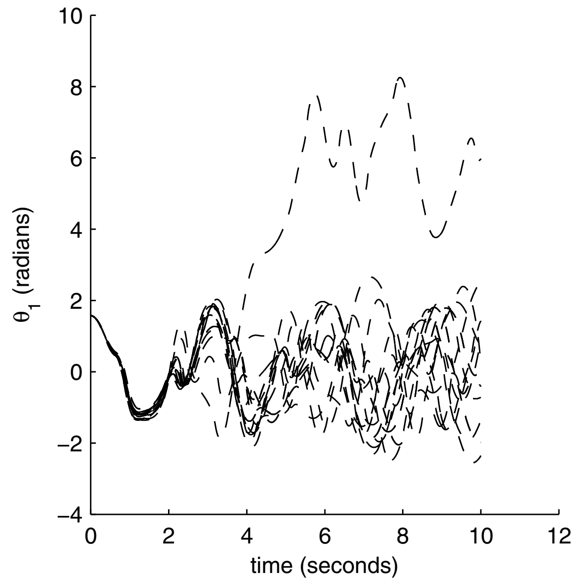

Figure 9: double pendulum θ_1 vs. time with multiplicative noise and $\rho = 0.0$

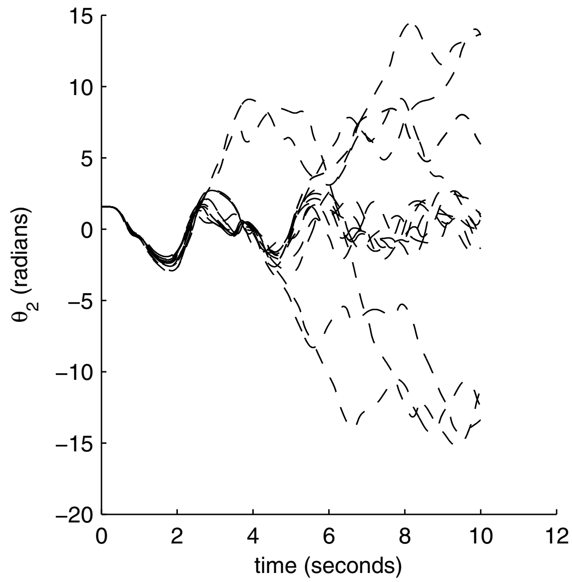

Figure 10: double pendulum θ_2 vs. time with multiplicative noise and $\rho = 0.0$

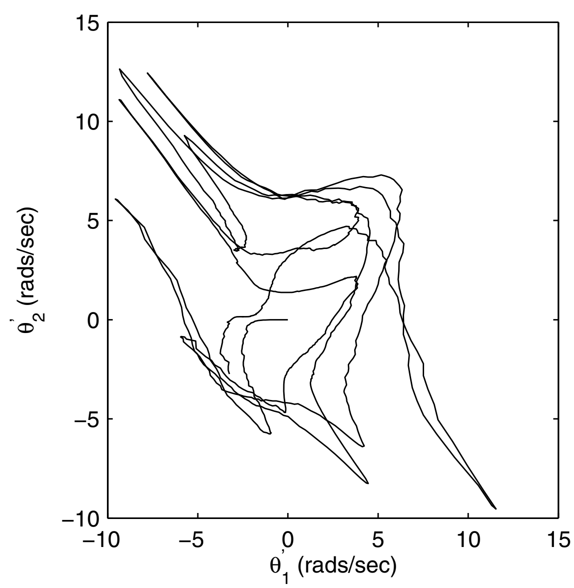

Figure 11: double pendulum $\dot{\theta}_1$ vs. $\dot{\theta}_2$ with multiplicative noise and $\rho = 1.0$

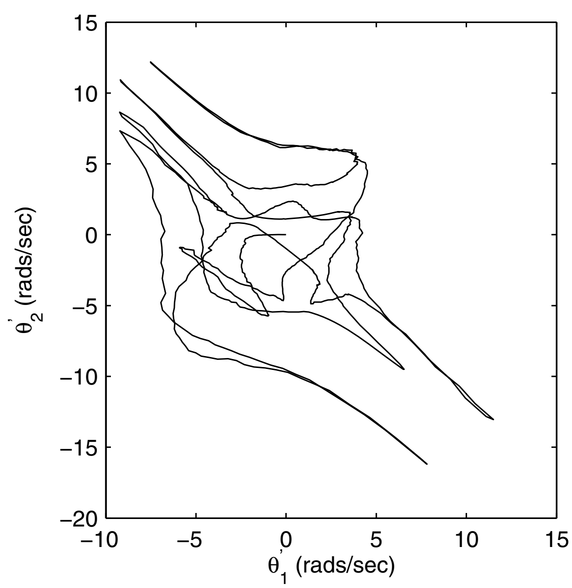

Figure 12: double pendulum $\dot{\theta}_1$ vs. $\dot{\theta}_2$ with multiplicative noise and $\rho = 0.0$

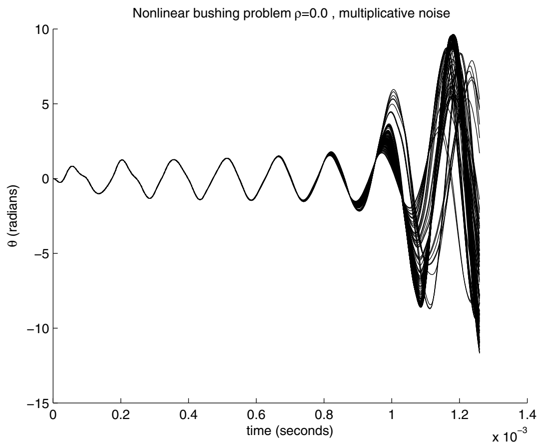

Figure 13: Bushing: sample paths of θ vs time with multiplicative noise and $\rho = 0.0$

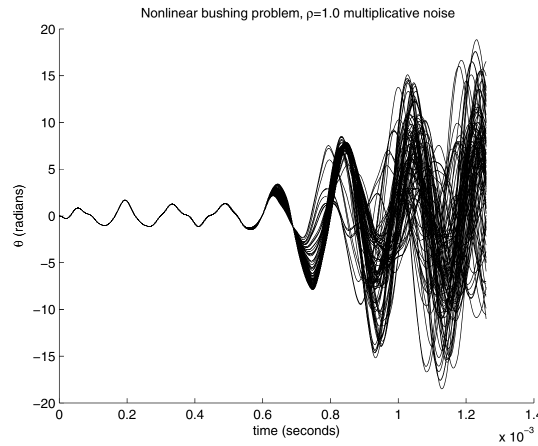

Figure 14: Bushing: sample paths of θ vs time with multiplicative noise and $\rho = 1.0$

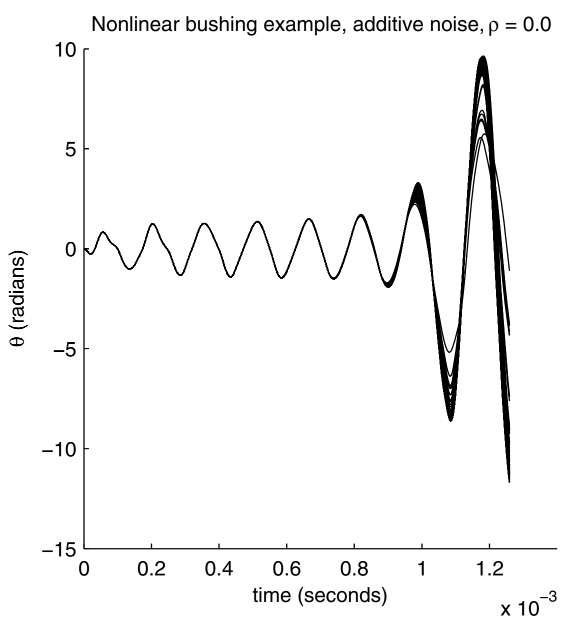

Figure 15: Bushing: sample paths of θ vs time with additive noise and $\rho = 0.0$

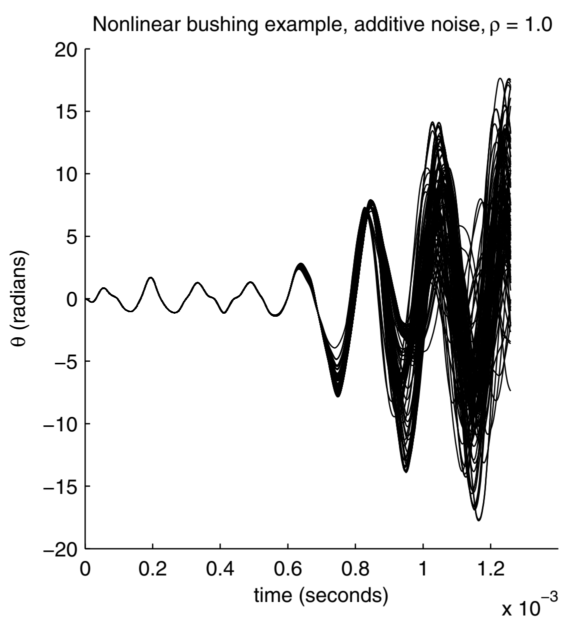

Figure 16: Bushing sample paths of θ vs time with additive noise and $\rho = 1.0$

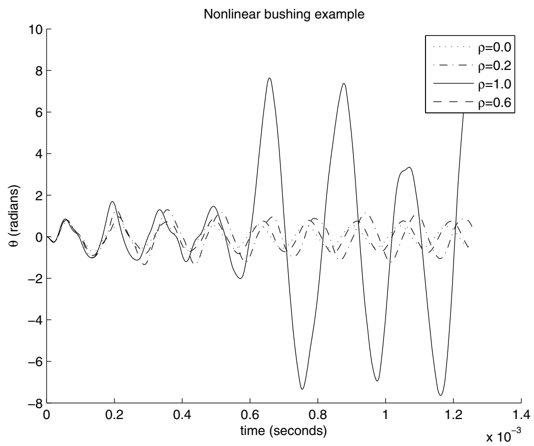

Figure 17: Bushing: same sample path of θ vs time with multiplicative noise for different choices of ρ

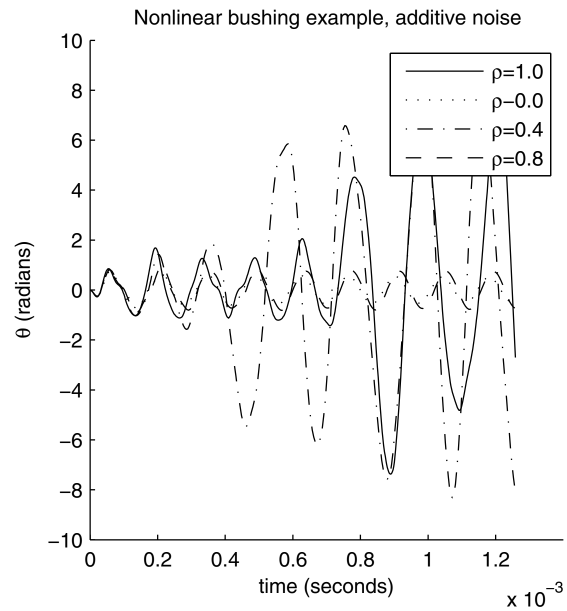

Figure 18: Bushing: same sample path of θ vs time with additive noise for different choices of ρ

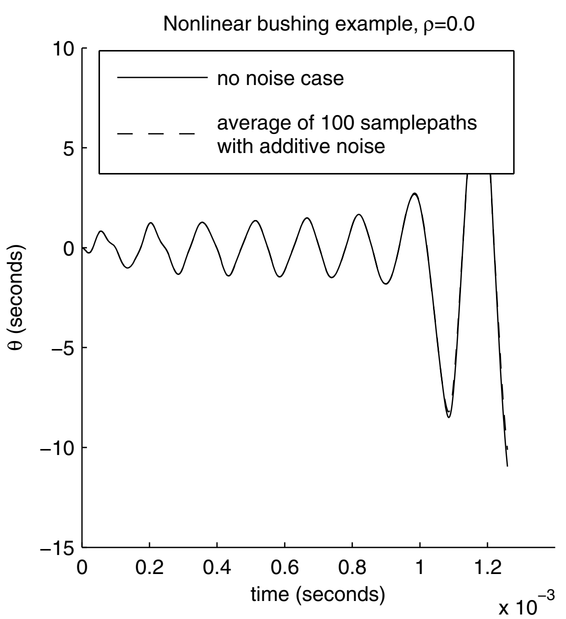

Figure 19: Bushing: "fully dissipated" θ vs time for additive noise

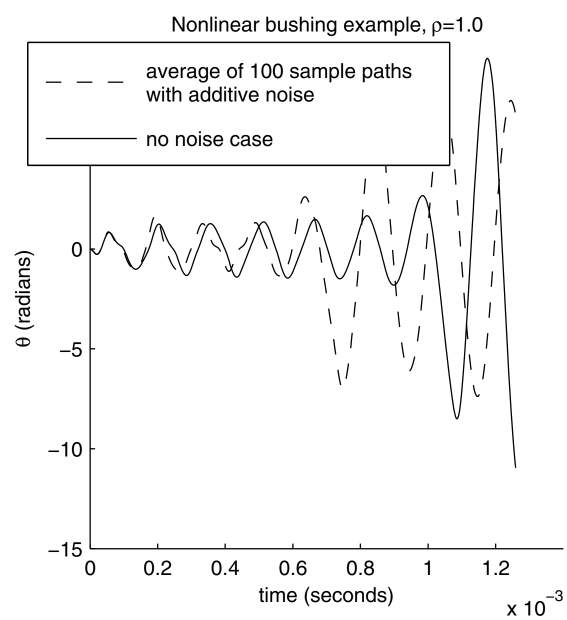

Figure 20: Bushing: θ vs time for additive noise and no dissipation

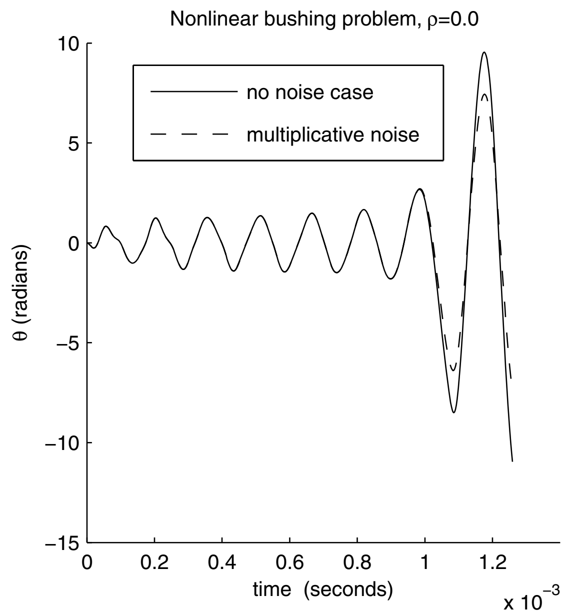

Figure 21: Bushing: "fully dissipated" θ vs time for multiplicative noise

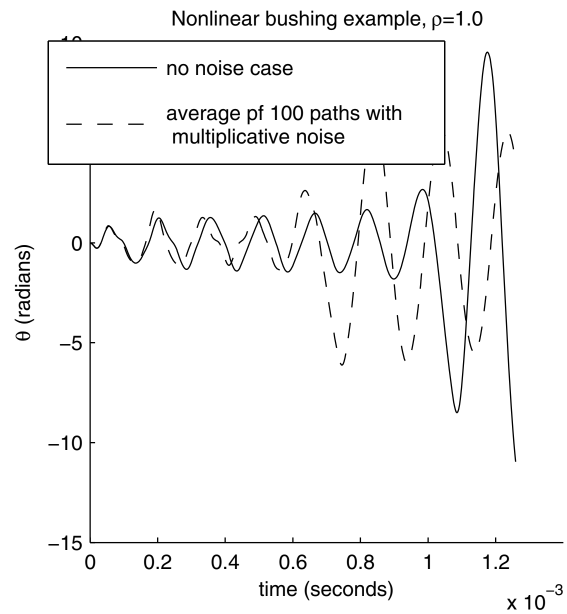

Figure 22: Bushing: θ vs time for multiplicative noise and no dissipation

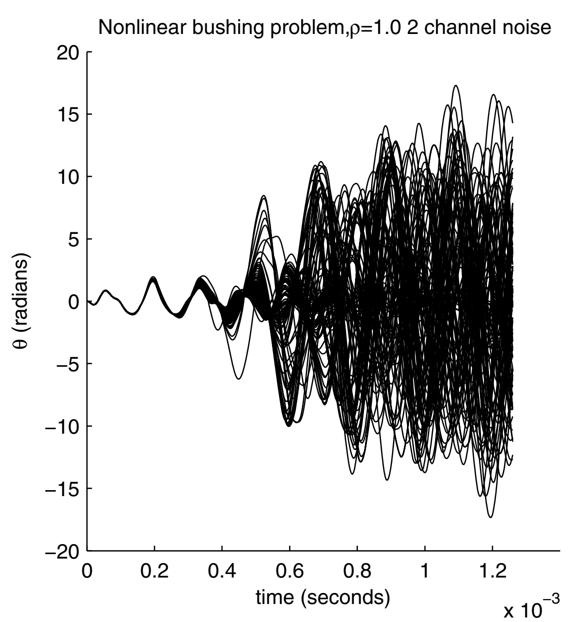

Figure 23: Bushing: θ vs time, 2-channel noise, $\rho = 1.0$

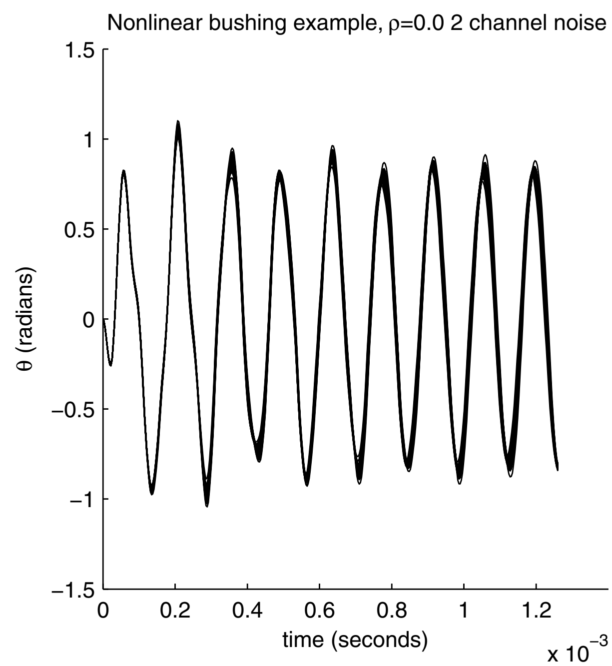

Figure 24: Bushing: θ vs time, 2-channel noise, $\rho = 0.0$

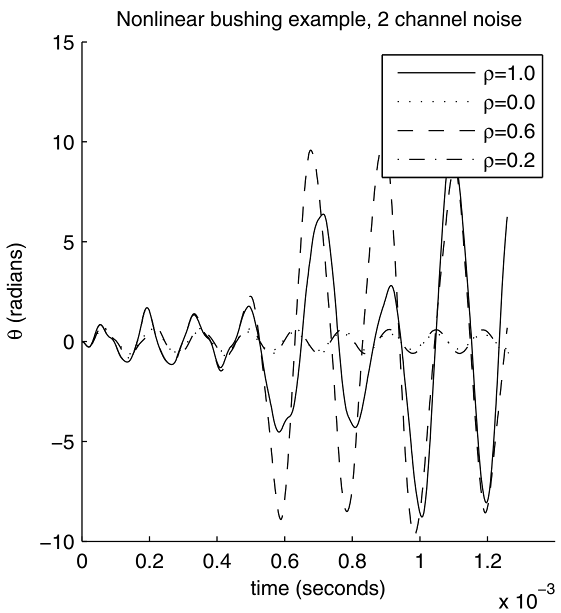

Figure 25: Bushing: sample paths over the same Wiener path θ vs time for different choices of ρ, 2-channel noise

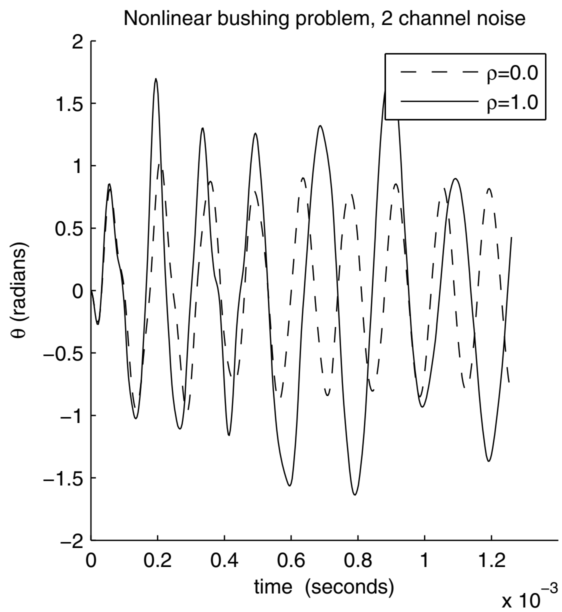

Figure 26: Bushing: average θ vs time, 2-channel noise

Noisy Nonlinear Bushing: Typical Sample Path $T = 0.0002\pi$ s, Newton Tolerance = 1E − 4 Single Channel Multiplicative Noise		
ρ	No. of Time Steps (T/h)	Total Number of Newton iterations
0.0	100	549
	300	2094
	600	4499
0.2	100	671
	300	2366
	600	Failed
0.4	100	800
	300	3395
	600	Failed
0.6	100	1037
	300	Failed
	600	6254
0.8	100	1561
	300	Failed
	600	5976
1.0	100	Failed
	300	3737
	600	5778

7.4 Stiff spring pendulum

The deterministic version of a stiff spring pendulum model is given in Cartesian coordinates as

$$\ddot{x} + x\lambda = 0 \tag{36a}$$

$$\ddot{y} + y\lambda - 1.0 = 0 \tag{36b}$$

$$\lambda = K\frac{\sqrt{x^2+y^2} - 1.0}{\sqrt{x^2+y^2}} \tag{36c}$$

where the stiff spring of unit nominal length and stiffness K is attached to the center of mass of the pendulum. The deterministic model is forced by Wiener Process in the Itô sense with multiplicative noise. Using initial conditions $x, y, \dot{x}, \dot{y} = 0.9, 0.0, 0.0, 0.0$ at $t = 0$ almost surely, $K = 2000.0$ and tolerance of the Newton iterations as 10^{-7}, the simulation was carried out for various choices of ρ and h. The diffusion coefficient

$$B_{multiplicative} := \begin{pmatrix} 0.2\dot{x} \\ 0.4\dot{y} \end{pmatrix}$$

was used for modeling the multiplicative noise. The sample paths are shown in figures 27-30 for the single channel multiplicative noise.

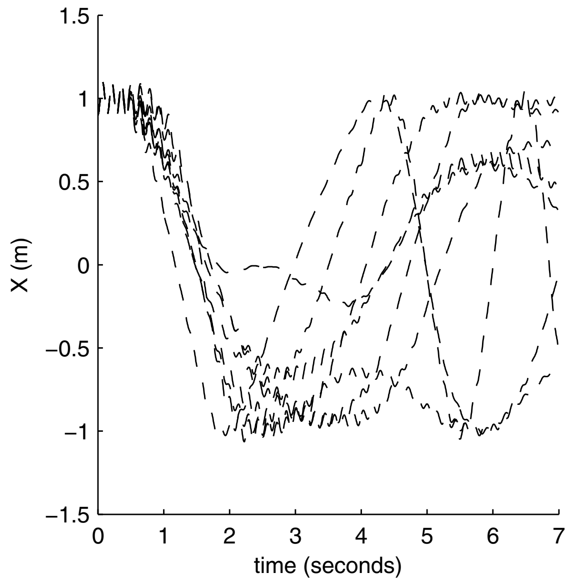

Figure 27: Stiff spring pendulum: X vs time with multiplicative noise and ρ=0.0

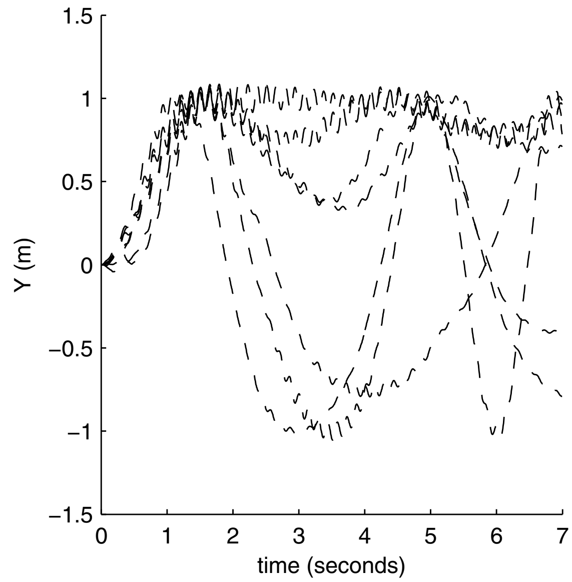

Figure 28: Stiff spring pendulum: Y vs time with multiplicative noise and ρ=0.0

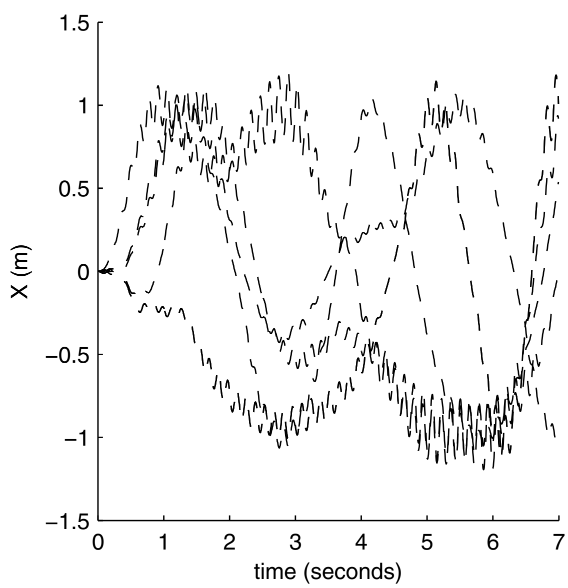

Figure 29: Stiff spring pendulum: X vs time with multiplicative noise and ρ=1.0

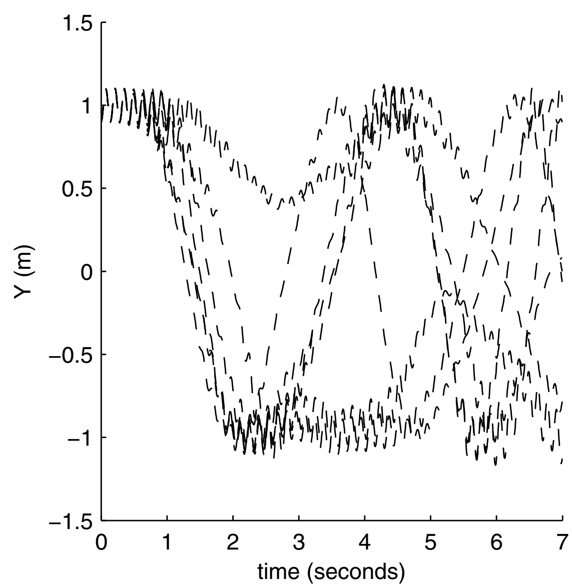

Figure 30: Stiff spring pendulum: Y vs time with multiplicative noise and ρ=1.0

Stiff Spring Pendulum: Typical Sample Path		
$T = 4\ s$, Newton Tolerance = 1E − 11 Single Channel Multiplicative Noise		
ρ	h	Total Number of Newton iterations
0.0	0.01	1602
	0.03	640
	0.08	291
0.2	0.01	1603
	0.03	643
	0.08	296
0.4	0.01	1603
	0.03	644
	0.08	313
0.6	0.01	1608
	0.03	655
	0.08	331
0.8	0.01	1654
	0.03	677
	0.08	370
1.0	0.01	1725
	0.03	701
	0.08	319

8 Discussions

A stiff system integrator for deterministic systems has been extended to stochastic dynamics. The analysis shows the limitations on step size in various asymptotic stability requirements and also how the choice of parameter ρ is affected by noise. The method is suitable for small to medium noise applications. It may be seen that with a choice of $\gamma = \frac{1}{2}$, $\alpha_f = \alpha_m = 1$ the method behaves as stochastic Newmark-Beta scheme. At $\rho = 1$ it is same as the average acceleration Newmark-Beta method. The method can be extended to singular mass matrix with some extra computations. A rank-revealing permutation of the rows and corresponding columns of mass matrix may be done to obtain a non-singular submatrix to which the present method may be applied. However, the remaining rows of the mass matrix would correspond to hidden algebraic equations which are satisfied at each time step to solve for the remaining (algebraic) variables. The system would correspond to a Differential-Algebraic Equation (DAE) system which can be solved if its index is lower than 2. Such computational issues are discussed in Raha and Petzold (2001).

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