The Trefftz Boundary Method in Viscoelasticity

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Abstract: In this paper, the Trefftz method is applied to solve linear viscoelasticity problems in the time domain, using Trefftz elastic series and considering the viscoelastic components in each time domain as fictitious body forces. The direct application of the Trefftz method to elastic problems is typically constrained to those cases in which the Navier equation is homogeneous. In the presence of body forces, the method of the particular solution or the method of the generalized particular solution should be used, depending on whether the body forces are constant or not inside the considered domain. Many viscoelasticity problems with or without aging can be solved by applying the elastic Trefftz series. To show the accuracy of the proposed formulation, some examples are solved and the results compared with those available in the literature.

Keyword: Viscoelasticity, Trefftz Method, Particular Solution Method

1 Introduction

Boundary methods have become quite popular in recent years because they are able to provide a complete solution in terms of boundary values only, and are also computationally efficient. The Trefftz method is a boundary-type solution procedure using T-complete functions [Herrera, (1984)] that satisfy the governing equation. It was first proposed by Trefftz in 1926 [Trefftz, (1926)] to analyze problems of bars under stress. The method includes a direct or indirect formulation [Kita, (1995)]. In the indirect formulation, the solution is approximated by a superposition of T-complete functions with unknown coefficients, which are determined to satisfy the boundary conditions. In the direct formulation, the weighted residual expression is derived from the differential equation by introducing T-complete functions as the weight functions. In recent years, the Trefftz method has been used to solve two-dimensional elasticity problems using the indirect method [Portela and Charafi, (1999)], and linear elasticity problems using the direct method [Sladek et al., (2000)]. Two types of material behavior can be considered in linear viscoelasticity: one that includes aging, present in polymers, polymeric composites and concrete and another one that does not age [Creus, (1986)]. To analyze problems of viscoelastic materials without aging by the Trefftz method, the elasticity problem associated with the Laplace transform domain could be solved and then the results can be numerically inverted to obtain the solution in the time domain. Conversely, the Laplace transform could be applied to the elastic Trefftz series to obtain the viscoelastic series. For viscoelastic materials with aging, the Laplace transform cannot be calculated because the viscoelastic constitutive equation does not allow a convolution representation [Creus, (1986)]. To develop a general enough formulation in the time domain that allows the resolution of viscoelastic problems with or without aging; an analogy between the body forces and the inelastic deformation gradient [Lin, (1968)] is proposed in this paper.

The direct application of the Trefftz method to elasticity problems is limited to those cases in which the Navier equation is homogeneous. In the presence of some particular body forces, the Particular Solution Method could be used [Sokolnikof, (1968)]. A generalization of the particular solution method is needed when pseudo body forces are present (inelastic, viscoelastic problems) because the governing equation is not ho-

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mogeneous.

Under these conditions, the case of viscoelastic materials in which the Poisson's ratio remains constant in time,;nd no body forces, the governing equation is homogeneous simplifying the solution. In order to solve problems governed by non-homogeneous differential equations with the Trefftz method, the non-homogeneous term was approximated by other series expansions using global approximating functions [Kita et al., (2003)], [Cho et al. (2004)], and the solution will usually be accurate if an appropriate number of internal nodes is used to calculate the parameters of the series [Partridge and Sensale, (1997)], what makes the procedure lose the "boundary-only" characteristic and the computational method increases considerably.

In the generalized particular solution method proposed in this paper, the non-homogeneous terms are calculated from a series expansion obtained from the T-complete functions. A generic expression of the particular solutions corresponding to each of these functions is chosen so that is not necessary to approximate it by a new series expansion. This allows maintaining the "boundary-only" characteristic of the Trefftz method.

This paper is composed by six sections: introduction; formulation of the Trefftz method applied to elasticity with and without body forces; fundamentals of viscoelasticity; the analogy between the body forces and the inelastic deformation gradient using the Trefftz method for viscoelastic materials with and without aging; numerical examples; and conclusions.

2   The Trefftz method in elasticity

The Trefftz method is a general method to solve homogeneous partial differential equations where the solution function is approximated by a linear combination of T-complete functions [Kita and Kamiya, (1995)]. The coefficients of these functions can be determined among others by the collocation method [Herrera, (1984)]. A group of regular T-complete functions can be built so that they satisfy the equation that governs the problem of a linear elastic body not subjected to body

forces; It is demonstrated that this infinite series of Trefftz functions is complete since any solution of the equation that governs the problem can be written as a linear combination of those functions. It has been proven [Qinq-Hua, (2000)] that satisfying the completeness condition guarantees the convergence of the method.

2.1   Resolution in absence of body forces

Be Ω a bounded region with its boundary $\partial\Omega$ given by $\partial\Omega_u$ and $\partial\Omega_p$, so that $\partial\Omega = \partial\Omega_u \cup \partial\Omega_p$, where the Navier elasticity equation [Hetnarski and Ignaczak, (2004)] is solved for the $\vec{u}$ displacements in an isotropic elastic body with shear modulus μ, and bulk modulus K without body forces.

$$\left(\frac{3K+\mu}{3}\right)\nabla(\nabla\cdot\vec{u}) + \mu\nabla^2\vec{u} = 0$$

$$\Leftrightarrow$$

$$\Im(\vec{u}) = 0, \quad \text{in } \Omega$$

$$\vec{u} = \vec{u}_d \quad \text{in } \partial\Omega$$

$$T\vec{n} = \vec{p} = \vec{p}_d \quad \text{in } \partial\Omega_p$$

$$(1)$$

being $\vec{n}$ the outside unit normal to $\partial\Omega$.

An approximate solution of the elasticity problem can be obtained by approximating the displacement vector by the truncated series:

$$u_1 = \sum_{i=1}^{NT} \hat{u}_{1i}a_i = \hat{\mathbf{u}}_1\,\mathbf{a}^T,$$

$$u_2 = \sum_{i=1}^{NT} \hat{u}_{2i}a_i = \hat{\mathbf{u}}_2\,\mathbf{a}^T$$

$$(2)$$

$$\sigma_{11} = \sum_{i=1}^{NT} \hat{\sigma}_{11i}a_i = \hat{\sigma}_{11}\,\mathbf{a}^T,$$

$$\sigma_{22} = \sum_{i=1}^{NT} \hat{\sigma}_{22i}a_i = \hat{\sigma}_{22}\,\mathbf{a}^T$$

$$\sigma_{12} = \sum_{i=1}^{NT} \hat{\sigma}_{12i}a_i = \hat{\sigma}_{12}\mathbf{a}^T$$

$$(3)$$

where u_1 and u_2 are the components of the displacement vector in a given base; σ_{11}, σ_{12}, σ_{22} represent the components of the array associated with the stress tensor in that base; a_i is a group of unknown parameters; and NT is the total number of considered Trefftz functions. All the functions $\hat{u}_{1i}$ and $\hat{u}_{2i}$ are solutions of the homogeneous

equation (1), and $\hat{\sigma}_{11i}$, $\hat{\sigma}_{22i}$ and $\hat{\sigma}_{12i}$ are the corresponding stresses.

If these solutions guarantee the completeness (T-complete functions according to Herrera's definition [Herrera ,(1984)]), the global solution of the problem is guaranteed. The accuracy of this approximation depends on the value of NT. For an interior domain, the T-complete functions are [Sensale, Sensale Rodríguez and Herskovits, (2005)]:

$$2G\hat{u}_1 = -\alpha_o + \sum_{i=1}^{N} \left\{ r^i \left[\kappa C(i) - iC(i-2) \right] \alpha_i \right.$$
$$\left. - r^i \left[\kappa S(i) - iS(i-2) \right] \beta_i - r^i C(i) \gamma_i + r^i S(i) \delta_i \right\} \tag{4}$$

$$2G\hat{u}_2 = \beta_o + \sum_{i=1}^{N} \left\{ r^i \left[\kappa S(i) + iS(i-2) \right] \alpha_i \right.$$
$$\left. + r^i \left[\kappa C(i) + iC(i-2) \right] \beta_i + r^i S(i) \gamma_i + r^i C(i) \delta_i \right\} \tag{5}$$

$$\hat{\sigma}_{11} = \sum_{i=1}^{N} \left\{ r^{i-1} \left[2iC(i-1) - i(i-1)C(i-3) \right] \alpha_i \right.$$
$$- r^{i-1} \left[2iS(i-1) - i(i-1)S(i-3) \right] \beta_i$$
$$\left. - r^{i-1} iC(i-1)\gamma_i + r^{i-1} iS(i-1)\delta_i \right\} \tag{6}$$

$$\hat{\sigma}_{12} = \sum_{i=1}^{N} \left\{ r^{i-1} i(i-1)S(i-3)\alpha_i \right.$$
$$+ r^{i-1} i(i-1)C(i-3)\beta_i$$
$$\left. + r^{i-1} iS(i-1)\gamma_i + r^{i-1} iC(i-1)\delta_i \right\} \tag{7}$$

$$\hat{\sigma}_{22} = \sum_{i=1}^{N} \left\{ r^{i-1} \left[2iC(i-1) + i(i-1)C(i-3) \right] \alpha_i \right.$$
$$- r^{i-1} \left[2iS(i-1) + i(i-1)S(i-3) \right] \beta_i$$
$$\left. - r^{i-1} iS(i-1)\gamma_i + r^{i-1} iC(i-1)\delta_i \right\} \tag{8}$$

where: $C(i) = \cos(i\theta)$ y $S(i) = \sin(i\theta)$

2.1.1 Collocation method

The weak formulation of the weighted residuals method corresponding to an elasticity problem can be expressed [Qinq-Hua, (2000)] as:

$$\int_{\partial\Omega_u} \varpi_u(\vec{u} - \vec{u}_d)d\Gamma + \int_{\partial\Omega_p} \varpi_p(\vec{p} - \vec{p}_d)d\Gamma = 0 \tag{9}$$

The weight functions ϖ_u and ϖ_p can be chosen arbitrarily, leading to different techniques of the Trefftz method. When these functions are defined through the Dirac Delta function, as:

$$\varpi_u = \varpi_p = \delta(P - P_i) \tag{10}$$

where P_i is the collocation point placed in the boundary, the collocation technique can thus be applied and (9) and (10) lead to:

$$\vec{u}(P_i) = \hat{\mathbf{u}}\mathbf{a}^T = \vec{u}_d(P_i) \qquad \forall\, P_i \in \partial\Omega_u$$
$$\vec{p}(P_i) = \hat{\mathbf{p}}\mathbf{a}^T = \vec{p}_d(P_i) \qquad \forall\, P_i \in \partial\Omega_p \tag{11}$$

The previous equations (11) can be written as:

$$K_{ij}a_j = b_i \quad \Leftrightarrow \quad \mathbf{Ka} = \mathbf{b} \tag{12}$$

where the unknown a_j represent the coefficients of the jth term of the expansion, while the b_i terms are given by:

$$b_i = \vec{u}_d(P_i) \quad \text{or} \quad b_i = \vec{p}_d(P_i) \tag{13}$$

In this paper, the limits of the expansions (2) and (3) are set by minimizing the functional:

$$F = \frac{1}{L_{Max}} \sum_{i=1}^{M_u} \|\vec{u}(P_i) - \vec{u}_d(P_i)\|$$
$$+ \frac{1}{\mu} \sum_{i=1}^{M_p} \|\vec{p}(P_i) - \vec{p}_d(P_i)\| \tag{14}$$

where M_u and M_p are the number of boundary points placed in the regions $\partial\Omega_u$ y $\partial\Omega_p$ respectively μ the shear modulus and L_{Max} is the maximum length between two points. In viscoelasticity, M_u and M_p would be determined for the initial time and would hold for the other times of the domain.

2.2 Resolution in presence of body forces

Considering the solution to the Navier equation with a body force term $\vec{b}$:

$$\left(\frac{3K+\mu}{3}\right)\nabla(\nabla\cdot\vec{u})+\mu\nabla^2\vec{u}+\vec{b}=0$$

$$\Leftrightarrow$$

$$\Im(\vec{u})+\vec{b}=0, \quad \text{in } \Omega$$

$$\vec{u}=\vec{u}_d \quad \text{in } \partial\Omega_u$$

$$T\vec{n}=\vec{p}=\vec{p}_d \quad \text{in } \partial\Omega_p$$

(15)

A direct way of solving (15) avoiding the non-homogeneous term is by exchanging variables in a way that the domain term disappears. This can be accomplished by adding a particular solution to a new variable [Sokolnikof (1968)]:

$$\vec{u}=\vec{\hat{u}}+\vec{u}^{(p)} \quad \text{and} \quad \vec{p}=\vec{\hat{p}}+\vec{p}^{(p)}$$

(16)

where the particular solution $\vec{u}^{(p)}$ satisfies (15), while $\vec{\hat{u}}$ satisfies (1) with boundary conditions derived from (16). A homogeneous problem is now obtained, and the Trefftz method can be used

$$\Im(\vec{\hat{u}})=0, \quad \text{in } \Omega$$

$$\vec{\hat{u}}=\vec{u}_d-\vec{u}^{(p)} \quad \text{in } \partial\Omega_u$$

and

$$\vec{\hat{p}}=\vec{p}_d-\vec{p}^{(p)} \quad \text{in } \partial\Omega_f$$

(17)

When the acceleration of gravity is in the x_2-direction, the body force vector is given by:

$$b_1=0, \quad b_2=-\rho g$$

(18)

where ρ is the mass density and g is the acceleration of gravity. A group of particular solutions corresponding to this problem were introduced; Sokolnikof (1968)]:

$$u_1^{(p)}(P)=\frac{-\rho g(3K-2\mu)}{4\mu(3K+\mu)}(x_1x_2)$$

(19)

$$u_2^{(p)}(P)=$$
$$\frac{\rho g}{8\mu(3K+\mu)}\left[(3K+4\mu)x_2^2+(3K-2\mu)x_1^2\right]$$

(20)

$$p_1^{(p)}(x)=0$$

(21)

$$p_2^{(p)}(x)=\rho gx_2n_2$$

(22)

In a similar way, there are particular solutions [Partridge and Sensale, (1997)] that allow the approximation of terms corresponding to general body forces for which the determination of an analytical formulation of the particular solutions is not possible.

3 Fundamentals of viscoelasticity

We consider bodies of a general linear aging viscoelastic material for which the constitutive relations may be written [Creus, (1986)] as:

$$\boldsymbol{\varepsilon}(t)=\int_{\tau_o}^{t}(t,\tau)d\boldsymbol{\sigma}(\tau)=\int_{\tau_o}^{t}\mathbf{D}(t,\tau)\dot{\boldsymbol{\sigma}}(\tau)d\tau$$

$$=\mathbf{D}(t,\tau)*\boldsymbol{\sigma}(\tau) \quad (23)$$

where $\boldsymbol{\varepsilon}$ is the infinitesimal strain tensor, $\boldsymbol{\sigma}$ is the stress tensor, $\mathbf{D}(t,\tau)$ is the time-dependent creep function, to be experimentally determined, and the last expression is merely an equivalent operational notation. For each tensor component, the elastic and inelastic parts can be isolated:

$$\mathbf{D}(t,\tau)=\mathbf{E}^{-1}(t)+\mathbf{C}(t,\tau)$$

(24)

where $\mathbf{E}(t)$ is the time-dependent elastic tensor and $\mathbf{C}(t,\tau)$ is the creep compliance. In the case of an isotropic material, only two independent functions are required for the elastic part and two for the delayed part. Decomposing stresses and strains into spherical and deviatoric terms:

$$\boldsymbol{\sigma}=\mathbf{s}+\sigma_o\mathbf{I_d}, \qquad \sigma_o=\frac{1}{3}\text{tr}(\boldsymbol{\sigma})$$

$$\boldsymbol{\varepsilon}=\mathbf{e}+\varepsilon_o\mathbf{I_d}, \qquad \varepsilon_o=\frac{1}{3}\text{tr}(\boldsymbol{\varepsilon})$$

(25)

where $\mathbf{I_d}$ is the identity tensor. For the elastic part, it can be written:

$$\varepsilon_o^{(e)}(t)=\frac{1}{3}K^{-1}(t)\sigma_o(t)$$

$$\mathbf{e}^{(e)}(t)=\frac{1}{2}\mu^{-1}(t)\mathbf{s}(t)$$

(26)

where $K(t)$ y $\mu(t)$ are the bulk modulus and shear modulus respectively. For the delayed part, it is:

$$\varepsilon_o^{(v)} = \frac{1}{3}C_K * \sigma_o(t) \quad \mathbf{e}^{(v)} = \frac{1}{2}C_\mu * \mathbf{s}(t) \qquad (27)$$

where $C_K(t, \tau)$ and $C_\mu(t, \tau)$ are the corresponding creep compliances.

Two particular cases are of importance in structural applications and will receive special attention in this paper.

1) For some materials (concrete, for example), experience has shown that $C_\mu(t, \tau)$ is proportional to $C_K(t, \tau)$ for all times. This indicates that the Poisson's ratio is a constant, which;s referred as the synchronous approximation in viscoelasticity [Pipkins, (1972)] and:

$$KC_K = \mu C_\mu = EC_E \qquad (28)$$

2) For some other materials (polymers and rubbers), the creep compliance under shear may be several orders of magnitude larger than the creep compliance under volumetric strain. Then, it can be assumed that the material has no delayed volumetric strains.

4 The Trefftz method in viscoelasticity

In section 2, the Trefftz Method in elasticity was analyzed in order to extend it to viscoelasticity. The analogy between the body forces and the inelastic strain gradient [Lin, (1968)] can be considered to analyze solids under inelastic strains with the methods applied to elastic materials. For a time t, the equation that governs the viscoelastic problems is:

$$\left(\frac{3K + \mu}{3}\right)\nabla(\nabla \cdot \vec{u}) + \mu\nabla^2\vec{u} + (\vec{b} + \vec{b}^{(v)}) = 0 \quad (29)$$

with boundary conditions:

$$\vec{u} = \vec{u}_d \text{ in } \partial\Omega_u \quad \text{and} \quad \vec{p} = \vec{p}_d - \vec{p}^{(v)} \text{ in } \partial\Omega_p \quad (30)$$

where the *equivalent body force* is given by:

$$\vec{b}^{(v)} = -\left(\frac{3K - 2\mu}{3}\nabla tr(\boldsymbol{\varepsilon}^{(v)}) + 2\mu\nabla \cdot \boldsymbol{\varepsilon}^{(v)}\right) \quad (31)$$

and the *equivalent surface traction* is given by:

$$\vec{p}^{(v)} = \left(\frac{3K - 2\mu}{3}tr(\boldsymbol{\varepsilon}^{(v)})\mathbf{I_d} + 2\mu\boldsymbol{\varepsilon}^{(v)}\right)\vec{n} \qquad (32)$$

These vectors are calculated from the viscoelastic strain tensor by the equations:

$$\boldsymbol{\sigma}^{(v)} = \mathbf{E}\boldsymbol{\varepsilon}^{(v)}$$
$$\vec{p}^{(v)} = \boldsymbol{\sigma}^{(v)}\vec{n} \qquad (33)$$
$$\vec{b}^{(v)} = -\nabla \cdot \boldsymbol{\sigma}^{(v)}$$

An important result is that the equivalent body force and the equivalent surface traction depend on the body forces, surface tractions, the first invariant of the stress tensor and its gradient at time t. To demonstrate this, it can be seen that:

$$\boldsymbol{\sigma}^{(v)} = \mathbf{E}\boldsymbol{\varepsilon}^{(v)} = 3K\varepsilon_o^{(v)}\mathbf{I_d} + 2\mu\mathbf{e}^{(v)}$$
$$= K(C_K * \sigma_o)\mathbf{I_d} + \mu(C_\mu * \mathbf{s}) \quad (34)$$

By replacing the deviatoric stress tensor $\mathbf{s}$, with $\boldsymbol{\sigma} - \sigma_o\mathbf{I_d}$, it is obtained:

$$\boldsymbol{\sigma}^{(v)} = K(C_K * \sigma_o)\mathbf{I_d} + \mu(C_\mu * \boldsymbol{\sigma}) - \mu(C_\mu * \sigma_o)\mathbf{I_d}$$
$$\vec{p}^{(v)} = \boldsymbol{\sigma}^{(v)}\vec{n}$$
$$= K(C_K * \sigma_o)\vec{n} + \mu(C_\mu * \vec{p}) - \mu(C_\mu * \sigma_o)\vec{n} \quad (35)$$
$$\vec{b}^{(v)} = -\nabla \cdot \boldsymbol{\sigma}^{(v)}$$
$$= -K(C_K * \nabla\sigma_o) + \mu(C_\mu * \vec{b}) + \mu(C_\mu * \nabla\sigma_o)$$

Replacing (28) in (35):

$$\vec{b}^{(v)} = E(C_E * \vec{b}) \qquad (36)$$

It is seen that in the case of synchronic approximation, the equivalent body force vanishes when the regular body forces are disregarded or approximated as an external load. This result simplifies the analysis of applications where the applied body forces can be disregarded or approximated as an external load, because the equation that governs the problem is homogeneous, and thus the Trefftz method can be applied directly. The numerical integration of viscoelasticity with reference to the constitutive relations defined by equation (23) was made by the Algorithmic Internal Variables [Simo and Hughes, (1998)].

The fundamental idea of the algorithm is to transform the constitutive relations given by equation (23) into a two step recurrence formula that involves the internal variables in the boundary nodes. From a computational point of view, this scheme avoids the need to store the entire history of deformations in all the boundary nodes, which would be unnecessary with a direct integration of equation (23).

This method is applicable when the creep function consisting of a linear combination of functions of time possesses the semi-group property. This property holds for the exponential function $e^{\frac{t+\Delta t}{a}} = e^{\frac{\Delta t}{a}} e^{\frac{t}{a}}$ for any constants Δt and a in $\mathbb{R}$ and $t \in \mathbb{R}$ [Simo and Hughes, (1998)].

The implementation of the Trefftz method in viscoelasticity, will be considered for three cases:

4.1 Synchronic approximation without body forces

In this implementation, to reduce the number of degrees of freedom of the problem, an algorithm of the state variables [Creus, (1986)] will be considered for the viscoelastic method with a series of parallel Kelvin elements. For this case:

$$\vec{p}^{(v)}(t) = \vec{p}_0^{(v)}(t) = E(C_E * \vec{p}(t)) \qquad (37)$$

Splitting the time in n intervals, each one of length Δt, and assuming $\vec{p}(t)$ as constant inside each interval, it is easy to demonstrate [Creus, (1986)] that for a Kelvin element of constants E_1 and θ_1:

$$\vec{p}^{(v)}(t + \Delta t) = \vec{p}^{(v)}(t)e^{-\frac{\Delta t}{\theta_1}} + \frac{E}{E_1}(1 - e^{-\frac{\Delta t}{\theta_1}})\vec{p}(t) \qquad (38)$$

The generalization to n Kelvin elements is immediate;ue to the fact that they are in parallel. It is enough to add the state variables $\vec{p}^{(v)}(t)$ of each element. To which a particular solution $\vec{p}_0^{(p)}(x_1, x_2, t_n)$ will correspond, given by an analogue expression in function of $\vec{p}_0(x_1, x_2)$.

To solve this viscoelastic problem in a time t is reduced, then, to solve an elastic problem ruled by the equations:

$$\Im(\vec{u}(x_1, x_2, t)) = 0 \quad \text{in } \Omega$$
$$\vec{u}(x_1, x_2, t) = \vec{u}_d(x_1, x_2, t) \quad \text{in } \partial\Omega_u$$
$$\vec{p}(x_1, x_2, t) = \vec{p}_d(x_1, x_2, t) - \vec{p}^{(v)}(x_1, x_2, t) \quad \text{in } \partial\Omega_p \qquad (39)$$

where $\vec{p}^{(v)}(t)$ is given by the equations (38).

4.2 Synchronic approximation with body forces

In this second case, besides having to calculate $\vec{p}^{(v)}(t)$ as in the previous case, $\vec{b}_0^{(v)}(t)$ needs to be calculated. The following properties will be considered:

$$\vec{b}(x_1, x_2, t) = \vec{b}_0(x_1, x_2)f_0(t),$$

and $\qquad (40)$

$$\vec{b}_0^{(v)}(x_1, x_2, t_0) = 0$$

With these values and using equation (40) for $\vec{b}_0^{(v)}(t)$, it can be demonstrated through mathematical induction that for a time $t_n = t_0 + n\Delta t$, where t_0 is the initial time, and Δt is the step increment:

$$\vec{b}_0^{(v)}(x_1, x_2, t_n) = \vec{b}_0(x_1, x_2)f_n(t_n) \qquad (41)$$

This last equation along with the fact that the Navier operator $\Im$ is linear allow the determination of a particular solution $\vec{u}_0^{(p)}(x_1, x_2, t_n)$ for the Navier equation when body force act according to (40). If $\vec{u}_0(x_1, x_2)$ is a particular solution for $\vec{b}_0(x_1, x_2)$:

$$\Im(\vec{u}_0(x_1, x_2)) = -\vec{b}_0(x_1, x_2) \qquad (42)$$

Multiplying both sides of the equation (42) by $f_n(t_n)$, and applying the linearity of the Navier operator $\Im$, it is obtained:

$$f_n(t_n)\Im(\vec{u}_0(x_1, x_2)) = \Im(\vec{u}_0(x_1, x_2)f_n(t_n))$$
$$= -\vec{b}_0(x_1, x_2)f_n(t_n) = -\vec{b}_0^{(v)}(x_1, x_2, t_n) \qquad (43)$$

leading to the desired particular solution $\vec{u}_0^{(p)}(x_1, x_2, t_n)$ as:

$$\vec{u}_0^{(p)}(x_1, x_2, t_n) = \vec{u}_0(x_1, x_2)f_n(t_n)$$
$$= \vec{u}_0^{(p)}(x_1, x_2, t_{n-1})e^{-\frac{\Delta t}{\theta_1}}$$
$$+ \frac{E}{E_1}\left(1 - e^{-\frac{\Delta t}{\theta_1}}\right)\vec{u}_0(x_1, x_2)f_0(t_{n-1}) \qquad (44)$$

and its particular solution $\vec{p}_0^{(p)}(x_1,x_2,t_n)$ will correspond to a similar expression in function of $\vec{p}_0(x_1,x_2)$. The resolution of this viscoelasticity problem at time t, is reduced to the resolution of an elasticity problem governed by the equations:

$$\Im(\vec{u}(x_1,x_2,t)) = 0 \quad \text{in } \Omega$$

$$\vec{u}(x_1,x_2,t) = \vec{u}_d(x_1,x_2,t) - \vec{u}_0^{(p)}(x_1,x_2,t) \text{ in } \partial\Omega_u$$

(45)

$$\vec{p}(x_1,x_2,t) = \vec{p}_d(x_1,x_2,t) - \vec{p}_0^{(p)}(x_1,x_2,t)$$
$$- \vec{p}^{(v)}(x_1,x_2,t) \text{ in } \partial\Omega_p$$

where $\vec{p}^{(v)}(t)$ and $\vec{u}^{(p)}(t)$ are given by the equations (38) and (43) respectively.

4.3 General case

In this last case, $\vec{p}^{(v)}(t)$ and $\vec{b}^{(v)}(t)$ have to be calculated through the equations (35).

4.3.1 Calculation of $\vec{p}^{(v)}(t)$

Naming:

$$\vec{p}^{(v)}(t) = \vec{p}_0^{(v)}(t) + \vec{p}_K^{(v)}(t) - \vec{p}_\mu^{(v)}(t)$$
$$\vec{p}_0^{(v)}(t) = \mu(C_\mu * \vec{p}(t))$$
$$\vec{p}_K^{(v)}(t) = K(C_K * \sigma_0(t))\vec{n}$$
$$\vec{p}_\mu^{(v)}(t) = \mu(C_\mu * \sigma_0(t))\vec{n}$$

(46)

and according to the algorithm of the state variables:

$$\vec{p}_0^{(v)}(t+\Delta t) = \vec{p}_0^{(v)}(t)e^{-\frac{\Delta t}{\theta_\mu}} + \frac{\mu}{\mu_1}\left(1 - e^{-\frac{\Delta t}{\theta_\mu}}\right)\vec{p}_0(t)$$

$$\vec{p}_K^{(v)}(t+\Delta t) = \vec{p}_K^{(v)}(t)e^{-\frac{\Delta t}{\theta_K}} + \frac{K}{K_1}\left(1 - e^{-\frac{\Delta t}{\theta_K}}\right)\sigma_0(t)\vec{n}$$

$$\vec{p}_\mu^{(v)}(t+\Delta t) = \vec{p}_\mu^{(v)}(t)e^{-\frac{\Delta t}{\theta_\mu}} + \frac{\mu}{\mu_1}\left(1 - e^{-\frac{\Delta t}{\theta_\mu}}\right)\sigma_0(t)\vec{n}$$

(47)

In order to calculate the last two equations (47), there is a need to calculate the first invariant of the stress tensor. Applying the Trefftz method at each time t, it is possible to calculate the invariant in every point of the domain with the equations (48), valid for the plane stress; for the plane

strains, those equations shall be multiplied by the respective factor:

$$\sigma_o(x_1,x_2,t) \cong \tilde{\sigma}_o(a(t),x_1,x_2)$$
$$= \sum_{j=1}^{NT} a_j(t)\sigma_{o_j}^*(x_1,x_2)$$
$$= \mathbf{a}(t)^T \boldsymbol{\sigma}^*(x_1,x_2)$$
$$= \sum_{i=1}^{N} 4ir^{i-1}[C(i-1)\alpha_i(t) + S(i-1)\beta_i(t)]$$

(48)

From this expression and by means of the viscoelastic operators, the equivalent body forces vector $\vec{p}^{(v)}(t)$ can be determined.

4.3.2 Calculation of $\vec{b}^{(v)}(t)$

To calculate $\vec{b}^{(v)}(t)$ a generalization of the particular solution method is considered, namely:

$$\vec{b}^{(v)}(t) = \vec{b}_0^{(v)}(t) - \vec{b}_K^{(v)}(t) + \vec{b}_\mu^{(v)}(t)$$
$$\vec{b}_0^{(v)}(t) = \mu(C_\mu * \vec{b}(t))$$
$$\vec{b}_K^{(v)}(t) = K(C_K * \nabla\sigma_0(t))$$
$$\vec{b}_\mu^{(v)}(t) = \mu(C_\mu * \nabla\sigma_0(t))$$

(49)

and according to the state variables algorithm:

$$\vec{b}_0^{(v)}(t+\Delta t) = \vec{b}_0^{(v)}(t)e^{-\frac{\Delta t}{\theta_\mu}} + \frac{\mu}{\mu_1}\left(1 - e^{-\frac{\Delta t}{\theta_\mu}}\right)\vec{b}_0(t)$$

$$\vec{b}_K^{(v)}(t+\Delta t) = \vec{b}_K^{(v)}(t)e^{-\frac{\Delta t}{\theta_K}} + \frac{K}{K_1}\left(1 - e^{-\frac{\Delta t}{\theta_K}}\right)\nabla\sigma_0(t)$$

$$\vec{b}_\mu^{(v)}(t+\Delta t) = \vec{b}_\mu^{(v)}(t)e^{-\frac{\Delta t}{\theta_\mu}} + \frac{\mu}{\mu_1}\left(1 - e^{-\frac{\Delta t}{\theta_\mu}}\right)\nabla\sigma_0(t)$$

(50)

The first of the equations (50) can be solved as described in section 4.2, obtaining a particular solution in the displacement $\vec{u}_0^{(p)}(x_1,x_2,t)$, given by the equation (44) changing E by μ. To work with the two remaining equations let's consider the respective gradients in equation (48) to obtain:

$$\nabla\sigma_o(x_1,x_2,t) \cong \nabla\tilde{\sigma}_o(x_1,x_2,t)$$
$$= \sum_{j=1}^{NT} a_j(t)\nabla\sigma_{o_j}^*(x_1,x_2)$$
$$= \mathbf{a}(t)^T \nabla\boldsymbol{\sigma}^*(x_1,x_2)$$

(51)

Being:

$$\tilde{\sigma}_{o,1} =$$

$$\sum_{i=1}^{N} 4i(i-1)r^{i-2}[C(i-2)\alpha_i(t) - S(i-2)\beta_i(t)]$$

$$\tilde{\sigma}_{o,2} =$$

$$-\sum_{i=1}^{N} 4i(i-1)r^{i-2}[S(i-2)\alpha_i(t) - C(i-2)\beta_i(t)]$$

$$(52)$$

By looking at equation (51) and the fact that $\vec{b}^{(K)}(t_0) = 0$ and $\vec{b}^{(\mu)}(t_0) = 0$, the problem corresponds to the the conditions for the body forces of section 4.2. Therefore, there are particular solutions $\vec{u}_K^{(p)}(x_1, x_2, t)$ and $\vec{u}_\mu^{(p)}(x_1, x_2, t)$ corresponding to the pseudo-body forces $\vec{b}_K^{(v)}(t)$ and $\vec{b}_\mu^{(v)}(t)$ given by:

$$\vec{u}_K^{(p)}(t + \Delta t) =$$

$$\vec{u}_K^{(p)}(t)e^{-\frac{\Delta t}{\theta_K}} + \frac{K}{K_1}(1 - e^{-\frac{\Delta t}{\theta_K}}) \sum_{j=1}^{n} a_j(t)\vec{u}_j^{(p^*)}$$

$$(53)$$

$$\vec{u}_\mu^{(p)}(t + \Delta t) =$$

$$\vec{u}_\mu^{(p)}(t)e^{-\frac{\Delta t}{\theta_\mu}} + \frac{\mu}{\mu_1}(1 - e^{-\frac{\Delta t}{\theta_\mu}}) \sum_{j=1}^{n} a_j(t)\vec{u}_j^{(p^*)}$$

where $\vec{u}_j^{(p^*)}$ is a particular solution of the Navier equation when applied a body force equal to $\nabla \sigma_{o_j}^*$. When a body force equal to each of the terms in equation (51) is applied to the Navier equation, it is possible to demonstrate that the respective particular solutions are:

$$\Im\left(\vec{u}_{4n}^{(p^*)}\right) = 4n(n-1)r^{n-2}[C(n-2)\vec{e}_1$$
$$- S(n-2)\vec{e}_2]$$

$$\Rightarrow$$

$$\vec{u}_{4n}^{(p^*)} \cdot \vec{e}_1 = \frac{n(n-1)r^n}{2\mu(3K+\mu)}\left[\frac{7\mu + 3K}{n-1}C(n-2)\right.$$
$$\left. - \frac{3K+\mu}{2}C(n-4)\right]$$

$$\vec{u}_{4n}^{(p^*)} \cdot \vec{e}_2 = -\frac{n(n-1)r^n}{2\mu(3K+\mu)}\left[\frac{7\mu + 3K}{n-1}S(n-2)\right.$$
$$\left. + \frac{3K+\mu}{2}S(n-4)\right]$$

$$\Im(\vec{u}_{4n+1}^{(p^*)})) = 4n(n-1)r^{n-2}[S(n-2)\vec{e}_1$$
$$+ C(n-2)\vec{e}_2]$$

$$\Rightarrow$$

$$\vec{u}_{4n+1}^{(p^*)} \cdot \vec{e}_1 = \frac{n(n-1)r^n}{2\mu(3K+\mu)}\left[\frac{7\mu + 3K}{n-1}S(n-2)\right.$$
$$\left. + \frac{3K+\mu}{2}S(n-4)\right]$$

$$\vec{u}_{4n+1}^{(p^*)} \cdot \vec{e}_2 = \frac{n(n-1)r^n}{2\mu(3K+\mu)}\left[\frac{7\mu + 3K}{n-1}C(n-2)\right.$$
$$\left. - \frac{3K+\mu}{2}C(n-4)\right]$$

$$(54)$$

These equations are valid for values of n different from 1 and 2, which yield the corresponding equations relatively easily.

Similar formulas are obtained for the surface tractions.

It is thus concluded that in order to solve the general viscoelasticity problem at time t, it is needed to solve an elasticity problem governed by the equations:

$$\Im(\vec{u}(x_1, x_2, t)) = 0 \quad \text{in } \Omega$$

$$\vec{u}(x_1, x_2, t) = \vec{u}_d(x_1, x_2, t) - \vec{u}^{(p)}(x_1, x_2, t) \quad \text{in } \partial\Omega_u$$

$$\vec{p}(x_1, x_2, t) = \vec{p}_d(x_1, x_2, t) - \vec{p}^{(p)}(x_1, x_2, t)$$
$$- \vec{p}^{(v)}(x_1, x_2, t) \quad \text{in } \partial\Omega_p$$

$$(55)$$

where $\vec{p}^{(v)}(t)$ is given by the formulas developed in section 4.3.1, while $\vec{u}^{(p)}(t) = \vec{u}_0^{(p)}(t) - \vec{u}_K^{(p)}(t) + \vec{u}_\mu^{(p)}(t)$ is given by three terms from the equations (44) and (53) respectively.

In the generalized particular solution method proposed in this paper, the non-homogeneous terms are calculated from a series expansion obtained from the T-complete functions. The particular solution corresponding to each of these functions can be found without approximation with new polynomials as in other formulations of the Trefftz method [Kita, Ikoda and Kamiya, (2003)].

This allows keeping the characteristics of the Trefftz method of only working in the boundary.
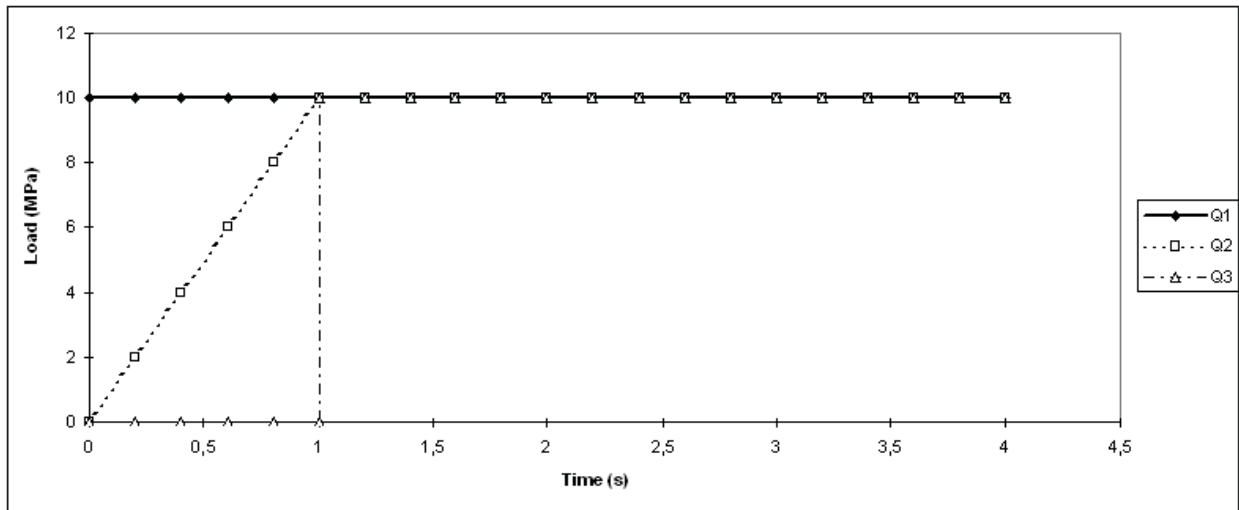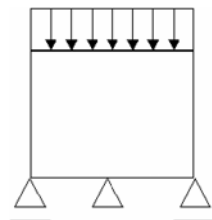
Figure 1: Considered load histories.

Figure 2: Square plate of a viscoelastic material.

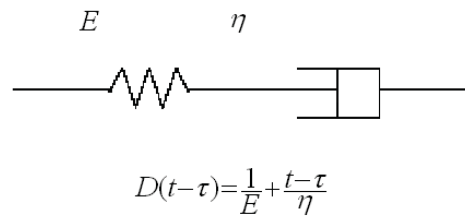$$D(t-\tau)=\frac{1}{E}+\frac{t-\tau}{\eta}$$

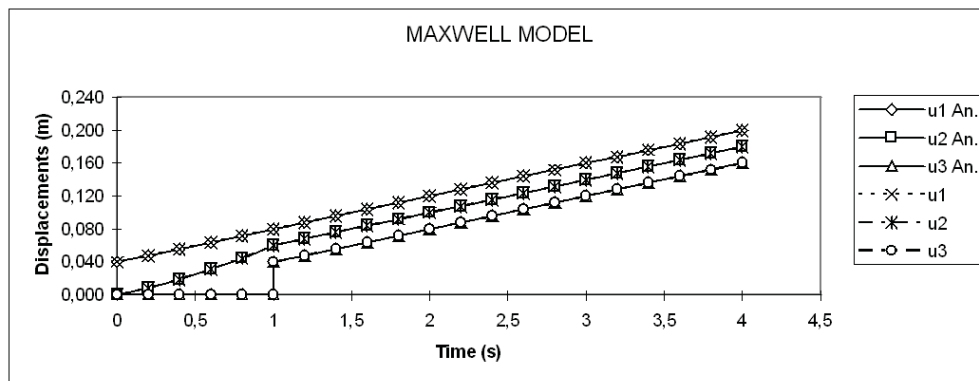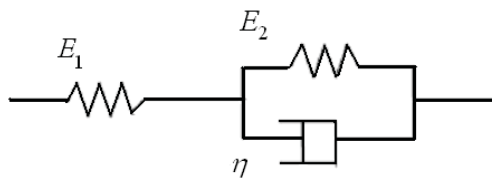Figure 3: Schematic and constitutive equation of Maxwell model.

Figure 4: Results obtained for a viscoelastic material that responds to a Maxwell model in its longitudinal mode.

$$D(t-\tau)=\frac{1}{E_1}+\frac{1}{E_2}(1-e^{\frac{t-\tau}{\eta}})$$

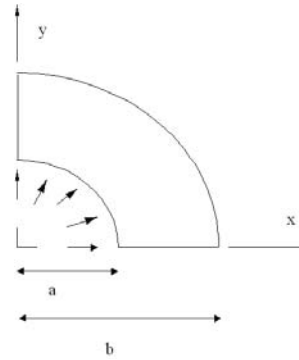Figure 5: Schematic and constitutive equation of Boltzmann model.

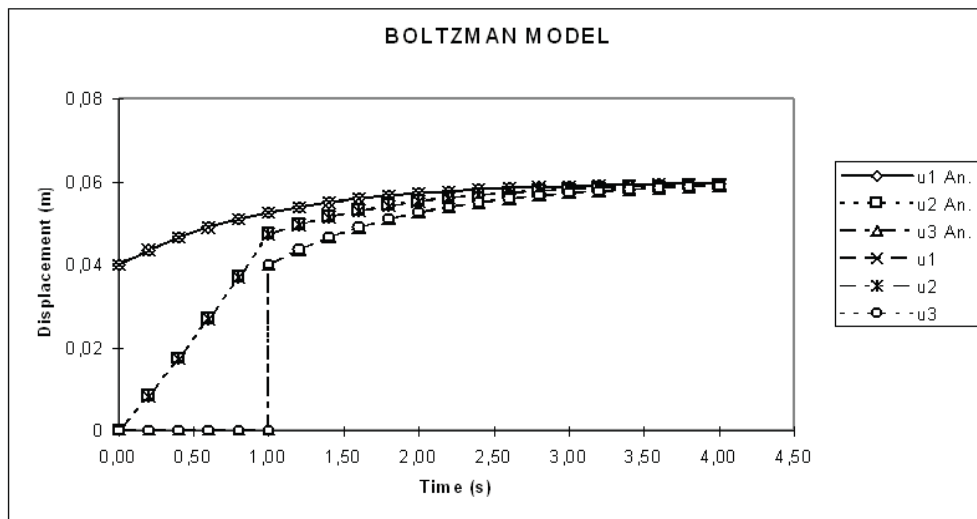Figure 7: One-quarter hollow cylinder subject to constant internal pressure.

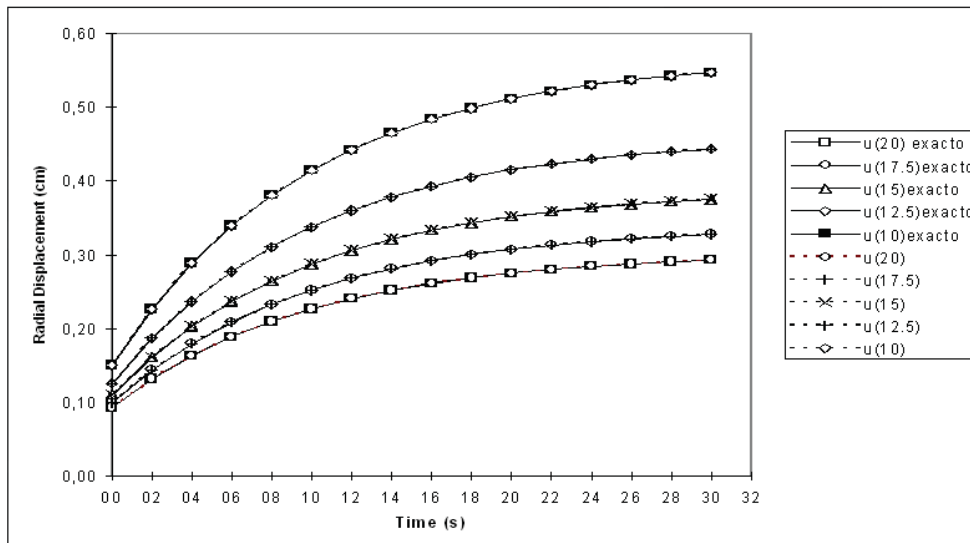Figure 6: Results obtained for a viscoelastic material that responds to a Boltzmann model.

Figure 8: Radial displacement of the hollow cylinder.

5 Examples

5.1 Fading memory

Three load histories as shown in Figure 1 are applied to a two-meter-side square plate.

The plate in Figure 2 was modelled with 12 nodes, taking double nodes in the corners and considering;he collocation method with 16 terms in the Trefftz series.

At first, it was assumed that a viscoelastic material (Figure 3) responds according to a Maxwell model in its longitudinal mode with $E_1 = 500$ MPa , $\eta_1 = 50$ MPa/s, and with a time-independent Poisson's ratio of 0.25.

In Figure 4, the obtained results are compared with the analytical solution for three load histories .

Similarly, the same loads are applied to a Boltzman model with $E_1 = 500$ MPa , $E_2 = 1000$ MPa , $\eta_2 = 100$ MPa/s and $v = 0.25$ (Figure 5).

In Figure 6, the obtained results are compared with the analytical solution.

A time step equal to 0.01 s was considered in both models.

It can be seen that, when the time increases in the Maxwell model, the deformation histories are different, while the Boltzmann model erases the details of the loading process. This is a characteristic behaviour of stable non-aging viscoelastic materials and is related to the concept of fading memory [Creus, (1986)].

5.2 Hollow cylinder subject to constant internal pressure

A thick hollow cylinder of internal radius, a, equal to 10 cm and external radius, b, equal to 20 cm is subject to an internal pressure $P(t) = P_o H(t)$, with $P_o = 10$ MPa , and $H(t)$ being the Heaviside function [Sim and Kwak, (1988)]. Due to symmetry, only one-quarter of the cylinder was considered in Figure 7. It was discretized using 126 nodes and 132 terms for the expansion.

A viscoelastic material having a Boltzmann model for the shear component and elastic model for the dilatational component was considered.

The constants were: $G_1 = 480$ MPa; $G_2 = 160$ MPa; $\eta_2 = 1600$ MPa/s, $K = 1280$ MPa. A time step of $\Delta t = 1$s was considered.

The numerical solution for radii of 20 cm, 17.5cm, 15 cm, 12.5 cm, and 10 cm are plotted along with the exact solution and shown in Figure 8.

5.3 Flat plate subjected to prescribed displacements

A flat plate was subjected to displacements that vary linearly along the considered boundary applied at zero time, and then kept constant as shown in Figure 9. The plate was discretized using 24 nodes and 36 terms in the expansion.
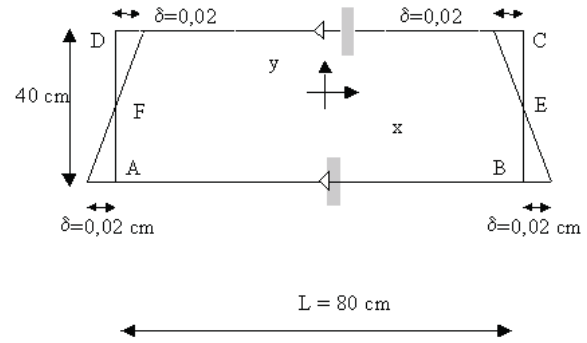

Figure 9: Flat plate under prescribed displacements.

A Dischinger model, as shown in Figure 10, is considered for the material in the shear mode with constants $\mu(t) = 8571$ MPa, $\gamma = 0.026$ day^{-1}, $\gamma\eta_0 = 35,667$ MPa, and linear elasticity in the dilatation mode with constant $K = 10,000$ MPa.

$$E(\tau) \qquad \eta(t) = \eta_0\, e^{\gamma t}$$

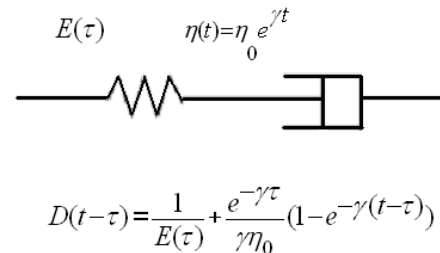$$D(t-\tau) = \frac{1}{E(\tau)} + \frac{e^{-\gamma\tau}}{\gamma\eta_0}(1 - e^{-\gamma(t-\tau)})$$

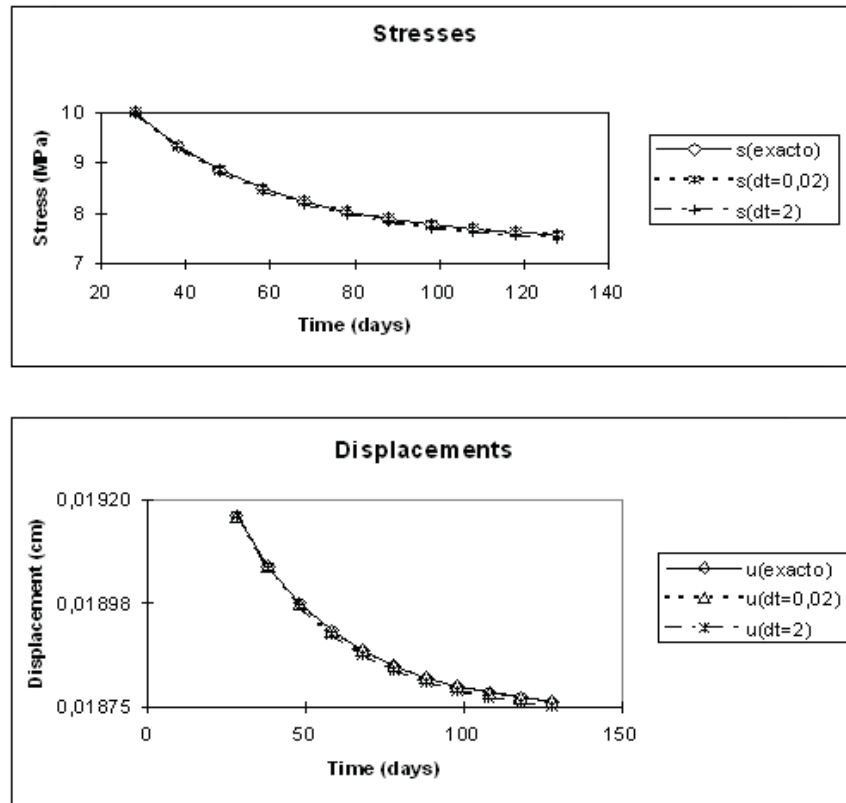Figure 10: Schematic and constitutive equation of Dischinger model.

Figure 11: Results for flat plate under constant prescribed displacements.

In Figure 11, σ_x at point A and displacements u_y at point E are compared with analytical results using integration intervals of $\Delta t = 2$ days and $\Delta t = 0.02$ days.

was non-homogeneous at every time step. The results obtained by the authors for several viscoelastic materials show the good performance of the method.

6   Conclusions

A generalization of the Trefftz method for analyzing problems of viscoelastic materials, with or without aging, was presented in this paper. It was shown that the elastic Trefftz method allows the solution of synchronic problems, in which the Poisson's ratio is constant, where the governing equation is homogeneous in the absence of body forces. Through the analogy between the pseudo body forces and the inelastic deformations gradient, the method of the particular solution was extended to the solution of general viscoelastic problems in which the Poisson's ratio changes with time. The Trefftz method is typically used on the boundary but it was proven here to work in spite of the the fact that the differential equation

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