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An Alternative Approach to Boundary Element Methods via the Fourier Transform

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Abstract: In general, the use of Boundary Element Methods (BEM) is restricted to physical cases for which a fundamental solution can be obtained. For simple differential operators (e.g. isotropic elasticity) these special solutions are known in their explicit form. Hence, the realization of the BEM is straight forward. For more complicated problems (e.g. anisotropic materials), we can only construct the fundamental solution numerically. This is normally done before the actual problem is tackled; the values of the fundamental solutions are stored in a table and all values needed later are interpolated from these entries. The drawbacks of this approach lie in the high amount of storage capacity, which is required, and in numerically errors due to interpolation especially near the singularity of the fundamental solution. Hence, an alternative BEM, the Fourier BEM, was proposed in Duddeck (2002) which is based on boundary integral equations (BIE) obtained via Fourier transform. It can be applied to all problems as long as the differential operator is linear and has constant coefficients.
The first step to derive the Fourier transformed BIE consists in a rigorous mathematical formulation via distribution theory, which was developed by Schwartz (1950/51) at the end of the 1940s and which is still the mathematical basis for the treatment of partial differential equations, e.g. Hörmander (1990). In the context of BEM, this theory offers a straightforward approach towards the discussion of singularities normally encountered in the BIE. Distribution theory is able to handle all kind of singularities (jumps, weak, strong and hyper singular values) occurring in the BEM formulations and it is the adequate approach for the discussion of the corresponding integrations. In fact it can be shown by this approach, cf. Duddeck (2002), that all strong and hyper singular components are vanishing. In addition, the distribution theory enlarges the applicability of Fourier transform leading to alternative formulations for linear differential equations with constant coefficients. All differentiations are converted to multiplications; the differential operator becomes a simple algebraic expression, which can easily be inverted. This inverse differential operator is the Fourier transform of the fundamental solution.
In the approach discussed here, this Fourier fundamental solution and not the fundamental solution itself is taken for the computation of all entries to the BEM-matrices. Based on Parsevals formula, which states the equivalence of energy expressions in the Fourier and the original space, alternative BIE can be derived in the Fourier space leading to the same entries for the matrices. Thus for the Fourier BEM, every term should be established in the Fourier space. Because a Galerkin approach leads to symmetric matrices and does not require a second integration in the Fourier BEM, this approach was preferred to the conventional collocation BEM. The trial and the test functions can be easily transformed to the Fourier space as long as they are defined on straight elements. Otherwise a numerical approach can be selected.
In this paper, the method is applied to thin plate problems according to Kirchhoff's theory. The differential operator is of fourth order leading to highly singular integral equations. Although these singularities are quite complex, it can be shown easily that all strong and hyper singular terms are vanishing in both, the original and the Fourier transformed space. In the small example, all integrals were solved analytically, thus - in contrast to other publications, e.g. Maucher and Hartmann (1999) – no numerical errors, i.e. artificial oscillations, are occurring at the corners of a rectangular plate.

keyword: Boundary Element Methods, Fourier Transform, Fundamental Solutions, Kirchhoff Plates.

1 Introduction

In the literature, BEM models are mostly discussed for isotropic plates. Collocation approaches can be found for example in Antes and Panagiotopoulos (1992); Beskos

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(1991). The Galerkin BIE were presented in Frangi and Bonnet (1998). Plates on Winkler foundations are treated by a collocation method in Jahn (1998). All these approaches were developed for isotropic plates in the original space. Anisotropic plates are regarded for example in Zhao (1995) where a collocation approach was chosen and the differential equation of fourth order was converted into two PDEs of second order. Hence, a fundamental solution can easily be found. The disadvantage is the resulting vectorial character of the equation.

The Fourier BEM proposed in Duddeck (2002) generalizes the traditional Galerkin BEM such that physical cases where the fundamental solution is not known explicitly can be treated. First applications in linear elasticity and heat transfer can be found in Duddeck and Geisenhofer (2002); Duddeck (2001). Here, the method is transferred to isotropic and anisotropic plate problems. In the first part, the model for the bending of plates is analyzed which is due to Kirchhoff and which is valid for thin plates where shear deformations can be neglected. In the second part, the fundamental solution for anisotropic Kirchhoff plates is given in the Fourier space. Applications to anisotropic cases will be published in a separate paper.

2 The principle of Fourier BEM

2.1 The Galerkin BIE for the Poisson equation

Because the Fourier BEM approach is rather new and establishes a fundamental different view on BEM, the main principles are demonstrated first for the simple example of the Poisson equation in an n-dimensional bounded domain $\Omega \subset \mathcal{R}^n$ with a boundary $\partial\Omega = \Gamma_u \cup \Gamma_t$:

$$\begin{aligned} -\Delta u(x) &= f(x), \qquad \Delta = \sum_j^n \partial^2/\partial x_j^2, x \in \Omega, \\ u(x) &= u_\Gamma(x), \qquad x \in \Gamma_u \subset \Gamma; \\ t(x) &= t_\Gamma(x), \qquad x \in \Gamma_t \subset \Gamma. \end{aligned} \tag{1}$$

u is the unknown quantity, f denotes known volume sources, and $u_\Gamma(x), t_\Gamma(x)$ are the known boundary values. The corresponding BIE is, e.g. Bonnet (1999):

$$\begin{aligned} \kappa(x)u(x) &= \int_\Omega f(y)U(x-y)\,\mathrm{d}y \\ &+ \sum_i^{N_\mathrm{t}} \mathrm{t}^i \int_\Gamma \phi_\mathrm{t}^i(y)U(x-y)\,\mathrm{d}\Gamma_y \\ &- \sum_i^{N_\mathrm{u}} \mathrm{u}^i \int_\Gamma \phi_\mathrm{u}^i(y)\mathcal{A}_\mathrm{t}^i U(x-y)\,\mathrm{d}\Gamma_y. \end{aligned} \tag{2}$$

κ is the free term, $\mathcal{A}_\mathrm{t}^i = -\partial_i = -\nu^i \cdot \nabla$ the boundary differential operator with ∇ as the Nabla symbol and ν^i as the normal vector of the i-th boundary element. $\phi_\mathrm{t}^i, \phi_\mathrm{u}^i$ are the i-th test and trial functions for u and $t = \mathcal{A}_\mathrm{t}^i u$ and $\mathrm{t}^i, \mathrm{u}^i$ are the known and unknown coefficients of the discretization. U is the fundamental solution. The Galerkin BIE is the obtained via a weigthing

$$\begin{aligned} \int_\Gamma \phi_\mathrm{t}^j(x)\kappa(x)u(x)\,\mathrm{d}\Gamma_x &= \int_\Gamma \phi_\mathrm{t}^j(x) \int_\Omega f(y)U(x-y)\,\mathrm{d}y\,\mathrm{d}\Gamma_x \\ &+ \sum_i^{N_\mathrm{t}} \mathrm{t}^i \int_\Gamma \phi_\mathrm{t}^j(x) \int_\Gamma \phi_\mathrm{t}^i(y)U(x-y)\,\mathrm{d}\Gamma_y\,\mathrm{d}\Gamma_x \\ &- \sum_i^{N_\mathrm{u}} \mathrm{u}^i \int_\Gamma \phi_\mathrm{t}^j(x) \int_\Gamma \phi_\mathrm{u}^i(y)\mathcal{A}_\mathrm{t}^i U(x-y)\,\mathrm{d}\Gamma_y\,\mathrm{d}\Gamma_x. \end{aligned} \tag{3}$$

The inner integral is a convolution $(.*.)$, which leads in the Fourier space to a multiplication (all integrals are extended to infinity; values outside the original support are kept to be zero). The outer integral is a scalar product $<.,.>$, which is equal to a similar scalar product in the Fourier space. We have the two main theorems known for the Fourier transform, e.g. Hörmander (1990):

$$u(x) * \phi^i(x) \qquad \overset{\mathcal{F}}{\leftrightarrow} \qquad \hat{u}(\hat{x})\hat{\phi}^i(\hat{x}); \tag{4}$$

$$\left\langle \phi^j(x), u(x) \right\rangle = \frac{1}{(2\pi)^n} \left\langle \hat{\phi}^j(-\hat{x}), \hat{u}(\hat{x}) \right\rangle. \tag{5}$$

The symbol $\overset{\mathcal{F}}{\leftrightarrow}$ links an expression in the original space to the corresponding term in the Fourier transformed space. Thus, the transform of (3) leads to BIE, which refer only to the transformed fundamental solution $\hat{U}$:

$$\begin{aligned} \left\langle \hat{\phi}_\mathrm{t}^j(-\hat{x}), \hat{u}(\hat{x}) \right\rangle &= \left\langle \hat{\phi}_\mathrm{t}^j(-\hat{x}), \hat{f}(\hat{x})\hat{U}(\hat{x}) \right\rangle \\ &+ \sum_i^{N_\mathrm{t}} \mathrm{t}^i \left\langle \hat{\phi}_\mathrm{t}^j(-\hat{x}), \hat{\phi}_\mathrm{t}^i(\hat{x})\hat{U}(\hat{x}) \right\rangle \\ &- \sum_i^{N_\mathrm{u}} \mathrm{u}^i \left\langle \hat{\phi}_\mathrm{t}^j(-\hat{x}), \hat{\phi}_\mathrm{u}^i(\hat{x})\hat{\mathcal{A}}_\mathrm{t}^i \hat{U}(\hat{x}) \right\rangle. \end{aligned} \tag{6}$$

It is emphasized here, that the matrix entries obtained via these Fourier BIE are identical to those obtained normally with the standard BEM approach. The other BIE are treated in the same manner. Thus there is no difference between the two algebraic systems, that obtained via traditional BEM and via the Fourier BEM. For the displacement BIE discussed here, we obtain, cf. Bonnet (1999),

$$\sum_i K_u^{ij} u^i = F_u^j + \sum_i H_u^{ij} t^i - \sum_i G_u^{ij} u^i; \tag{7}$$

where the entries can be evaluated in the Fourier space

$$\begin{aligned}
F_u^j &= \frac{1}{(2\pi)^n} \left\langle \hat{\phi}_t^j(-\hat{x}), \hat{f}(\hat{x}) \hat{U}(\hat{x}) \right\rangle; \\
H_u^{ij} &= \frac{1}{(2\pi)^n} \left\langle \hat{\phi}_t^j(-\hat{x}), \hat{\phi}_t^i(\hat{x}) \hat{U}(\hat{x}) \right\rangle; \\
G_u^{ij} &= \frac{1}{(2\pi)^n} \left\langle \hat{\phi}_t^j(-\hat{x}), \hat{\phi}_u^i(\hat{x}) \hat{\mathcal{A}}_t^i \hat{U}(\hat{x}) \right\rangle; \\
K_u^{ij} &= \frac{1}{(2\pi)^n} \left\langle \hat{\phi}_t^j(-\hat{x}), \hat{\phi}^i(\hat{x}) \hat{U}(\hat{x}) \right\rangle;
\end{aligned}$$

or in the original space

$$\begin{aligned}
F_u^j &= \left\langle \phi_t^j(x), f(x) * U(x) \right\rangle; \\
H_u^{ij} &= \left\langle \phi_t^j(x), \phi_t^i(x) * U(x) \right\rangle; \\
G_u^{ij} &= \left\langle \phi_t^j(x), \phi_u^i(x) * \mathcal{A}_t^i U(x) \right\rangle; \\
K_u^{ij} &= \left\langle \phi_t^j(x), \phi^i(x) * U(x) \right\rangle;
\end{aligned}$$

The latter can be used only if the fundamental solution U is known while the first can be computed in all cases as long as the differential operators is linear and has constant coefficients. The evaluations in the Fourier space consist of only one integration of oscillant integrands and the formulations in the original space require a double integration. In the following, this principle is transferred to thin plate theory.

3 The isotropic thin plate

The midsurface of the plate with a uniform thickness h is situated in the (x_1, x_2)-plane, see Fig.1. $w(x); x = x_1, x_2$ denotes the out-of-plane bending displacement. We define the unit outward normal $\nu = (\nu_1, \nu_2)^T$ and the unit tangent $\tau = (\tau_1, \tau_2)^T = (-\nu_2, \nu_1)^T$. The moment m_{kl} and the shear components q_k are constructed from the stress tensor σ_{kl} by

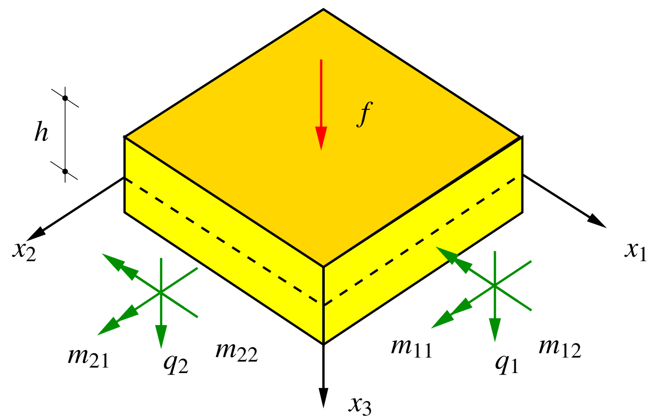

Figure 1 : Plate forces and moments

$$\begin{aligned}
m_{kl} &= \int_{-h/2}^{h/2} \sigma_{kl}(x, x_3) x_3 \, \mathrm{d}x_3, \\
q_k &= \int_{-h/2}^{h/2} \sigma_{k3}(x, x_3) \, \mathrm{d}x_3.
\end{aligned} \tag{8}$$

The moment m_{kl} can be related to the vertical displacement w by:

$$m_{kl} = -K_{klmn} \partial_{mn} w \qquad \overset{\mathcal{F}}{\leftrightarrow} \qquad \hat{m}_{kl} = K_{klmn} \hat{x}_m \hat{x}_n \hat{w} \tag{9}$$

and the shear forces q_k by:

$$q_k = \partial_l m_{kl} = -D \partial_{kll} w \overset{\mathcal{F}}{\leftrightarrow} \hat{q}_k = i \hat{x}_l \hat{m}_{kl} = i D \hat{x}_k \hat{x}_l \hat{x}_l \hat{w};$$

with the stiffness of the plate as:

$$K_{klmn} = D[(1-\bar{\nu}) \delta_{km} \delta_{ln} + \bar{\nu} \delta_{kl} \delta_{mn}], \; D = \frac{E h^3}{12(1-\bar{\nu}^2)},$$

where summation has to be applied for repeated indices $k, l, m, n = 1, 2$. $E, \bar{\nu}, D$ are the Young's modulus for elasticity, the Poisson's coefficient, and the flexural rigidity of the plate, respectively. δ_{kl} is Kronecker's symbol.

The Fourier transform is done with respect to all coordinates. $(\hat{\cdot})$ denotes a quantity in the Fourier space, thus $\hat{x}_k$ are the wave numbers. ∂_{kl} is the short notation for the partial differential operator $\partial^2 / (\partial x_k \partial x_l)$.

The differential equation for the bending of the isotropic Kirchhoff plate is

$$D \Delta\Delta w = f \qquad \overset{\mathcal{F}}{\leftrightarrow} \qquad D(\hat{x}_1^2 + \hat{x}_2^2)^2 \hat{w} = \hat{f}, \tag{10}$$

with f as transversal load per unit area and Δ as the Laplace operator. Thus the differential operator is in the Fourier space

$$\hat{\mathcal{P}}(\hat{x}) = D(\hat{x}_1^2+\hat{x}_2^2)^2. \tag{11}$$

The transformed fundamental solution $\hat{W}$ is obtained as the inverse of $\hat{\mathcal{P}}(\hat{x})$:

$$\hat{W}(\hat{x}) = \frac{1}{D|\hat{x}|^4} = \frac{1}{D(\hat{x}_1^2+\hat{x}_2^2)^2}, \tag{12}$$

with $|\hat{x}| = \sqrt{\hat{x}_1^2+\hat{x}_2^2}$.

3.1 The boundary differential operators

The boundary quantities which are relevant in the context of BEM are the deflection w, the normal slope φ_ν, the normal bending moment m_ν, and the equivalent Kirchhoff shear $q_\nu = \nu_k q_k + \mathrm{d}m_T/\mathrm{d}s$ (we assume that the normal vector ν is piecewise constant)

$$w \overset{\mathcal{F}}{\leftrightarrow} \hat{w}; \tag{13}$$

$$\varphi_\nu = \nu_k \partial_k w \overset{\mathcal{F}}{\leftrightarrow} \hat{\varphi}_\nu = i\nu_k \hat{x}_k \hat{w};$$

$$m_\nu = \nu_k \nu_l m_{kl} \overset{\mathcal{F}}{\leftrightarrow} \hat{m}_\nu = \nu_k \nu_l \hat{m}_{kl};$$

$$q_\nu = \nu_k q_k + \tau_k \partial_k m_{lm} \nu_l \tau_m$$
$$\overset{\mathcal{F}}{\leftrightarrow} \quad \hat{q}_\nu = \nu_k \hat{q}_k + i\tau_k \hat{x}_k \hat{m}_{lm} \nu_l \tau_m,$$

where the twisting moment for the Kirchhoff shear is defined as $m_\mathrm{T} = m_{kl}\nu_k\tau_l$, and the differentiation with respect to the arc length s of the boundary is $\mathrm{d}/\mathrm{d}s = \tau_k\partial_k$. At the i-th corner, there is a particular corner force f_c^i which is related to a jump of the twisting moment m_T around the corner

$$f_\mathrm{c}^i(x = x_\mathrm{c}^i) \quad = \quad m_\mathrm{T}^+(x = x_\mathrm{c}^i) - m_\mathrm{T}^-(x = x_\mathrm{c}^i). \tag{14}$$

This corner force is for arbitrary angles

$$f_\mathrm{c}^i = \mathcal{A}_\mathrm{c}^i w \qquad \overset{\mathcal{F}}{\leftrightarrow} \qquad \hat{f}_\mathrm{c}^i = \hat{\mathcal{A}}_\mathrm{c}^i \hat{w}, \tag{15}$$

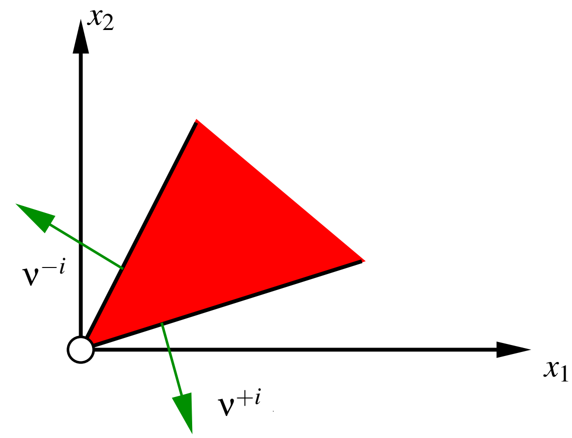

Figure 2 : Definition of the normal vectors ν^{+i}, ν^{-i} for the corner term.

with the differential operator (see Fig. 2 for the definition of ν^{+i}, ν^{-i})

$$\mathcal{A}_\mathrm{c}^i = (1-\overline{\nu})D(\nu_1^{+i}\nu_2^{+i} - \nu_1^{-i}\nu_2^{-i})(\partial_{11}-\partial_{22})$$
$$+2D(\nu_1^{+i}\nu_1^{+i} - \nu_2^{+i}\nu_2^{+i} - \nu_1^{-i}\nu_1^{-i} + \nu_2^{-i}\nu_2^{-i})\partial_{12}.$$

$$\overset{\mathcal{F}}{\leftrightarrow} \hat{\mathcal{A}}_\mathrm{c}^i = -(1-\overline{\nu})D(\nu_1^{+i}\nu_2^{+i} - \nu_1^{-i}\nu_2^{-i})(\hat{x}_1^2-\hat{x}_2^2)$$
$$+2D(\nu_1^{+i}\nu_1^{+i} - \nu_2^{+i}\nu_2^{+i} - \nu_1^{-i}\nu_1^{-i} + \nu_2^{-i}\nu_2^{-i})\hat{x}_1\hat{x}_2.$$

Thus, the three boundary differential operators $\mathcal{A}_{\varphi,\mathrm{m,q}}^i$ for a boundary element with the normal ν^i and the corner differential operator $\mathcal{A}_\mathrm{c}^i$ are defined as follows:

For φ_ν :

$$\mathcal{A}_\varphi^i = \nu_k^i \partial_k \qquad \overset{\mathcal{F}}{\leftrightarrow} \qquad \hat{\mathcal{A}}_\varphi^i = i\nu_k \hat{x}_k; \tag{16}$$

for m_ν :

$$\mathcal{A}_\mathrm{m}^i = -K_{klmn}\nu_k^i\nu_l^i\partial_{mn} \qquad \overset{\mathcal{F}}{\leftrightarrow} \qquad \hat{\mathcal{A}}_\mathrm{m}^i = K_{klmn}\nu_k^i\nu_l^i\hat{x}_m\hat{x}_n;$$

for q_ν :

$$\begin{aligned} \mathcal{A}_\mathrm{q}^i \quad &= \quad -D\nu_k^i\partial_{kll} - K_{klmn}\tau_p^i\nu_k^i\tau_l^i\partial_{mnp} \\ \overset{\mathcal{F}}{\leftrightarrow} \quad \hat{\mathcal{A}}_\mathrm{q}^i \quad &= \quad iD\nu_k^i\hat{x}_k\hat{x}_l\hat{x}_l + iK_{klmn}\tau_p^i\nu_k^i\tau_l^i\hat{x}_m\hat{x}_n\hat{x}_p; \end{aligned}$$

and for f_c :

$$\mathcal{A}_\mathrm{c}^i \quad \text{see (15)} \quad \overset{\mathcal{F}}{\leftrightarrow} \quad \hat{\mathcal{A}}_\mathrm{c}^i \quad \text{see (15)}.$$

The boundary quantity and its transform are obtained via $\mathcal{A}_\mathrm{k}^i w$ or $\hat{\mathcal{A}}_\mathrm{k}^i \hat{w}$ $(k = \varphi, m, q, c)$.

For any well-posed problem half of the boundary data must be given. For each pair of dual variables (dual in the sense that the pairs lead to work terms) either one of these two variables or the relation between both (a Robin type boundary condition) must be prescribed. There are four different types of boundary conditions :

$$\begin{aligned} w(x) &= w_\Gamma(x) && \text{for} \quad x \in \Gamma_\mathrm{w}; \\ \varphi_\nu(x) &= \varphi_{\nu\Gamma}(x) && \text{for} \quad x \in \Gamma_\varphi; \\ m_\nu(x) &= m_{\nu\Gamma}(x) && \text{for} \quad x \in \Gamma_\mathrm{m}; \\ q_\nu(x) &= q_{\nu\Gamma}(x) && \text{for} \quad x \in \Gamma_\mathrm{q}. \end{aligned} \tag{17}$$

In addition, we have to prescribe for each corner point x_c^i either the jump of the twisting moment $f_\mathrm{c}^i(x_\mathrm{c}^i)$ or the displacement $w(x_\mathrm{c}^i)$ at this point. For establishing the BIE, the following derivatives of the fundamental solution W are required, cf. Beskos (1991) for the original space: The fundamental slope Φ_ν (with $\ln r_0^2 = 1$)

$$\Phi_\nu = \mathcal{A}_\varphi^i W = \frac{\nu_k^i x_k}{4\pi D} \ln|x| \quad \overset{\mathcal{F}}{\leftrightarrow} \quad \hat{\Phi}_\nu = \hat{\mathcal{A}}_\varphi^i \hat{W} = \frac{i\nu_k \hat{x}_k}{D(\hat{x}_1^2 + \hat{x}_2^2)^2}, \tag{18}$$

the fundamental normal moment:

$$\begin{aligned} M_\nu = \mathcal{A}_\mathrm{m}^i W = &-\frac{1}{8\pi}\,[2(1+\overline{\nu})\ln|x| + \\ &+ (3+\overline{\nu})(\nu^i \cdot \nabla|x|)^2 + (1+3\overline{\nu})(\tau^i \cdot \nabla|x|)^2] \\ \overset{\mathcal{F}}{\leftrightarrow} \quad \hat{M}_\nu = \hat{\mathcal{A}}_\mathrm{m}^i \hat{W} = &\frac{K_{klmn}\nu_k^i \nu_l^i \hat{x}_m \hat{x}_n}{D(\hat{x}_1^2 + \hat{x}_2^2)^2}, \end{aligned} \tag{19}$$

and the fundamental Kirchhoff shear

$$\begin{aligned} Q_\nu = \mathcal{A}_\mathrm{q}^i W = &-\frac{1}{4\pi|x|}\left[2\nu^i \cdot \nabla|x| + \right. \\ &+ (1-\overline{\nu})(\nu^i \cdot \nabla|x| - \kappa|x|)((\nu^i \cdot \nabla|x|)^2 - \\ &\left. - (\tau^i \cdot \nabla|x|)^2)\right] \end{aligned} \tag{20}$$

$$\overset{\mathcal{F}}{\leftrightarrow} \quad \hat{Q}_\nu = \hat{\mathcal{A}}_\mathrm{q}^i \hat{W} = \frac{i\nu_k^i \hat{x}_k \hat{x}_l \hat{x}_l}{(\hat{x}_1^2 + \hat{x}_2^2)^2} + \frac{K_{klmn}\tau_p^i \nu_k^i \tau_l^i \hat{x}_m \hat{x}_n \hat{x}_p}{D(\hat{x}_1^2 + \hat{x}_2^2)^2}.$$

The fundamental corner force is

$$F_\mathrm{c} = \mathcal{A}_\mathrm{c}^i W \quad \overset{\mathcal{F}}{\leftrightarrow} \quad \hat{F}_\mathrm{c} = \hat{\mathcal{A}}_\mathrm{c}^i \hat{W}, \tag{21}$$

The evaluation of the derivatives of the fundamental solution in the Fourier space is very easy compared to that in the original space.

3.2 *The symmetric Galerkin BIE*

The Somigliana identity as a weak form equivalent to (10) is, cf. Frangi and Bonnet (1998); Jahn (1998),

$$\begin{aligned} \kappa(x)w(x) = &\int_\Omega f(y)W(x-y)\,\mathrm{d}\Omega \\ &+ \int_\Gamma q_\nu(y)W(x-y)\,\mathrm{d}\Gamma_y - \int_\Gamma m_\nu(y)\Phi_\nu(x-y)\,\mathrm{d}\Gamma_y \\ &+ \int_\Gamma \varphi_\nu(y)M_\nu(x-y)\,\mathrm{d}\Gamma_y - \int_\Gamma w(y)Q_\nu(x-y)\,\mathrm{d}\Gamma_y \\ &+ \sum_i f_\mathrm{c}(y_\mathrm{c}^i)W(x-y_\mathrm{c}^i) - \sum_i w(y_\mathrm{c}^i)F_\mathrm{c}(x-y_\mathrm{c}^i). \end{aligned} \tag{22}$$

According to Beskos (1991), the free term is $\kappa = \triangle\psi/(2\pi)$, i.e. the percentage of the total angle 2π. The Galerkin version is obtained by additional weighting with test functions ϕ_q^j (q is the dual variable of w). It is in distributional notation ($< .,. >$ denotes the scalar product and $*$ is the symbol for convolution)

$$\begin{aligned} \langle \phi_\mathrm{q}^j, w_\chi \rangle = &\langle \phi_\mathrm{q}^j, f * W \rangle + \langle \phi_\mathrm{q}^j, q_\nu * W \rangle - \langle \phi_\mathrm{q}^j, m_\nu * \Phi_\nu \rangle \\ &= \langle \phi_\mathrm{q}^j, \varphi_\nu * M_\nu \rangle - \langle \phi_\mathrm{q}^j, w * Q_\nu \rangle \\ &+ \sum_i \langle \phi_\mathrm{q}^j, f_\mathrm{c}(y_\mathrm{c}^i)W(x-y_\mathrm{c}^i) \rangle \\ &- \sum_i \langle \phi_\mathrm{q}^j, w(y_\mathrm{c}^i)F_\mathrm{c}(x-y_\mathrm{c}^i) \rangle. \end{aligned} \tag{23}$$

The boundary factor κ is obtained implicitely by $w_\chi = \chi(x)w(x)$ with χ as the cutoff-distribution of the domain, cf. Duddeck (2002),

$$\chi(x) := \begin{cases} 1 & \ldots \quad x \in \Omega \\ \kappa(x) & \ldots \quad x \in \Gamma \\ 0 & \ldots \quad x \notin \overline{\Omega} = \Omega \cup \Gamma. \end{cases} \tag{24}$$

We introduce now the discretizations for all boundary quantities. Because of the high order of the differential operator (order four) we have to respect certain continuity requirements for the trial functions. The approximation of the deflection w must be C^1 at the nodes (the tangential derivative must be continuous) and the trial functions for the slope should be C^0, i.e. continuous. Therefore, the boundary values are approximated by Hermite polynomials, cf. Frangi and Bonnet (1998), $(.)' = \mathrm{d}(.)/\mathrm{d}s = \tau_k^i \partial_k(.)$ is the tangential derivative,

$$w(x) \approx \sum_i \mathrm{w}^i \phi_\mathrm{w}^i(x) + \mathrm{p}^i \phi_\varphi^i(x). \tag{25}$$

The functions $\phi_\mathrm{w}^i, \phi_\varphi^i$ are constructed from the trial functions for the reference element (L_e^i is the length of the i-th element)

$$\phi_{\mathrm{w}}^0 = \begin{cases} x_1^2(3-2x_1) & \text{for the } (i-1)\text{-th element} \\ (x_1-1)^2(1+2x_1) & \text{for the } i\text{-th element} \\ 0 & \text{otherwise} \end{cases}$$

$$\phi_{\varphi}^0 = \begin{cases} x_1^2(x_1-1)L_{\mathrm{e}}^{i-1} & \text{for the } (i-1)\text{-th element} \\ x_1(x_1-1)^2 L_{\mathrm{e}}^i & \text{for the } i\text{-th element} \\ 0 & \text{otherwise.} \end{cases}$$

These functions have the interpolation properties

$$\phi_{\mathrm{w}}^0(x=x^l) = \delta_{kl} \quad \frac{\mathrm{d}}{\mathrm{d}s}\phi_{\mathrm{w}}^0(x=x^l) = 0 \tag{26}$$

$$\phi_{\varphi}^0(x=x^l) = 0 \quad \frac{\mathrm{d}}{\mathrm{d}s}\phi_{\varphi}^0(x=x^l) = \delta_{kl}.$$

The other quantities of the boundary, i.e. m_v and q_v, are approximated by piecewise linear polynomials

$$\phi_{\mathrm{m}}^0 = \phi_{\mathrm{q}}^0 = \begin{cases} x_1 & \text{for the } (i-1)\text{-th element} \\ 1-x_1 & \text{for the } i\text{-th element} \\ 0 & \text{otherwise} \end{cases} \tag{27}$$

These trial functions are in distributional notation (H is the Heaviside-distribution and δ denotes the Dirac-distribution):

$$\phi_{\mathrm{w}}^0 = x_1^2(3-2x_1)\mathrm{H}(x_1)\mathrm{H}(1-x_1)\delta(x_2)$$
$$\overset{\mathcal{F}}{\leftrightarrow} \hat{\phi}_{\mathrm{w}}^0 = \frac{-12+6i\hat{x}_1+i\hat{x}_1^3\mathrm{e}^{-i\hat{x}_1}+6i\hat{x}_1\mathrm{e}^{-i\hat{x}_1}+12\mathrm{e}^{-i\hat{x}_1}}{\hat{x}_1^4};$$

$$\phi_{\mathrm{w}}^0 = (x_1-1)^2(1+2x_1)\mathrm{H}(x_1)\mathrm{H}(1-x_1)\delta(x_2)$$
$$\overset{\mathcal{F}}{\leftrightarrow} \hat{\phi}_{\mathrm{w}}^0 = \frac{12-i\hat{x}_1^3-6i\hat{x}_1-6i\hat{x}_1\mathrm{e}^{-i\hat{x}_1}-12\hat{x}_1\mathrm{e}^{-i\hat{x}_1}}{\hat{x}_1^4}. \tag{28}$$

For the slope trial functions we get

$$\phi_{\varphi}^0 = x_1^2(x_1-1)L_{\mathrm{e}}^{i-1}\mathrm{H}(x_1)\mathrm{H}(1-x_1)\delta(x_2)$$
$$\overset{\mathcal{F}}{\leftrightarrow} \hat{\phi}_{\varphi}^0 = L_{\mathrm{e}}^{i-1}\frac{6+\hat{x}_1^2\mathrm{e}^{-i\hat{x}_1}-4i\hat{x}_1\mathrm{e}^{-i\hat{x}_1}-6\mathrm{e}^{-i\hat{x}_1}-2i\hat{x}_1}{\hat{x}_1^4};$$

$$\phi_{\varphi}^0 = x_1(x_1-1)^2 L_{\mathrm{e}}^j\mathrm{H}(x_1)\mathrm{H}(1-x_1)\delta(x_2)$$
$$\overset{\mathcal{F}}{\leftrightarrow} \hat{\phi}_{\varphi}^0 = L_{\mathrm{e}}^j\frac{6-\hat{x}_1^2-4i\hat{x}_1-2i\hat{x}_1\mathrm{e}^{-i\hat{x}_1}-6\mathrm{e}^{-i\hat{x}_1}}{\hat{x}_1^4}. \tag{29}$$

The transformations of the linear trial functions are

$$\phi_{\mathrm{m,q}}^0 = x_1\mathrm{H}(x_1)\mathrm{H}(1-x_1)\delta(x_2)$$
$$\overset{\mathcal{F}}{\leftrightarrow} \quad \hat{\phi}_{\mathrm{m,q}}^0 = \frac{-1+i\hat{x}_1\mathrm{e}^{-i\hat{x}_1}+\mathrm{e}^{-i\hat{x}_1}}{\hat{x}_1^2} \tag{30}$$

$$\phi_{\mathrm{m,q}}^0 = (1-x_1)\mathrm{H}(x_1)\mathrm{H}(1-x_1)\delta(x_2)$$
$$\overset{\mathcal{F}}{\leftrightarrow} \quad \hat{\phi}_{\mathrm{m,q}}^0 = \frac{1-\mathrm{e}^{i\hat{x}_1}-i\hat{x}_1}{\hat{x}_1^2}.$$

With the exception of the deflection itself, all boundary quantities are dependent on the normal vector, they are discontinuous at corner points. Hence we define two different nodal values at these corners.

The treatment of the corner force f_c and the corner deflection w_c in the Fourier space is enabled by defining particular corner trial functions $\phi_{\mathrm{c}}^i = \delta(x-x_{\mathrm{c}}^i)$

$$f_\mathrm{c} = \mathrm{f}^i\,\delta(x-x_\mathrm{c}^i) \quad \overset{\mathcal{F}}{\leftrightarrow} \quad \hat{f}_\mathrm{c} = \mathrm{f}^i\,\mathrm{e}^{-i<x_\mathrm{c}^i,\hat{x}>}; \tag{31}$$
$$w_\mathrm{c} = \mathrm{w}^i\,\delta(x-x_\mathrm{c}^i) \quad \overset{\mathcal{F}}{\leftrightarrow} \quad \hat{w}_\mathrm{c} = \mathrm{w}^i\,\mathrm{e}^{-i<x_\mathrm{c}^i,\hat{x}>}.$$

The contributions of the corner terms to (23) can be written as

$$\sum_i \left\langle \phi_\mathrm{q}^j, f_\mathrm{c}(y_\mathrm{c}^i)W(x-y_\mathrm{c}^i)\right\rangle = \sum_i \mathrm{f}^i\left\langle \phi_\mathrm{q}^j, \phi_\mathrm{c}^i * W\right\rangle$$
$$\overset{\mathcal{F}}{\leftrightarrow} \quad \frac{1}{(2\pi)^2}\sum_i \mathrm{f}^i\left\langle \hat{\phi}_\mathrm{q}^j(-\hat{x}), \hat{\phi}_\mathrm{c}^i(\hat{x})\hat{W}(\hat{x})\right\rangle. \tag{32}$$

These discretizations result finally in the discretized Galerkin BIE

$$\begin{aligned}\left\langle \phi_\mathrm{q}^j, w_\chi\right\rangle &= \left\langle \phi_\mathrm{q}^j, f_\chi * W\right\rangle + \sum_i^{N_\mathrm{q}} \mathrm{q}^i\left\langle \phi_\mathrm{q}^j, \phi_\mathrm{q}^i * W\right\rangle \\ &- \sum_i^{N_\mathrm{m}} \mathrm{m}^i\left\langle \phi_\mathrm{q}^j, \phi_\mathrm{m}^i * \Phi_\mathrm{v}\right\rangle \\ &+ \sum_i^{N_\varphi} \mathrm{p}^i\left\langle \phi_\mathrm{q}^j, \phi_\varphi^i * M_\mathrm{v}\right\rangle - \sum_i^{N_\mathrm{w}} \mathrm{w}^i\left\langle \phi_\mathrm{q}^j, \phi_\mathrm{w}^i * Q_\mathrm{v}\right\rangle \\ &+ \sum_i^{N_\mathrm{c}} \mathrm{f}^i\left\langle \phi_\mathrm{q}^j, \phi_\mathrm{c}^i * W\right\rangle - \sum_i^{N_\mathrm{c}} \mathrm{w}^i\left\langle \phi_\mathrm{q}^j, \phi_\mathrm{c}^i * F_\mathrm{c}\right\rangle.\end{aligned} \tag{33}$$

with $\left\langle \phi_\mathrm{q}^j, w_\chi\right\rangle = \left\langle \phi_\mathrm{q}^j, \chi p_\mathrm{q}^j\right\rangle$; p_q^j is the polynomial defined for the test function ϕ_q^j. The kernels of this BIE can be highly singular, cf. Frangi and Bonnet (1998) for the regularization. A more formal presentation of this Galerkin BIE is

$$
\begin{aligned}
\left\langle \phi_{\mathrm{q}}^j, w_\chi \right\rangle &= \left\langle \phi_{\mathrm{q}}^j, f_\chi * W \right\rangle + \sum_i^{N_{\mathrm{q}}} \mathrm{q}^i \left\langle \phi_{\mathrm{q}}^j, \phi_{\mathrm{q}}^i * W \right\rangle \\
&- \sum_i^{N_{\mathrm{m}}} \mathrm{m}^i \left\langle \phi_{\mathrm{q}}^j, \phi_{\mathrm{m}}^i * \mathcal{A}_\varphi^i W \right\rangle \\
&+ \sum_i^{N_\varphi} \mathrm{p}^i \left\langle \phi_{\mathrm{q}}^j, \phi_\varphi^i * \mathcal{A}_{\mathrm{m}}^i W \right\rangle \\
&- \sum_i^{N_{\mathrm{w}}} \mathrm{w}^i \left\langle \phi_{\mathrm{q}}^j, \phi_{\mathrm{w}}^i * \mathcal{A}_{\mathrm{q}}^i W \right\rangle \\
&+ \sum_i^{N_{\mathrm{c}}} \mathrm{f}^i \left\langle \phi_{\mathrm{q}}^j, \phi_{\mathrm{c}}^i * W \right\rangle - \sum_i^{N_{\mathrm{c}}} \mathrm{w}^i \left\langle \phi_{\mathrm{q}}^j, \phi_{\mathrm{c}}^i * \mathcal{A}_{\mathrm{c}}^i W \right\rangle ,
\end{aligned} \tag{34}
$$

which finds its Fourier equivalent in (after cancelling the factor $(2\pi)^{-2}$)

$$
\begin{aligned}
\left\langle \hat{\phi}_{\mathrm{q}}^j(-\hat{x}), \hat{w}_\chi \right\rangle &= \left\langle \hat{\phi}_{\mathrm{q}}^j(-\hat{x}), \hat{f}_\chi \hat{W} \right\rangle \\
&+ \sum_i^{N_{\mathrm{q}}} \mathrm{q}^i \left\langle \hat{\phi}_{\mathrm{q}}^j(-\hat{x}), \hat{\phi}_{\mathrm{q}}^i \hat{W} \right\rangle \\
&- \sum_i^{N_{\mathrm{m}}} \mathrm{m}^i \left\langle \hat{\phi}_{\mathrm{q}}^j(-\hat{x}), \hat{\phi}_{\mathrm{m}}^i \hat{\mathcal{A}}_\varphi^i \hat{W} \right\rangle \\
&+ \sum_i^{N_\varphi} \mathrm{p}^i \left\langle \hat{\phi}_{\mathrm{q}}^j(-\hat{x}), \hat{\phi}_\varphi^i \hat{\mathcal{A}}_{\mathrm{m}}^i \hat{W} \right\rangle \\
&- \sum_i^{N_{\mathrm{w}}} \mathrm{w}^i \left\langle \hat{\phi}_{\mathrm{q}}^j(-\hat{x}), \hat{\phi}_{\mathrm{w}}^i \hat{\mathcal{A}}_{\mathrm{q}}^i \hat{W} \right\rangle \\
&+ \sum_i^{N_{\mathrm{c}}} \mathrm{f}^i \left\langle \hat{\phi}_{\mathrm{q}}^j(-\hat{x}), \hat{\phi}_{\mathrm{c}}^i \hat{W} \right\rangle \\
&- \sum_i^{N_{\mathrm{c}}} \mathrm{w}^i \left\langle \hat{\phi}_{\mathrm{q}}^j(-\hat{x}), \hat{\phi}_{\mathrm{c}}^i \hat{\mathcal{A}}_{\mathrm{c}}^i \hat{W} \right\rangle .
\end{aligned} \tag{35}
$$

For a symmetric Galerkin method, additional BIEs are required which will be given in the following for the Fourier space. If needed they can be transferred easily to the original space. The BIE for the normal slope is obtained by applying the adjoint of $\hat{\mathcal{A}}_\varphi^j = -\hat{\mathcal{A}}_\varphi^j$ on (35) and by choosing the dual test function. Here we need a moment test function ϕ_{m}^j. The BIE is then

$$
\begin{aligned}
-\left\langle \hat{\phi}_{\mathrm{m}}^j(-\hat{x}), \hat{\mathcal{A}}_\varphi^j \hat{w}_\chi \right\rangle &= -\left\langle \hat{\phi}_{\mathrm{m}}^j(-\hat{x}), \hat{f}_\chi \hat{\mathcal{A}}_\varphi^j \hat{W} \right\rangle \\
&- \sum_i^{N_{\mathrm{q}}} \mathrm{q}^i \left\langle \hat{\phi}_{\mathrm{m}}^j(-\hat{x}), \hat{\phi}_{\mathrm{q}}^i \hat{\mathcal{A}}_\varphi^j \hat{W} \right\rangle \\
&+ \sum_i^{N_{\mathrm{m}}} \mathrm{m}^i \left\langle \hat{\phi}_{\mathrm{m}}^j(-\hat{x}), \hat{\phi}_{\mathrm{m}}^i \hat{\mathcal{A}}_\varphi^j \hat{\mathcal{A}}_\varphi^i \hat{W} \right\rangle \\
&- \sum_i^{N_\varphi} \mathrm{p}^i \left\langle \hat{\phi}_{\mathrm{m}}^j(-\hat{x}), \hat{\phi}_\varphi^i \hat{\mathcal{A}}_\varphi^j \hat{\mathcal{A}}_{\mathrm{m}}^i \hat{W} \right\rangle \\
&+ \sum_i^{N_{\mathrm{w}}} \mathrm{w}^i \left\langle \hat{\phi}_{\mathrm{m}}^j(-\hat{x}), \hat{\phi}_{\mathrm{w}}^i \hat{\mathcal{A}}_\varphi^j \hat{\mathcal{A}}_{\mathrm{q}}^i \hat{W} \right\rangle \\
&- \sum_i^{N_{\mathrm{c}}} \mathrm{f}^i \left\langle \hat{\phi}_{\mathrm{m}}^j(-\hat{x}), \hat{\phi}_{\mathrm{c}}^i \hat{\mathcal{A}}_\varphi^j \hat{W} \right\rangle \\
&+ \sum_i^{N_{\mathrm{c}}} \mathrm{w}^i \left\langle \hat{\phi}_{\mathrm{m}}^j(-\hat{x}), \hat{\phi}_{\mathrm{c}}^i \hat{\mathcal{A}}_\varphi^j \hat{\mathcal{A}}_{\mathrm{c}}^i \hat{W} \right\rangle .
\end{aligned} \tag{36}
$$

The other BIE are obtained analoguously. By using the operator notation introduced in (16), the following scheme can be established for the total system

$$
\begin{aligned}
\left\langle \hat{\phi}^j(-\hat{x}), \hat{\mathcal{B}} \hat{w}_\chi \right\rangle &= \left\langle \hat{\phi}^j(-\hat{x}), \hat{f}_\chi \hat{\mathcal{B}} \hat{W} \right\rangle \\
&+ \sum_i \mathrm{u}^i \left\langle \hat{\phi}^j(-\hat{x}), \hat{\phi}^i \hat{\mathcal{A}} \hat{W} \right\rangle ,
\end{aligned} \tag{37}
$$

with

$$
\begin{aligned}
\hat{\phi}^j &= \begin{pmatrix} \hat{\phi}_{\mathrm{q}}^j, & \hat{\phi}_{\mathrm{m}}^j, & \hat{\phi}_\varphi^j, & \hat{\phi}_{\mathrm{w}}^j, & \hat{\phi}_{\mathrm{c}}^j, & \hat{\phi}_{\mathrm{c}}^j \end{pmatrix} \\
\hat{\phi}^i &= \begin{pmatrix} \sum \hat{\phi}_{\mathrm{q}}^i, & \sum \hat{\phi}_{\mathrm{m}}^i, & \sum \hat{\phi}_\varphi^i, & \sum \hat{\phi}_{\mathrm{w}}^i, & \sum \hat{\phi}_{\mathrm{c}}^i, & \sum \hat{\phi}_{\mathrm{c}}^i \end{pmatrix} \\
\mathrm{u}^i &= \begin{pmatrix} \mathrm{q}^i, \mathrm{m}^i, \mathrm{p}^i, \mathrm{w}^i, \mathrm{f}^i, \mathrm{w}^i \end{pmatrix} ,
\end{aligned}
$$

the matrix $\hat{\mathcal{A}}$ is equal to:

$$
\begin{bmatrix}
I & -\hat{\mathcal{A}}_\varphi^i & \hat{\mathcal{A}}_{\mathrm{m}}^i & -\hat{\mathcal{A}}_{\mathrm{q}}^i & I & -\hat{\mathcal{A}}_{\mathrm{c}}^i \\
-\hat{\mathcal{A}}_\varphi^j & \hat{\mathcal{A}}_\varphi^j \hat{\mathcal{A}}_\varphi^i & -\hat{\mathcal{A}}_\varphi^j \hat{\mathcal{A}}_{\mathrm{m}}^i & \hat{\mathcal{A}}_\varphi^j \hat{\mathcal{A}}_{\mathrm{q}}^i & -\hat{\mathcal{A}}_\varphi^j & \hat{\mathcal{A}}_\varphi^j \hat{\mathcal{A}}_{\mathrm{c}}^i \\
\hat{\mathcal{A}}_{\mathrm{m}}^j & -\hat{\mathcal{A}}_{\mathrm{m}}^j \hat{\mathcal{A}}_\varphi^i & \hat{\mathcal{A}}_{\mathrm{m}}^j \hat{\mathcal{A}}_{\mathrm{m}}^i & -\hat{\mathcal{A}}_{\mathrm{m}}^j \hat{\mathcal{A}}_{\mathrm{q}}^i & \hat{\mathcal{A}}_{\mathrm{m}}^j & -\hat{\mathcal{A}}_{\mathrm{m}}^j \hat{\mathcal{A}}_{\mathrm{c}}^i \\
-\hat{\mathcal{A}}_{\mathrm{q}}^j & \hat{\mathcal{A}}_{\mathrm{q}}^j \hat{\mathcal{A}}_\varphi^i & -\hat{\mathcal{A}}_{\mathrm{q}}^j \hat{\mathcal{A}}_{\mathrm{m}}^i & \hat{\mathcal{A}}_{\mathrm{q}}^j \hat{\mathcal{A}}_{\mathrm{q}}^i & -\hat{\mathcal{A}}_{\mathrm{q}}^j & \hat{\mathcal{A}}_{\mathrm{q}}^j \hat{\mathcal{A}}_{\mathrm{c}}^i \\
I & -\hat{\mathcal{A}}_\varphi^i & \hat{\mathcal{A}}_{\mathrm{m}}^i & -\hat{\mathcal{A}}_{\mathrm{q}}^i & I & -\hat{\mathcal{A}}_{\mathrm{c}}^i \\
-\hat{\mathcal{A}}_{\mathrm{c}}^j & \hat{\mathcal{A}}_{\mathrm{c}}^j \hat{\mathcal{A}}_\varphi^i & -\hat{\mathcal{A}}_{\mathrm{c}}^j \hat{\mathcal{A}}_{\mathrm{m}}^i & \hat{\mathcal{A}}_{\mathrm{c}}^j \hat{\mathcal{A}}_{\mathrm{q}}^i & -\hat{\mathcal{A}}_{\mathrm{c}}^j & \hat{\mathcal{A}}_{\mathrm{c}}^j \hat{\mathcal{A}}_{\mathrm{c}}^i
\end{bmatrix} ;
$$

while we get for $\hat{\mathcal{B}}$:

$$
\begin{pmatrix} I & -\hat{\mathcal{A}}_\varphi^j & \hat{\mathcal{A}}_{\mathrm{m}}^j & -\hat{\mathcal{A}}_{\mathrm{q}}^j & I & -\hat{\mathcal{A}}_{\mathrm{c}}^j \end{pmatrix} .
$$

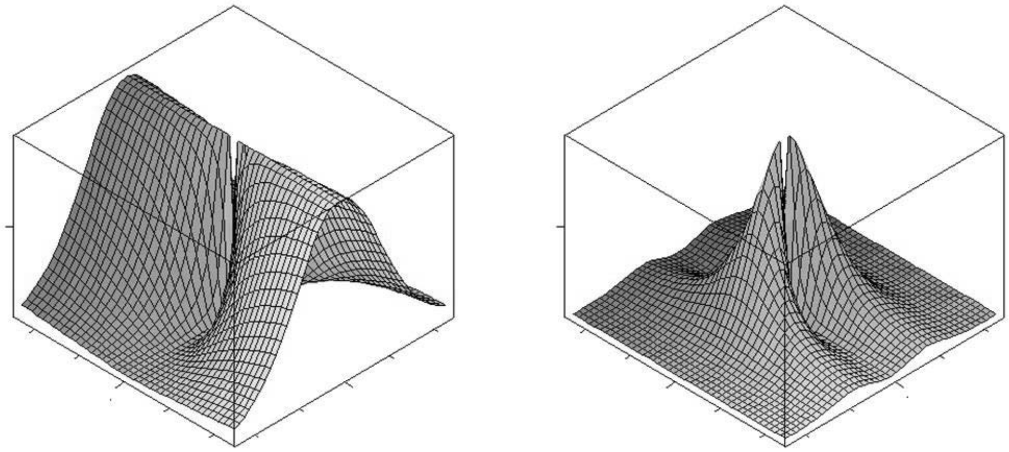

Figure 3 : An example of a non-vanishing integrand leading to a strong singularity (left) and the corresponding term if all terms, especially the free term on the left-hand side of the BIE, are taken into account.

I is the identity operator such that $I\hat{W}=\hat{W}$. The transformed terms in the operator matrix $\hat{\mathcal{A}}$ of (37) are obtained by simple multiplication. The scalar product on the left-hand side of (37) can be used for regularization purposes in the original as well as in the transformed space. The differentiation $\mathcal{B}w_\chi=\mathcal{B}\{w(x)\chi(x)\}$ should be done carefully taking into account that the product of the deflection with the cutoff-distribution of the domain has to be evaluated.

3.3 Regularization

As in the traditional BEM, the integrals of the Fourier BIE can be singular. Due to the fact that the Fourier transform shifts local singularities to global singularities and vice versa, the procedures developed in the standard approach cannot be applied directly (a local singularity is a singularity due to a single point and a global singularity is a singularity due to an integration of a non-vanishing integrand at infinity). As in the traditional approach, weak, strong and hyper singular values are encountered in the Fourier BEM.

By the means of a rigorous distributional discussion, it was shown in Duddeck (2002) that the free terms of the left-hand side of the BIE cancel out with all strong and hyper singular terms on the right-hand side. The non-vanishing integrands have only to be evaluated together. Figure 3 shows an example of a non-vanishing integrand (left) and the corresponding vanishing term if all contributions are summed up (right).

4 Example for isotropic thin plates

4.1 Clamped square plate

For a clamped square plate with $\Omega=[0,1]\times[0,1]$ and with two elements at each side, the system (37) can be reduced because of $w=0,\varphi_\nu=0$ on the total boundary and due to $w_c=0$ for all corner points. The right-hand side of the Fourier BIE is

$$\begin{pmatrix}\hat{\phi}^j_q(-\hat{x})\\ \hat{\phi}^j_m(-\hat{x})\\ \hat{\phi}^j_c(-\hat{x})\end{pmatrix}^T\begin{bmatrix} I & -\hat{\mathcal{A}}^i_\varphi & I\\ -\hat{\mathcal{A}}^j_\varphi & \hat{\mathcal{A}}^j_\varphi\hat{\mathcal{A}}^i_\varphi & -\hat{\mathcal{A}}^j_\varphi\\ I & -\hat{\mathcal{A}}^i_\varphi & I\end{bmatrix}\begin{pmatrix}\sum_i^{N_q}\hat{\phi}^i_q\hat{W}\\ \sum_i^{N_m}\hat{\phi}^i_m\hat{W}\\ \sum_i^{N_c}\hat{\phi}^i_c\hat{W}\end{pmatrix}.$$

For the volume forces f we get

$$\begin{pmatrix}\hat{\phi}^j_q(-\hat{x})\\ \hat{\phi}^j_m(-\hat{x})\\ \hat{\phi}^j_c(-\hat{x})\end{pmatrix}^T\begin{pmatrix} I\\ -\hat{\mathcal{A}}^j_\varphi\\ I\end{pmatrix}\hat{f}_\chi\hat{W}. \tag{38}$$

The free term on the left-hand side is zero because of $w=\varphi_\nu=0$ along the boundary. Linear trial and test functions are chosen for the normal moment and the Kirchhoff shear forces at the boundary. For the corner forces we have the four trial functions

$$\begin{aligned}
\phi^1_c &= \delta(x_1)\delta(x_2) &&\overset{\mathcal{F}}{\leftrightarrow} && \hat{\phi}^2_c=1;\\
\phi^2_c &= \delta(x_1-1)\delta(x_2) &&\overset{\mathcal{F}}{\leftrightarrow} && \hat{\phi}^2_c=\mathrm{e}^{-i\hat{x}_1};\\
\phi^3_c &= \delta(x_1-1)\delta(x_2-1) &&\overset{\mathcal{F}}{\leftrightarrow} && \hat{\phi}^3_c=\mathrm{e}^{-i(\hat{x}_1+\hat{x}_2)};\\
\phi^4_c &= \delta(x_1)\delta(x_2-1) &&\overset{\mathcal{F}}{\leftrightarrow} && \hat{\phi}^4_c=\mathrm{e}^{-i\hat{x}_2}.
\end{aligned}$$

Therefore, we get the following system of equations:

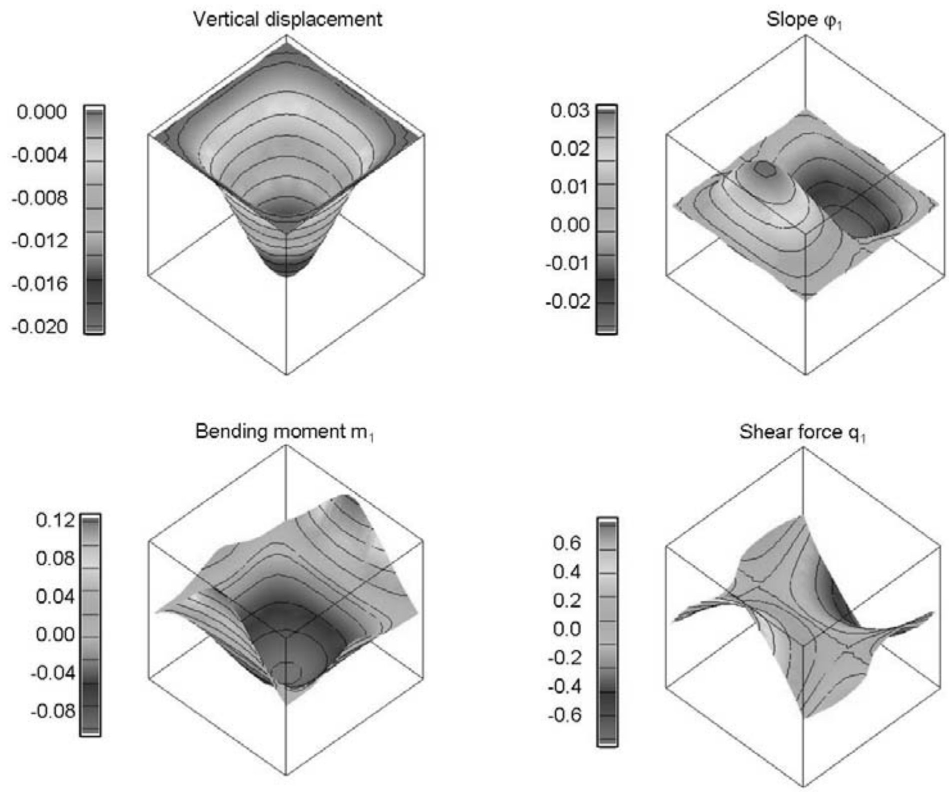

Figure 4 : The clamped thin plate under uniform loading (4×4 elements)

For $j = 1 \dots N_\mathrm{q}$:

$$0 = \left\langle \hat{\phi}_\mathrm{q}^j, \hat{f}_\chi \hat{W} \right\rangle + \sum_i^{N_\mathrm{q}} \mathrm{q}^i \left\langle \hat{\phi}_\mathrm{q}^j, \hat{\phi}_\mathrm{q}^i \hat{W} \right\rangle - \sum_i^{N_\mathrm{m}} \mathrm{m}^i \left\langle \hat{\phi}_\mathrm{q}^j, \hat{\phi}_\mathrm{m}^i \hat{\mathcal{A}}_\varphi^i \hat{W} \right\rangle + \sum_i^{N_\mathrm{c}} \mathrm{f}_c^i \left\langle \hat{\phi}_\mathrm{q}^j, \hat{\phi}_\mathrm{c}^i \hat{W} \right\rangle;$$

for $j = 1 \dots N_\mathrm{m}$:

$$0 = -\left\langle \hat{\phi}_\mathrm{m}^j, \hat{f}_\chi \hat{\mathcal{A}}_\varphi^j \hat{W} \right\rangle - \sum_i^{N_\mathrm{q}} \mathrm{q}^i \left\langle \hat{\phi}_\mathrm{m}^j, \hat{\phi}_\mathrm{q}^i \hat{\mathcal{A}}_\varphi^j \hat{W} \right\rangle + \sum_i^{N_\mathrm{m}} \mathrm{m}^i \left\langle \hat{\phi}_\mathrm{m}^j, \hat{\phi}_\mathrm{m}^i \hat{\mathcal{A}}_\varphi^j \hat{\mathcal{A}}_\varphi^i \hat{W} \right\rangle - \sum_i^{N_\mathrm{c}} \mathrm{f}_c^i \left\langle \hat{\phi}_\mathrm{m}^j, \hat{\phi}_\mathrm{c}^i \hat{\mathcal{A}}_\varphi^j \hat{W} \right\rangle;$$

for $j = 1 \dots N_\mathrm{c}$:

$$0 = -\left\langle \hat{\phi}_\mathrm{c}^j, \hat{f}_\chi \hat{W} \right\rangle - \sum_i^{N_\mathrm{q}} \mathrm{q}^i \left\langle \hat{\phi}_\mathrm{c}^j, \hat{\phi}_\mathrm{q}^i \hat{W} \right\rangle + \sum_i^{N_\mathrm{m}} \mathrm{m}^i \left\langle \hat{\phi}_\mathrm{c}^j, \hat{\phi}_\mathrm{m}^i \hat{\mathcal{A}}_\varphi^i \hat{W} \right\rangle - \sum_i^{N_\mathrm{c}} \mathrm{f}_c^i \left\langle \hat{\phi}_\mathrm{c}^j, \hat{\phi}_\mathrm{c}^i \hat{W} \right\rangle.$$

A uniform loading

$$f = \mathrm{H}(x_1)\mathrm{H}(2-x_1)\mathrm{H}(x_2)\mathrm{H}(2-x_2) \quad \overset{\mathcal{F}}{\leftrightarrow} \quad \hat{f} = -\frac{(1-\mathrm{e}^{-2i\hat{x}_1})(1-\mathrm{e}^{-2i\hat{x}_2})}{\hat{x}_1 \hat{x}_2} \tag{39}$$

was applied. All integrations were computed analytically in the Fourier space. For the moment as well as for the shear force, the values at the corner are theoretical zero which is approximatively fulfilled even by a coarse mesh. The displacements w, the slope $\varphi_1 = \partial_1 w$, the moment m_{11}, and the shear force q_v in the interior are shown in Fig.4.

As a second example, a clamped plate ($a \times a = 2 \times 2$ m) is regarded subjected to a single unit point load $P = 1$ in the center. The results are given in Fig.5. The maximum vertical deflection in the center of the plate directly under the load is $w_\mathrm{max} = 0.02046m$, which can be compared to the analytical value given by Timoshenko and Woinowsky-Krieger (1959) of $w_\mathrm{max,analyt.} = 0.0056Pa^2/D = 0.0224$ (the stiffness is assumed to be $D = 1$). For the coarse grid of 4×4 elements, this result is reasonable.

It is emphasized here, that these results, which repre-

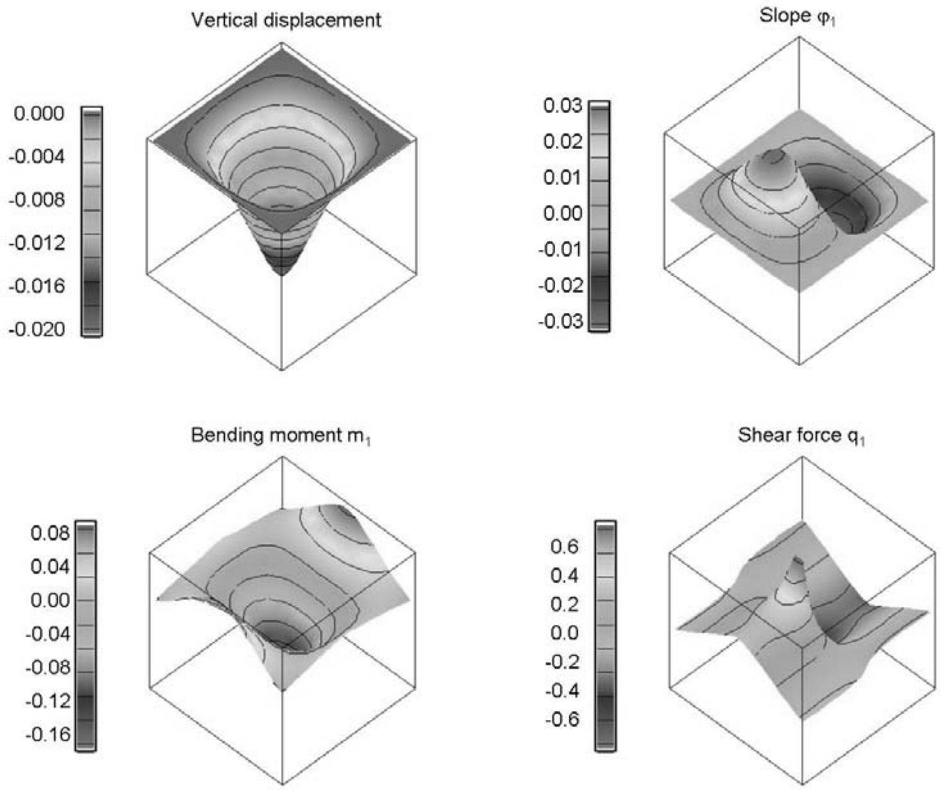

Figure 5 : The clamped thin plate subjected to a single Dirac force at the center (4×4 elements)

sent the real physical values are obtained directly in the Fourier space without any inverse Fourier transform.

4.2 Verification of the example

The results of the example of a clamped square plate are verified by comparing the matrix entries obtained via the newly established Fourier BEM approach with those originating from a standard BEM evaluation to be found in Frangi and Bonnet (1998). An example of a linear test and trial function is in the original and the Fourier space:

$$\phi_q^1 = (1-x_1)\mathrm{H}(x_1)\mathrm{H}(1-x_1)\delta(x_2)$$
$$\overset{\mathcal{F}}{\leftrightarrow}\ \hat{\phi}_q^1 = \frac{1-\mathrm{e}^{-i\hat{x}_1}+i\hat{x}_1}{\hat{x}_1^2} \tag{40}$$

The fundamental solutions used in both approaches are (r_0 is an arbitrary term), cf. Duddeck (2002); Frangi and Bonnet (1998):

$$W(x) = \frac{|x|^2}{8\pi D}(\ln|x| - \ln|r_0|)$$
$$\overset{\mathcal{F}}{\leftrightarrow}\ \hat{W}(\hat{x}) = \frac{1}{D(\hat{x}_1^2+\hat{x}_2^2)^2}. \tag{41}$$

An exemplary analytical integration of one entry of the matrix leads in the single layer case for the original space to (we have chosen $r_0 = \sqrt{\mathrm{e}}$)

$$\begin{aligned}
H^{11} &= \langle \phi_q^1, \phi_q^1 * W \rangle \\
&= \frac{1}{8D\pi}\int_0^1 (1-x_1)\int_0^1 (1-y_1)(x_1-y_1)^2 \times \\
&\quad \times \ln\left(\sqrt{\frac{(x_1-y_1)^2}{\mathrm{e}}}\right)\mathrm{d}y_1\,\mathrm{d}x_1 \\
&= \frac{1}{D\pi}\int_0^1 \left[-\frac{13}{1152} - \frac{13x_1^2}{192} - \frac{x_1^3\ln|x_1-1|}{24}\right. \\
&\quad + \frac{\ln|x_1-1|^2}{192} + \frac{13x_1}{288} + \frac{x_1^4\ln|x_1-1|}{96} + \frac{x_1^3}{96} \\
&\quad - \frac{x_1\ln|x_1-1|^2}{48} + \frac{x_1^2\ln|x_1-1|^2}{32} \\
&\quad \left. + \frac{x_1^3\ln|x_1|}{24} - \frac{x_1^4\ln|x_1|}{96}\right]\mathrm{d}x_1 = -\frac{5}{1152D\pi}.
\end{aligned}$$

The corresponding integral in the Fourier space is computed as

$$
\begin{aligned}
H^{11} &= \frac{1}{(2\pi)^2}\left\langle \hat{\phi}^1_{\mathrm{q}}(-\hat{x}), \hat{\phi}^1_{\mathrm{q}}(\hat{x})\hat{W}(\hat{x})\right\rangle \\
&= \frac{1}{(2\pi)^2}\int_{\mathcal{R}^2} \hat{\phi}^1_{\mathrm{q}}(-\hat{x})\hat{\phi}^1_{\mathrm{q}}(\hat{x})\hat{W}(\hat{x})\,\mathrm{d}\hat{x} \\
&= \frac{1}{D4\pi^2}\int_{\mathcal{R}^2} \frac{2-2\hat{x}_1\sin\hat{x}_1+\hat{x}_1^2-2\cos\hat{x}_1}{\hat{x}_1^4(\hat{x}_1^4+2\hat{x}_1^2\hat{x}_2^2+\hat{x}_2^4)}\,\mathrm{d}\hat{x}_1\,\mathrm{d}\hat{x}_2 \\
&= \frac{1}{2\pi D}\int_{\mathcal{R}^1}\Bigg[\frac{1}{3\hat{x}_2^4}+\frac{5\mathrm{sgn}\hat{x}_2}{2\hat{x}_2^7}+\frac{5\mathrm{e}^{\hat{x}_2}}{4\hat{x}_2^7}-\frac{5\mathrm{e}^{-\hat{x}_2}}{4\hat{x}_2^7} \\
&+\frac{5\mathrm{sgn}\hat{x}_2\mathrm{e}^{\hat{x}_2}}{4\hat{x}_2^6}-\frac{5\mathrm{sgn}\hat{x}_2\mathrm{e}^{-\hat{x}_2}}{4\hat{x}_2^6}-\frac{5\mathrm{sgn}\hat{x}_2\mathrm{e}^{-\hat{x}_2}}{4\hat{x}_2^7} \\
&-\frac{5\mathrm{e}^{-\hat{x}_2}}{4\hat{x}_2^6}-\frac{5\mathrm{e}^{\hat{x}_2}}{4\hat{x}_2^6}+\frac{\mathrm{e}^{\hat{x}_2}}{4\hat{x}_2^5}-\frac{5\mathrm{sgn}\hat{x}_2\mathrm{e}^{\hat{x}_2}}{4\hat{x}_2^7}-\frac{\mathrm{e}^{\hat{x}_2}}{4\hat{x}_2^5} \\
&-\frac{\mathrm{sgn}\hat{x}_2\mathrm{e}^{-\hat{x}_2}}{4\hat{x}_2^5}-\frac{\mathrm{sgn}\hat{x}_2\mathrm{e}^{\hat{x}_2}}{4\hat{x}_2^5}-\frac{3\mathrm{sgn}\hat{x}_2}{4\hat{x}_2^5}\Bigg]\,\mathrm{d}\hat{x}_2 \\
&= -\frac{5}{1152D\pi}.
\end{aligned}
$$

The kernel in the Fourier space is hyper singular and is regularized as discussed in section 3.3, i.e. the strong and hyper singular parts cancel with the free terms on the left-hand side of the BIE. The result shown here is only the part originating from the weak singular and the regular parts. The two results are identical, thus the procedure after establishing the matrices can be taken from the standard BEM approach and it is not discussed here.

5 Generalization and Outlook

5.1 The General Anisotropic Plate

The strain-stress relations for general anisotropic plates are with the flexibilities a_{kl} :

$$
\begin{aligned}
\varepsilon_{11} &= a_{11}\sigma_{11}+a_{12}\sigma_{22}+a_{16}\sigma_{12}; \\
\varepsilon_{22} &= a_{12}\sigma_{11}+a_{22}\sigma_{22}+a_{26}\sigma_{12}; \\
\varepsilon_{12} &= a_{16}/2\sigma_{11}+a_{26}/2\sigma_{22}+a_{66}/2\sigma_{12}. \qquad (42)
\end{aligned}
$$

They are linked to the corresponding rigidities D_{kl} by

$$
\begin{aligned}
D_{11} &= c_1(a_{22}a_{66}-a_{26}^2); \\
D_{22} &= c_1(a_{11}a_{66}-a_{16}^2); \\
D_{12} &= c_1(a_{16}a_{26}-a_{12}a_{66}); \\
D_{66} &= c_1(a_{11}a_{22}-a_{12}^2); \\
D_{16} &= c_1(a_{12}a_{26}-a_{22}a_{16}); \\
D_{26} &= c_1(a_{12}a_{16}-a_{11}a_{26}); \qquad (43)
\end{aligned}
$$

with

$$
\frac{1}{c_1} = \frac{12}{h^3}\ \det\begin{bmatrix} a_{11} & a_{12} & a_{16} \\ a_{12} & a_{22} & a_{26} \\ a_{16} & a_{26} & a_{66}\end{bmatrix}.
$$

The bending and twisting moments are

$$
\begin{aligned}
m_{11} &= -(D_{11}\partial_{11}+D_{12}\partial_{22}+2D_{16}\partial_{12})w; \\
m_{12} &= -(D_{16}\partial_{11}+D_{26}\partial_{22}+2D_{66}\partial_{12})w; \\
m_{22} &= -(D_{12}\partial_{11}+D_{22}\partial_{22}+2D_{26}\partial_{12})w. \qquad (44)
\end{aligned}
$$

and the shear forces are

$$
\begin{aligned}
q_1 &= -[D_{11}\partial_{111}+3D_{16}\partial_{112}+(D_{12}+2D_{66})\partial_{122}+D_{26}\partial_{222}]; \\
q_2 &= -[D_{16}\partial_{111}+(D_{12}+2D_{66})\partial_{112}+3D_{26}\partial_{122}+D_{22}\partial_{222}].
\end{aligned}
$$

The differential operator for general anisotropic thin plates is, cf. Albuquerque, Sollero, Venturini, and Aliabadi (2006); Lekhnitskii (1968),

$$
\begin{aligned}
\mathcal{P}(\partial) &= [D_{11}\partial_{1111}+4D_{16}\partial_{1112} \\
&\quad +2(D_{12}+2D_{66})\partial_{1122}+4D_{26}\partial_{1222}+D_{22}\partial_{2222}] \\
&\overset{\mathcal{F}}{\leftrightarrow} \hat{\mathcal{P}}(\hat{x}) = [D_{11}\hat{x}_1^4+4D_{16}\hat{x}_1^3\hat{x}_2 \\
&\quad +2(D_{12}+2D_{66})\hat{x}_1^2\hat{x}_2^2+4D_{26}\hat{x}_1\hat{x}_2^3+D_{22}\hat{x}_2^4].
\end{aligned}
$$

Which leads to the Fourier fundamental solution

$$
\begin{aligned}
\hat{W} &= [D_{11}\hat{x}_1^4+4D_{16}\hat{x}_1^3\hat{x}_2 \\
&\quad +2(D_{12}+2D_{66})\hat{x}_1^2\hat{x}_2^2+4D_{26}\hat{x}_1\hat{x}_2^3+D_{22}\hat{x}_2^4]^{-1}. \qquad (45)
\end{aligned}
$$

The relevant boundary operators are in the Fourier space

$$
\hat{\mathcal{A}}^i_{\varphi} = i\nu^i_k\hat{x}_k;
$$

$$
\begin{aligned}
\hat{\mathcal{A}}^i_{\mathrm{m}} &= (\nu^i_1\nu^i_1D_{11}+2\nu^i_1\nu^i_2D_{16}+\nu^i_2\nu^i_2D_{12})\,\hat{x}_1^2 \\
&\quad +2(\nu^i_1\nu^i_1D_{16}+2\nu^i_1\nu^i_2D_{66}+\nu^i_2\nu^i_2D_{26})\hat{x}_1\hat{x}_2 \\
&\quad +(\nu^i_1\nu^i_1D_{12}+2\nu^i_1\nu^i_2D_{26}+\nu^i_2\nu^i_2D_{22})\,\hat{x}_2^2;
\end{aligned}
$$

$$
\begin{aligned}
\hat{\mathcal{A}}^i_{\mathrm{q}} &= i[\nu^i_1D_{11}(1+\nu^i_2\nu^i_2)-\nu^i_1\nu^i_2\nu^i_2D_{12} \\
&\quad +2\nu^i_2\nu^i_2\nu^i_2D_{16}]\hat{x}_1^3+i[-\nu^i_1\nu^i_1\nu^i_2D_{11} \\
&\quad +\nu^i_2(2-\nu^i_2\nu^i_2)D_{12}+4\nu^i_1D_{16}-2\nu^i_1\nu^i_2\nu^i_2D_{26} \\
&\quad +4\nu^i_2\nu^i_2\nu^i_2D_{66}]\hat{x}_1^2\hat{x}_2+i[-\nu^i_2\nu^i_2\nu^i_1D_{22} \\
&\quad +\nu^i_1(2-\nu^i_1\nu^i_1)D_{12}+4\nu^i_2D_{26}-2\nu^i_2\nu^i_1\nu^i_1D_{16} \\
&\quad +4\nu^i_1\nu^i_1\nu^i_1D_{66}]\hat{x}_2^2\hat{x}_1+i[\nu^i_2D_{22}(1+\nu^i_1\nu^i_1) \\
&\quad -\nu^i_2\nu^i_1\nu^i_1D_{12}+2\nu^i_1\nu^i_1\nu^i_1D_{26}]\hat{x}_2^3;
\end{aligned}
$$

$$\mathcal{A}_c^i = (\nu_1^{+i}\nu_2^{+i} - \nu_1^{-i}\nu_2^{-i})[(D_{11}-D_{12})\hat{x}_1^2 + (D_{12}-D_{22})\hat{x}_2^2] + (\nu_1^{+i}\nu_1^{+i} - \nu_2^{+i}\nu_2^{+i} - \nu_1^{-i}\nu_1^{-i} + \nu_2^{-i}\nu_2^{-i})[D_{16}\hat{x}_1^2 + D_{26}\hat{x}_2^2] - [2(\nu_1^{+i}\nu_2^{+i} - \nu_1^{-i}\nu_2^{-i})(D_{16}-D_{26}) + 4\nu_2^{+i}\nu_2^{+i}\nu_2^{-i}\nu_2^{-i}D_{66}]\hat{x}_1\hat{x}_2.$$

Once more, the Galerkin BIE can be obtained by inserting these operators and the anisotropic fundamental solution into the Galerkin BIE of the isotropic case.

6 Summary

A Fourier transformed approach to solve BEM problems was presented in this paper. It is based on the knowledge of only the Fourier transformed fundamental solution and not the fundamental solutions itself. This is advantageous in cases, where the differential operator is rather complex and no analytical expression for the fundamental solution has been established. All equations are defined and solved in the Fourier transformed space. This leads to the same matrix entries as in the usual BEM approach. As application, a Galerkin BEM for isotropic thin plates was discussed. Further work might transfer this method to orthotropic or general anisotropic plates with or without Winkler foundations. Thick plates can as well be tackled as geometrical and physical non-linear problems, cf. Duddeck (2002). Static and dynamic cases may be included. The range of studies found in the literature, e.g. the recent works of Baiz and Aliabadi (2006); Moraru (2006); Purbolaksono and Aliabadi (2005); Wen, Aliabadi, and Young (2002), can be enlarged.

The rigorous distributional approach used for the Fourier BEM shows clearly that all strong and hyper singular entries cancel. Thus only weak singular matrix entries have to be computed. To the authors opinion, this is general the case in all such-like problems.

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