Parallel iterative procedures for a computational electromagnetic modeling based on a nonconforming mixed finite element method

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Abstract: We present nonoverlapping domain decomposition methods for the approximation of both electromagnetic fields in a three-dimensional bounded domain satisfying absorbing boundary conditions. *A Seidel-type domain decomposition iterative method* is introduced based on a hybridization of a *nonconforming mixed finite element method.* Convergence results for the numerical procedure are proved by introducing a suitable pseudo-energy. The spectral radius of the iterative procedure is estimated and a method for choosing an optimal matching parameter is given. A red-black Seidel-type method which is readily parallelizable is also introduced and analyzed. Numerical experiments confirm that the presented algorithms are faster than the conventional *Jacobi-type ones*.

keyword: Nonoverlapping domain decomposition, Maxwell's equations, Transmission condition, Parallel iterative algorithm

1 Introduction

Computational electromagnetic modeling has many applications in engineering and industry and thus there have been increasing attention from both scientists and engineers: see, for instance, Ben Belgacem, Buffa, and Maday (2001); Boffi, Demkowicz, and Costabel (2003); Bouillault, Buffa, Maday, and Rapetti (2003); Gopalakrishnan, Pasciak, and Demkowicz (2004); Hiptmair and Schwab (2002); Hiptmair (2002); Hu and Zou (2003); Jose, Kanapady, and Tamma (2004); Monk (2003); Reitich and Tamma (2004); Toselli and Vasseur (2005); Volakis, Sertel, Jorgensen, and Kindt (2004), and the references therein.

In this work we are particularly interested in the numer-

ical approximation of scattered electromagnetic fields $E = E(x,\omega)$ and $H = H(x,\omega)$ which satisfy the time-harmonic Maxwell's equations at a given angular frequency ω. Absorbing boundary conditions are used to truncate the infinite, or at least huge, space domains into domains of reasonably smaller size (see Bendali and Halpern (1988); Berenger (1994, 1996); Feng (1999); Hanouzet and Sesqès (1990); Joly and Mercier (1989); Mur (1981); Sesquès (1990); Sheen (1997) and the references therein). However, the numerical treatment of three-dimensional time-harmonic Maxwell's equations still requires an efficient iterative procedure rather than a direct Gaussian elimination one for solving resulting matrix problems. Recently, several domain decomposition methods have been proposed to solve this problem; for instance, see Alonso and Valli (1997, 1999); Ben Belgacem, Buffa, and Maday (2001); Bouillault, Buffa, Maday, and Rapetti (2003); Collino, Delbue, Joly, and Piacentini (1997); Després (1991); Després, Joly, and Roberts (1992); Jr., Santos, and Sheen (2001); Hu and Zou (2003); Santos (1998); Santos and Sheen (1998); Toselli and Vasseur (2005).

In particular, following the idea in Lions (1988, 1990), Després (1991) and Després, Joly, and Roberts (1992) proposed an iterative method using nonoverlapping domain decompositions and a pseudo-energy with which convergence results were shown. The convergence results obtained by Després were weak in the sense no actual convergence rates are given. Later, Douglas Jr, Paes Leme, Roberts, and Wang (1993) applied this idea to second-order elliptic problems, and obtained, using mixed finite elements, an estimate for the spectral radius of the iterative operator which guarantees the actual rate of convergence of the scheme. By utilizing similar structures to mixed finite element spaces, such estimates for spectral radii have been obtained, based on hybridization of nonconforming finite elements, for elliptic, Helmholtz, and viscoelastic problems)Douglas Jr., Santos, Sheen, and Ye (1999); Ha, Santos, and Sheen (2002);

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Jr., Santos, and Sheen (2001).

Recently a nonconforming mixed finite element method has been proposed in Douglas Jr., Santos, and Sheen (2000) to compute E and H simultaneously. Error estimates using the were also given. In that paper, Jacobi-type domain decomposition iterative procedure based on hybridization has been analyzed with an actual rate convergence. The object of our current paper is to study *Seidel-type domain decomposition iterative procedures* for calculating both E and H satisfying the full Maxwell's equations, based on nonconforming mixed finite elements proposed in Douglas Jr., Santos, and Sheen (2000). In the domain decomposition methods the convergence speed depends on a choice of matching parameter β between the tangential electric and magnetic fields that appear in the interface condition on the common boundaries of domains Ω_j and Ω_k:

$$(\mathbf{v}_j \times H_j + \beta_{jk}\pi_\tau E_j) = -\mathbf{v}_k \times H_k + \beta_{jk}\pi_\tau E_k \quad \text{on} \quad \Gamma_{jk} \forall k.$$

In this paper we suggest a method of choice of matching parameter β. Since Seidel-type iterative method is not parallelizable directly, we employ red-black type iterative algorithm in numerical simulations.

Our approach is similar to the conforming version of domain decomposition scheme introduced in Collino and Joly (2000); Collino, Delbue, Joly, and Piacentini (1997), in the sense that the pseudo-energies calculated in our work are to be calculated on both sides of interfaces while those in Collino and Joly (2000); Collino, Delbue, Joly, and Piacentini (1997) are to be calculated on single sides of interfaces. We also give a simple proof of convergence of the iterative procedure which depends only on the resulting weak formulation of problems, and we will show that the speed of Seidel-type procedures is about twice of that of Jacobi-type ones. However, in Collino, Delbue, Joly, and Piacentini (1997) only one field, say E, of the two fields E and H is calculated in the second-order formulation which requires the calculation of $\mathbf{v} \times (\nabla \times E)$ at interfaces to update; indeed, in Collino, Delbue, Joly, and Piacentini (1997) the problem is reformulated in the mixed variational form to compute E and $\mathbf{v} \times (\nabla \times E)$ in subdomains and their boundaries. We also suggest a method to find iteratively an optimal choice of the matching parameter between the tangential components of E and H on each subdomain interface. The use of nonlocal boundary conditions, their higher-order approximations, or perfectly matched layers and relaxation

as in Berenger (1994, 1996); Collino, Delbue, Joly, and Piacentini (1997) will be a subject of interest to be developed.

In the formulation of finite element methods for Maxwell's equations with absorbing boundary conditions, and domain decomposition methods for Maxwell's equations, the tangential traces of $\mathbf{H}(\mathbf{curl};\Omega)$ are involved in many places. Based on proper understanding the tangential traces of $\mathbf{H}(\mathbf{curl};\Omega)$, the integration by parts formula for functions in $\mathbf{H}(\mathbf{curl};\Omega)$ in the sense of Sobolev spaces has been studied intensively in recent literatures Buffa, Costabel, and Sheen (2002) (see also Buffa and P. Ciarlet Jr. (2001); Buffa, Costabel, and Sheen (2002); Buffa (2001).) We will briefly survey these results in Section 2.

The organization of the paper is as follows. In Section 2, Maxwell's equations are described with absorbing boundary conditions, and then a weak formulation is given. In the next section the nonconforming mixed finite element method and its hybridization are given. Then in Section 4 a *Seidel-type iterative scheme* is proposed, and a suitable pseudo-energy is defined with which convergence and an estimate for the spectral radius of the iterative procedure are shown. A suggestion is made for an optimal choice of matching parameter between the tangential components of E and H for the interface condition is given. Also a *red-black Seidel type procedure* is introduced which is readily parallelizable. Finally in Section 5 some results from numerical experiments are presented to compare the analyses given in the previous sections.

2 The time-harmonic Maxwell's equations

2.1 *The model problem*

Let E and H denote the electric and magnetic fields for a given angular frequency ω. The time-harmonic Maxwell's equations are given by

$$(i\omega\varepsilon + \sigma)E - \nabla \times H = F, \tag{1a}$$

$$i\omega\mu H + \nabla \times E = G, \tag{1b}$$

where ε, μ, and σ denote the electric permittivity, magnetic permeability, and conductivity, respectively, which satisfy the following bound, for any $\xi \in \mathbf{C}^3$,

$$0 < \varepsilon_* |\xi|^2 \leq \left| \sum_{j,k} \varepsilon_{jk} \xi_j \overline{\xi_k} \right| \leq \varepsilon^* |\xi|^2, \tag{2a}$$

$$0 \le \sigma_* |\xi|^2 \le \left| \sum_{j,k} \sigma_{jk} \xi_j \overline{\xi_k} \right| \le \sigma^* |\xi|^2, \tag{2b}$$

$$0 < \mu_* |\xi|^2 \le \left| \sum_{j,k} \mu_{jk} \xi_j \overline{\xi_k} \right| \le \mu^* |\xi|^2. \tag{2c}$$

The free space equations are reduced to a truncated compact domain, so that a practical computational procedure can be defined, with suitable absorbing boundary conditions imposed on the truncated boundary. Let Ω be a unit cube and $\Gamma = \partial \Omega$ be its boundary, and then we will consider the following boundary value problem

$$(i\omega\varepsilon + \sigma)E - \boldsymbol{\nabla} \times H = F \quad \text{in} \quad \Omega, \tag{3a}$$

$$i\omega\mu H + \boldsymbol{\nabla} \times E = G \quad \text{in} \quad \Omega, \tag{3b}$$

$$\alpha\pi_\tau E + \mathbf{v} \times H = 0 \quad \text{on} \quad \Gamma, \tag{3c}$$

where $\mathbf{v}$ denote the unit outer normal to Γ, $\pi_\tau(\varphi) = \varphi - \mathbf{v}(\mathbf{v} \cdot \varphi) = -\mathbf{v} \times (\mathbf{v} \times \varphi)$ is the projection of the trace of φ on Γ. Assume that, for $\alpha = \alpha_R - i\alpha_I$, there exist $\alpha_{R_*}, \alpha_{R^*}, \alpha_{I_*}$, and α_{I^*}, with $0 < \alpha_{R_*} \le \alpha_{R^*}$ and $0 \le \alpha_{I_*} \le \alpha_{I^*}$ such that

$$0 < \alpha_{R_*} |\xi|^2 \le \left| \sum_{j,k} \alpha_{R_{jk}} \xi_j \overline{\xi_k} \right| \le \alpha_{R^*} |\xi|^2, \tag{4a}$$

$$0 \le \alpha_{I_*} |\xi|^2 \le \left| \sum_{j,k} \alpha_{I_{jk}} \xi_j \overline{\xi_k} \right| \le \alpha_{I^*} |\xi|^2. \tag{4b}$$

Condition (3c) is a general form of absorbing boundary conditions such that electromagnetic fields arriving at Γ at certain incident angles do not reflect Sheen (1997). For technical reasons to follow, we assume that α is a complex-valued scalar function which is Lipschitz-continuous on Γ.

2.2 Function spaces and preliminaries

For an open set Ω and a real number r, let $(H^r(\Omega), \|\cdot\|_{r,\Omega})$ indicate the usual complex-valued Sobolev space and its norm. In particular, $(H^0(\Omega), \|\cdot\|_{0,\Omega})$ denotes the usual $L^2(\Omega)$-space and its norm, $(L^2(\Omega), \|\cdot\|_{0,\Omega})$, with the associated inner product

$$(\varphi, \psi) = \int_\Omega \varphi \overline{\psi} d\mathbf{x}$$

Also, for a part γ of the boundary $\partial \Omega$ of Ω,

$$\langle \varphi, \psi \rangle_\gamma = \int_\gamma \varphi \overline{\psi} d\Omega$$

will mean the inner product on $L^2(\gamma)$ with associated norm $|\cdot|_{0,\gamma}$. The following spaces are standard:

$$\mathbf{H}(\mathbf{curl}; \Omega) = \{\varphi \in [L^2(\Omega)]^3 : \boldsymbol{\nabla} \times \varphi \in [L^2(\Omega)]^3\},$$

$$\mathbf{H}(\mathrm{div}; \Omega) = \{\varphi \in [L^2(\Omega)]^3 : \boldsymbol{\nabla} \cdot \varphi \in [L^2(\Omega)]\},$$

equipped with the natural norms

$$\|\varphi\|_{\mathbf{H}(\mathbf{curl}; \Omega)} = (\|\varphi\|_{0,\Omega}^2 + \|\boldsymbol{\nabla} \times \varphi\|_{0,\Omega}^2)^{1/2},$$

$$\|\varphi\|_{\mathbf{H}(\mathrm{div}; \Omega)} = (\|\varphi\|_{0,\Omega}^2 + \|\boldsymbol{\nabla} \cdot \varphi\|_{0,\Omega}^2)^{1/2}.$$

In the following analysis proper meaning of the traces of $\mathbf{H}(\mathbf{curl}; \Omega)$ and integration by parts with functions in the space $\mathbf{H}(\mathbf{curl}; \Omega)$ are very important. For this, we will give a brief review of the characterization of the space of tangential traces and tangential components for vector fields in $\mathbf{H}(\mathbf{curl}; \Omega)$ following Buffa, Costabel, and Sheen (2002). Set

$$\mathbf{V} = [H^{1/2}(\Gamma)]^3; \quad \mathbf{V}' = [H^{-1/2}(\Gamma)]^3,$$

$$H_*^{-s}(\Gamma) = \{v \in H^{-s}(\Gamma) | \langle v, 1 \rangle_{s,\Gamma} = 0\}, s \in [0,1],$$

$$\mathbf{L}_t^2(\Gamma) = \{\mathbf{v} \in [L^2(\Gamma)]^3 | \mathbf{v} \cdot \mathbf{v} = 0 \text{ on } \Gamma\},$$

where the space $\mathbf{L}_t^2(G)$ is identified with the space of fields belonging to the tangent bundle $T\Gamma$ of Γ almost everywhere. Also, let $H^{3/2}(\Gamma)$ be the trace space of $H^2(\Omega)$ endowed with the norm

$$\|\lambda\|_{3/2,\Gamma} = \inf_{\substack{v \in H^2(\Omega) \\ v|_G = \lambda}} \{\|v\|_{2,\Omega}\}.$$

Then the space $H^{-3/2}(\Gamma)$ is defined as the dual space of $H^{3/2}(\Gamma)$ with the pivot space $L^2(\Gamma)$.

Definition 2.1. *The "tangential component trace" mapping $\pi_\tau : \mathcal{D}(\overline{\Omega})^3 \to \mathbf{L}_t^2(\Gamma)$ and the "tangential trace" mapping $\gamma_\tau : \mathcal{D}(\overline{\Omega})^3 \to \mathbf{L}_t^2(\Gamma)$ are defined as $\mathbf{v} \mapsto -\mathbf{v} \times (\mathbf{v} \times \mathbf{v})$ and $\mathbf{v} \mapsto -\mathbf{v} \times \mathbf{v}$, respectively.*

Denoting by γ the standard trace operator on the product space $[H^1(\Omega)]^3 \to \mathbf{V}$ defined by $\gamma(\mathbf{v}) = \mathbf{v}|_\Gamma$ and by γ^{-1} its right inverse, we shall abuse the notations π_τ and γ_τ for the composite operators $\pi_\tau \circ \gamma^{-1}$ and $\gamma_\tau \circ \gamma^{-1}$, respectively. Due to the density of $[D(\overline{\Omega})]^3$ in $[L^2(\Gamma)]^3$, the operators π_τ and γ_τ can be extended to $[L^2(\Gamma)]^3$ linearly and continuously. Now set

$$\mathbf{V}_\gamma = \gamma_\tau(\mathbf{V}); \quad \mathbf{V}_\pi = \pi_\tau(\mathbf{V}),$$

which are Hilbert spaces endowed with the norms

$$\|\lambda\|_{\mathbf{V}_\gamma} = \inf_{\substack{v \in \mathbf{V} \\ \gamma_\tau(v)=\lambda}} \{\|v\|_\mathbf{V}\}; \quad \|\lambda\|_{\mathbf{V}_\pi} = \inf_{\substack{v \in \mathbf{V} \\ \pi_\tau(v)=\lambda}} \{\|v\|_\mathbf{V}\}.$$

Let $i_\pi : \mathbf{L}^2_t(\Gamma) \to \mathbf{L}^2(\Gamma)$ and $i_\gamma : \mathbf{L}^2_t(\Gamma) \to \mathbf{L}^2(\Gamma)$ be the adjoint operators of π_τ and γ_τ respectively. These operators are the identifications of tangent fields with 3D vector fields. Thanks to the Lipschitz assumption, a local system of orthonormal coordinates $(\tau_1, \tau_2, \mathbf{v})$ can be defined at almost every $\mathbf{x} \in \Gamma$. Here, τ_1 and τ_2 are two orthonormal vectors belonging to the tangent plane for almost every $\mathbf{x} \in \Gamma$, while $\mathbf{v}$ is the outer normal to Ω. Of course, the vectors τ_1 and τ_2 can also be considered as "tangent fields"(sections of the tangent bundle). The operators i_π and i_γ can be extended as isomorphisms in the following way:

$$i_\pi : \mathbf{V}'_\pi \to \left(\ker(\pi_\tau) \cap \mathbf{V} \right)^0, \tag{5a}$$

$$i_\gamma : \mathbf{V}'_\gamma \to \left(\ker(\gamma_\tau) \cap \mathbf{V} \right)^0, \tag{5b}$$

where $\cdot^0$ denotes the polar set (or, the annihilator).

Tangential gradient and curl operators are defined as usual using a localization argument:

$$\boldsymbol{\nabla}_\Gamma : H^1(\Gamma) \to \mathbf{L}^2_t(\Gamma); \quad \mathbf{curl}_\Gamma : H^1(\Gamma) \to \mathbf{L}^2_t(\Gamma),$$

and the corresponding adjoint operators, which are linear and continuous, are defined

$$\operatorname{div}_\Gamma : \mathbf{L}^2_t(\Gamma) \to H^{-1}_*(\Gamma); \quad \operatorname{curl}_\Gamma : \mathbf{L}^2_t(\Gamma) \to H^{-1}_*(\Gamma),$$

respectively. Moreover, the following operators are continuous:

$$\boldsymbol{\nabla}_\Gamma : H^{3/2}(\Gamma) \to \mathbf{V}_\pi; \quad \boldsymbol{\nabla}_\Gamma : H^{1/2}(\Gamma) \to \mathbf{V}'_\gamma;$$

$$\mathbf{curl}_\Gamma : H^{3/2}(\Gamma) \to \mathbf{V}_\gamma; \quad \mathbf{curl}_\Gamma : H^{1/2}(\Gamma) \to \mathbf{V}'_\pi;$$

satisfying

$$\|\lambda\|_{H^{1/2}(\Gamma)/\mathbf{C}} \le C\|\boldsymbol{\nabla}\lambda\|_{\mathbf{V}'_\gamma}; \quad \|\lambda\|_{H^{1/2}(\Gamma)/\mathbf{C}} \le C\|\boldsymbol{\nabla}\lambda\|_{\mathbf{V}'_\pi}.$$

Therefore their adjoint operators $\operatorname{div}_\Gamma : \mathbf{V}_\gamma \to H^{-1/2}_*(\Gamma)$ and $\mathbf{curl}_\Gamma : \mathbf{V}_\pi \to H^{-1/2}_*(\Gamma)$ are continuous and surjective. Based on these tangential operators, the Laplace-Beltrami operator $\Delta_\Gamma : H^1(\Gamma) \to H^{-1}_*(\Gamma)$ is defined by $\Delta_\Gamma v = \operatorname{div}_\Gamma \boldsymbol{\nabla}_\Gamma v$ for any $v \in H^1(\Gamma)$.

We are now ready to introduce the traces of $\mathbf{H}(\mathbf{curl}; \Omega)$. Set

$$\mathbf{H}^{-1/2}(\operatorname{div}_\Gamma; \Gamma) = \{\lambda \in \mathbf{V}'_\pi | \operatorname{div}_\Gamma(\lambda) \in H^{-1/2}(\Gamma)\},$$

$$\mathbf{H}^{-1/2}(\mathbf{curl}_\Gamma; \Gamma) = \{\lambda \in \mathbf{V}'_\gamma | \mathbf{curl}_\Gamma(\lambda) \in H^{-1/2}(\Gamma)\}$$

with the graph norms

$$\|\mathbf{v}\|_{\mathbf{H}^{-1/2}(\operatorname{div}_\Gamma;\Gamma)} = \sqrt{\|\mathbf{v}\|^2_{\mathbf{V}'_\pi} + \|\operatorname{div}_\Gamma(\mathbf{v})\|^2_{-1/2,\Gamma}},$$

$$\|\mathbf{v}\|_{\mathbf{H}^{-1/2}(\mathbf{curl}_\Gamma;\Gamma)} = \sqrt{\|\mathbf{v}\|^2_{\mathbf{V}'_\gamma} + \|\mathbf{curl}_\Gamma(\mathbf{v})\|^2_{-1/2,\Gamma}}.$$

Then the following theorem holds:

Theorem 2.1. *The operators* $\gamma_\tau : \mathbf{H}(\mathbf{curl}; \Omega) \to \mathbf{H}^{-1/2}(\operatorname{div}_\Gamma; \Gamma)$ *and* $\pi_\tau : \mathbf{H}(\mathbf{curl}; \Omega) \to \mathbf{H}^{-1/2}(\mathbf{curl}_\Gamma; \Gamma)$ *are linear, continuous, and surjective.*

The proof of the surjectivity, on the other hand, is based on the proof given by Tartar in Tartar (1997). Let

$$T := \{\xi \in \mathbf{V}' \mid \exists \eta \in H^{-1/2}(\Gamma) : \forall \phi \in H^2(\Omega) :$$
$$\mathbf{v}'\langle \xi, \gamma(\boldsymbol{\nabla}\phi)\rangle \rangle_\mathbf{V} = \langle \eta, \phi \rangle_{1/2,\Gamma}\}. \tag{6}$$

In Tartar (1997), the tangential trace operator is defined as "$\gamma_\tau : \mathbf{H}(\mathbf{curl}; \Omega) \to T, \mathbf{v} \mapsto \mathbf{v} \times \mathbf{v}$" and it is proven to be surjective by a localization argument. Here, our setting is different: the ranges of the operators π_τ and γ_τ defined above are Hilbert spaces of tangent fields. It is shown that the mapping i_π defined in (5) is indeed an isomorphism between T and $\mathbf{H}^{-1/2}(\operatorname{div}_\Gamma; \Gamma)$, *i.e.*,

$$i_\pi\left(\mathbf{H}^{-1/2}(\operatorname{div}_\Gamma; \Gamma)\right) \equiv T.$$

Set

$$\mathcal{H}(\Gamma) := \{p \in H^1(\Gamma)/\mathbf{R} \mid \Delta_\Gamma p \in H^{-1/2}_\star(\Gamma)\}.$$

The trace spaces of $\mathbf{H}(\mathbf{curl}; \Omega)$ have the following Hodge-type decomposition results:

Theorem 2.2.

$$\mathbf{H}^{-1/2}(\operatorname{div}_\Gamma; \Gamma) = \boldsymbol{\nabla}_\Gamma(\mathcal{H}(\Gamma)) \oplus \mathbf{curl}_\Gamma(H^{1/2}(\Gamma)),$$

$$\mathbf{H}^{-1/2}(\mathbf{curl}_\Gamma; \Gamma) = \mathbf{curl}_\Gamma(\mathcal{H}(\Gamma)) \oplus \boldsymbol{\nabla}_\Gamma(H^{1/2}(\Gamma)).$$

Based on the above decomposition results, a duality can be defined between $\mathbf{H}^{-1/2}(\operatorname{div}_\Gamma; \Gamma)$ and $\mathbf{H}^{-1/2}(\mathbf{curl}_\Gamma; \Gamma)$ with the pivot space $\mathbf{L}^2_t(\Gamma)$.

Theorem 2.3. *Let* $\mathbf{u} \in \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma;\Gamma)$ *and* $\mathbf{v} \in \mathbf{H}^{-1/2}(\mathbf{curl}_\Gamma;\Gamma)$ *be decomposed as:* $\mathbf{u} = \boldsymbol{\nabla}_\Gamma \alpha_u + \mathbf{curl}_\Gamma \beta_\mathbf{u}, \mathbf{v} = \boldsymbol{\nabla}_\Gamma \beta_\mathbf{v} + \mathbf{curl}_\Gamma \alpha_\mathbf{v}$ *with* $\beta_\mathbf{u}, \beta_\mathbf{v} \in H^{1/2}(\Gamma)$ *and* $\alpha_\mathbf{u}, \alpha_\mathbf{v} \in \mathcal{H}(\Gamma)$. *Then, we have*

$$\gamma\langle \mathbf{u},\mathbf{v}\rangle_\pi := -\langle \Delta_\Gamma \alpha_u, \beta_v\rangle_{1/2,\Gamma} + \langle \Delta_\Gamma \alpha_v, \beta_u\rangle_{1/2,\Gamma}. \tag{7}$$

Given $\mathbf{u} \in \mathbf{H}(\mathbf{curl};\Omega)$, recall the decompositions $\mathbf{u} = \boldsymbol{\Phi} + \boldsymbol{\nabla}p$ with $\boldsymbol{\Phi} \in [H^1(\Omega)]^3, p \in H^1(\Omega)$. Based on these decompositions, the integration by parts formula holds:

Theorem 2.4. *Given* $\mathbf{u},\mathbf{v} \in \mathbf{H}(\mathbf{curl};\Omega)$, *let* $\mathbf{u} = \boldsymbol{\Phi} + \boldsymbol{\nabla}p, \mathbf{v} = \Psi + \boldsymbol{\nabla}q$ *with* $\boldsymbol{\Phi},\boldsymbol{\Psi} \in [H^1(\Omega)]^3, p,q \in H^1(\Omega)$.

$$\int_\Omega \{\boldsymbol{\nabla}\times\mathbf{u}\cdot\overline{\mathbf{v}} - \mathbf{u}\cdot\boldsymbol{\nabla}\times\overline{\mathbf{v}}\}\, d\mathbf{x} = -\gamma\langle \gamma_\tau(\mathbf{u}), \pi_\tau(\mathbf{v})\rangle_\pi, \tag{8}$$

where the boundary term can be interpreted as

$$\begin{aligned}
\gamma\langle \gamma_\tau(\mathbf{u}), \pi_\tau(\mathbf{v})\rangle_\pi &= \int_\Gamma \gamma_\tau(\boldsymbol{\Phi})\cdot\overline{\pi_\tau(\boldsymbol{\Psi})} + v'_\gamma\langle \boldsymbol{\nabla}_\Gamma q, \gamma_\tau(\boldsymbol{\Psi})\rangle_{v_\gamma} \\
&+ v'_\pi\langle \mathbf{curl}_\Gamma p, \pi_\tau(\boldsymbol{\Psi})\rangle_{v_\pi} \\
&= \int_\Gamma \gamma_\tau(\boldsymbol{\Phi})\cdot\overline{\pi_\tau(\boldsymbol{\Psi})} - \langle \mathrm{div}_\Gamma\gamma_\tau(\boldsymbol{\Phi}), q\rangle_{1/2,\Gamma} \\
&+ \langle \mathbf{curl}_\Gamma\pi_\tau(\boldsymbol{\Psi}), p\rangle_{1/2,\Gamma}.
\end{aligned}$$

In what follows we will use the notation $\mathbf{v}\times\varphi$ to denote $-\gamma_\tau(\varphi)$ if φ is in the appropriate space $\mathbf{H}(\mathbf{curl};\Omega)$. Recall the classical Green's formula

$$\begin{aligned}
(\boldsymbol{\nabla}\times\varphi,\psi) - (\varphi,\boldsymbol{\nabla}\times\psi) &= \langle \mathbf{v}\times\varphi,\psi\rangle_{\partial\Omega} \\
&= \langle \mathbf{v}\times\varphi,\pi_\tau\psi\rangle_{\partial\Omega},
\end{aligned}$$

for all $\varphi \in \mathbf{H}(\mathbf{curl};\Omega), \psi \in [H^1(\Omega)]^3$. If both φ and ψ belong to $\mathbf{H}(\mathbf{curl};\Omega)$, $\mathbf{v}\times\varphi$ and $\pi_\tau\psi$ belong to $[H^{-1/2}(\partial\Omega)]^3$ and therefore Theorem 2.4 generalizes the above classical integration by formula.

Remark 2.1. *In Sheen (1992) the integration by parts formula for a Lipschitz domain* Ω *is proved directly by using a density argument with a different interpretation of the boundary integral term in* (8):

$$\begin{aligned}
(\boldsymbol{\nabla}\times\varphi,\psi) - (\varphi,\boldsymbol{\nabla}\times\psi) &= \langle \mathbf{v}\times\varphi,\psi\rangle_{\partial\Omega} \\
&= \langle \mathbf{v}\times\varphi,\pi_\tau\psi\rangle_{\partial\Omega}, \tag{9}
\end{aligned}$$

for all $\varphi,\psi \in \mathbf{H}(\mathbf{curl};\Omega)$, *where, the boundary integral term* $\langle \mathbf{v}\times\varphi,\pi_\tau\psi\rangle_{\partial\Omega}$ *is understood as* $\langle (\mathbf{v}\times\varphi)\cdot\overline{\pi_\tau\psi}, 1\rangle$, *the duality pairing between* $\mathbf{v}\times\varphi\cdot\overline{\pi_\tau\psi} \in \mathrm{Lip}(\partial\Omega)'$

and $1 \in \mathrm{Lip}(\partial\Omega)$. *Indeed, in Sheen (1992) the integration by parts formula* (8) *was proved for a general class of linear first order differential operators.* $\mathcal{L} = \sum_{j=1}^N \mathbf{A}_j(x)\frac{\partial}{\partial x_j}$, *where the* $\mathbf{A}_j$'s *are* $k\times k$ *matrices with uniformly Lipschitz–continuous components on* $\overline{\Omega}$ *and* $\mathcal{L}^*$ *be the formal adjoint of* $\mathcal{L}$ *given by* $\mathcal{L}^* = -\sum_{j=1}^N \frac{\partial}{\partial x_j}\mathbf{A}_j^*(x)$, *with* $\mathbf{A}_j^*$'s *being the adjoint matrices of* $\mathbf{A}_j$'s. *Then the following Hilbert space* $H(\mathcal{L};\Omega) = \{\mathbf{u} \in [L^2(\Omega)]^k; \mathcal{L}\mathbf{u} \in [L^2(\Omega)]^k\}$ *is endowed with the inner product and the norm*

$$(\mathbf{u},\mathbf{v})_{H(\mathcal{L};\Omega)} = (\mathbf{u},\mathbf{v}) + (\mathcal{L}\mathbf{u},\mathcal{L}\mathbf{v}),$$

$$\|\mathbf{u}\|_{H(\mathcal{L};\Omega)} = \left\{\|\mathbf{u}\|^2 + \|\mathcal{L}\mathbf{u}\|^2\right\}^{\frac{1}{2}}.$$

The analogues for $\mathcal{L}^*$ *holds. Denoting* $\mathbf{A_v} = \sum_{j=1}^N \nu_j\mathbf{A}_j$, *the main result of Sheen (1992) is given as follows: the map* $\{\mathbf{u},\mathbf{v}\} \mapsto \mathbf{A_v}\mathbf{u}\cdot\overline{\mathbf{v}}$ *from* $[\mathcal{D}(\overline{\Omega})]^k \times [\mathcal{D}(\overline{\Omega})]^k$ *into* $\mathrm{Lip}(\Gamma)'$ *can be extended by continuity to a continuous sesquilinear map from* $H(\mathcal{L};\Omega) \times H(\mathcal{L}^*;\Omega)$ *into* $\mathrm{Lip}(\Gamma)'$; *moreover, for all* $\mathbf{u} \in H(\mathcal{L};\Omega)$ *and* $\mathbf{v} \in H(\mathcal{L}^*;\Omega)$, *the following Green's formula holds:*

$$(\mathcal{L}\mathbf{u},\mathbf{v}) - (\mathbf{u},\mathcal{L}^*\mathbf{v}) = \{A_\nu\mathbf{u},\mathbf{v}\}_\Gamma \equiv {}_{\mathrm{Lip}(\Gamma)'}\langle \mathbf{A_v}\mathbf{u}\cdot\overline{\mathbf{v}}, 1\rangle_{\mathrm{Lip}(\Gamma)},$$

where ${}_{\mathrm{Lip}(\Gamma)'}\langle \cdot,\cdot\rangle_{\mathrm{Lip}(\Gamma)}$ *denotes the duality paring between* $\mathrm{Lip}(\Gamma)'$ *and* $\mathrm{Lip}(\Gamma)$.

An immediate result of Theorem 2.3 is useful:

Proposition 2.1. *Let* $\mathbf{u} \in \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma;\Gamma) \cap \mathbf{H}^{-1/2}(\mathbf{curl}_\Gamma;\Gamma)$ *with* $\mathbf{v}\cdot\mathbf{u} = 0$. *Then* $\mathbf{u} \in \mathbf{L}_t^2(\Gamma)$.

Proof. Suppose $\mathbf{u} \in \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma;\Gamma) \cap \mathbf{H}^{-1/2}(\mathbf{curl}_\Gamma;\Gamma)$. Then $\mathbf{u}$ has the following decomposition: $\mathbf{u} = \boldsymbol{\nabla}_\Gamma\alpha + \mathbf{curl}_\Gamma\beta$ and $\mathbf{u} = \boldsymbol{\nabla}_\Gamma\beta' + \mathbf{curl}_\Gamma\alpha'$ with $\alpha,\alpha' \in \mathcal{H}(\Gamma)$ and $\beta,\beta' \in H^{1/2}(\Gamma)$. Moreover, the duality (7) implies that

$$\langle \mathbf{u},\mathbf{u}\rangle_{0,\Gamma} = -\gamma\langle \mathbf{u},\mathbf{u}\rangle_\pi = -\langle \Delta_\Gamma\alpha, \beta'\rangle_{1/2,\Gamma} + \langle \Delta_\Gamma\alpha', \beta\rangle_{1/2,\Gamma},$$

which is bounded by the definition of duality. Thus, $\mathbf{u} \in \mathbf{L}_t^2(\Gamma)$. $\qquad\square$

2.3 Existence and uniqueness results

We begin with the following lemma:

Lemma 2.1. *If* $\{E,H\} \in [\mathbf{H}(\mathbf{curl};\Omega)]^2$ *satisfies* (3), *the boundary terms* $\alpha\pi_\tau E$ *and* $\mathbf{v}\times H$ *belong to* $\mathbf{L}_t^2(\Gamma)$.

Proof. Indeed, let $\widetilde{\alpha}$ be an invertible Lipschitz extension of α onto $\overline{\Omega}$ such that $\widetilde{\alpha} = \alpha$ on Γ. Then, taking the inner product of the equation (3a) with $(\widetilde{\alpha}^*)^{-1}E$ and applying (9) with using the boundary condition (3c), we obtain

$$
\begin{aligned}
0 &= (F, (\widetilde{\alpha}^*)^{-1}E) - ((i\omega\varepsilon + \sigma)E, (\widetilde{\alpha}^*)^{-1}E) \\
&\quad + (H, \boldsymbol{\nabla} \times (\widetilde{\alpha}^*)^{-1}E) \\
&= (F, (\widetilde{\alpha}^*)^{-1}E) - ((i\omega\varepsilon + \sigma)E, (\widetilde{\alpha}^*)^{-1}E) \\
&\quad + (\boldsymbol{\nabla} \times H, (\widetilde{\alpha}^*)^{-1}E) - \langle \mathbf{v} \times H, (\alpha^*)^{-1}\pi_\tau E \rangle_\Gamma \\
&= (F, (\widetilde{\alpha}^*)^{-1}E) - ((i\omega\varepsilon + \sigma)E, (\widetilde{\alpha}^*)^{-1}E) \\
&\quad + (\boldsymbol{\nabla} \times H, (\widetilde{\alpha}^*)^{-1}E) + \langle \pi_\tau E, \pi_\tau E \rangle_\Gamma.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
|\pi_\tau E|^2_{0,\Gamma} &\leq_\gamma \langle \pi_\tau E, \pi_\tau E \rangle_\pi \\
&\leq |(F, (\widetilde{\alpha}^*)^{-1}E)| + |((i\omega\varepsilon + \sigma)E, (\widetilde{\alpha}^*)^{-1}E)| \\
&\quad + |(\boldsymbol{\nabla} \times H, (\widetilde{\alpha}^*)^{-1}E)| \\
&\leq C \left[\|E\|^2_{0,\Omega} + \|H\|^2_{\mathbf{H}(\mathbf{curl};\Omega)} + \|F\|^2_{0,\Omega} \right].
\end{aligned}
$$

Therefore we see that $\pi_\tau E$ actually belongs to $\mathbf{L}^2_t(\Gamma)$. The boundary condition (3c) leads to $\mathbf{v} \times H \in \mathbf{L}^2_t(\Gamma)$. □

The existence and uniqueness results for Problem (3) without the term ε are given in Santos and Sheen (2000) with the use of integration by parts formula in the sense of Remark 2.1. However, it is straightforward to check all the arguments given there are still valid with the term ε included and the use of integration by parts formula in the sense of Theorem 2.4:

Theorem 2.1. *Assume that $\sigma_* > 0$. Let $F, G \in [L^2(\Omega)]^3$ and $\omega \neq 0$. Then, there exists a unique electromagnetic field $\{E, H\} \in [\mathbf{H}(\mathbf{curl};\Omega)]^2$ satisfying (3) with $\pi_\tau E, \mathbf{v} \times H \in \mathbf{L}^2_t(\Gamma)$. If, in addition, F and G belong to $H(div;\Omega)$ and ε, σ, and μ are Lipschitz-continuous on $\overline{\Omega}$, then E and H belong to $[H^{1/2}(\Omega)]^3$; more precisely, E and H belong to $[H(\mathbf{curl};\Omega) \cap H(div;\Omega)]$ with boundary values in $[L^2(\Gamma)]^3$*

In getting the weak formulation we will see the cross term $\langle \alpha\pi_\tau E, \pi_\tau\phi \rangle_\Gamma$ with $E, \phi \in \mathbf{H}(\mathbf{curl};\Omega)$. In order to provide a meaning to this term, the correct test function space for the weak problem will be

$$
\begin{aligned}
\mathbf{H}^*(\mathbf{curl};\Omega) = \{ &v \in \mathbf{H}(\mathbf{curl};\Omega) : \alpha\pi_\tau v = \mathbf{v} \times \phi \\
&\text{for some } \phi \in \mathbf{H}(\mathbf{curl};\Omega) \}.
\end{aligned}
$$

By applying (9), a *mixed weak formulation* of Problem (1) follows immediately: find $\{E, H\} \in \mathbf{H}^*(\mathbf{curl};\Omega) \times [L^2(\Omega)]^3$ such that

$$
\begin{aligned}
&((i\omega\varepsilon + \sigma)E, \phi) - (H, \boldsymbol{\nabla} \times \phi) + \langle \alpha\pi_\tau E, \pi_\tau\phi \rangle_\Gamma = (F, \phi), \\
&\quad \phi \in H^*(\mathbf{curl};\Omega),
\end{aligned} \tag{10a}
$$

$$
\begin{aligned}
&i\omega(\mu H, \psi) + (\boldsymbol{\nabla} \times E, \psi) = (G, \psi), \\
&\quad \psi \in [L^2(\Omega)]^3.
\end{aligned} \tag{10b}
$$

It is easy to see that if $\{E, H\}$ satisfies the above weak problem, it is then a solution of the differential equations (3) with the boundary condition (3c) in the sense of $\mathbf{H}^{-1/2}(\mathbf{curl}_\Gamma; \Gamma)$.

3   The nonconforming method

In this section we summarize some results on nonconforming mixed finite element space based on three-dimensional rectangular domain introduced in Douglas Jr., Santos, and Sheen (2000).

3.1   *Nonconforming mixed finite element space*

Let $\widehat{K}$ be the reference cube $[-1, 1]^3$ and let $\widehat{Q}(\widehat{K}) = \widehat{Q}_x \times \widehat{Q}_y \times \widehat{Q}_z$, where

$$
\widehat{Q}_x = \text{Span}\{1, y, z, (y^2 - \frac{5}{3}y^4) - (z^2 - \frac{5}{3}z^4)\},
$$

$$
\widehat{Q}_y = \text{Span}\{1, z, x, (z^2 - \frac{5}{3}z^4) - (x^2 - \frac{5}{3}x^4)\},
$$

$$
\widehat{Q}_z = \text{Span}\{1, x, y, (x^2 - \frac{5}{3}x^4) - (y^2 - \frac{5}{3}y^4)\}.
$$

Let m_i, $i = 1, \cdots, 6$, be the centroid of the ith face of $\widehat{K}$. Then, for $\phi \in \widehat{Q}(\widehat{K})$, we consider the following local degrees of freedom :

$$
\Sigma(\phi) = \{ (\pi_\tau\phi)(m_i) \mid i = 1, \cdots, 6 \}.
$$

Then, a local interpolant $\widehat{\pi} : H^2(\widehat{K}) \to \widehat{Q}(\widehat{K})$ is defined as follows:

$$
\pi_\tau(\widehat{\pi}\phi - \phi)(m_i) = 0, \quad i = 1, \cdots, 6. \tag{11}
$$

Let $\widehat{S}(\widehat{K}) = \widehat{S}_x \times \widehat{S}_y \times \widehat{S}_z$, where

$$
\widehat{S}_x = \text{Span}\{1, y - \frac{10}{3}y^3, z - \frac{10}{3}z^3\},
$$

$$\widehat{S}_y = \text{Span}\{1, z - \frac{10}{3}z^3, x - \frac{10}{3}x^3\},$$

$$\widehat{S}_z = \text{Span}\{1, x - \frac{10}{3}x^3, y - \frac{10}{3}y^3\},$$

and a local interpolant $\widehat{P} : \mathbf{H}(\mathbf{curl}; \widehat{K}) \to \widehat{S}(\widehat{K})$ be defined by the following rules: for $l = x, y, z$,

$$\int_{\widehat{K}} (\widehat{P}\psi_l - \psi_l)\, dxdydz = 0, \tag{12a}$$

$$\int_{\widehat{K}} \mathbf{\nabla} \times (\widehat{P}\psi_l - \psi_l)\, dxdydz = 0, \tag{12b}$$

for all $\psi = (\psi_x, \psi_y, \psi_z)$, Note that (12) provides the nine degrees of freedom needed to determine an element in $\widehat{S}(\widehat{K})$ and that

$$\mathbf{\nabla} \times \widehat{Q} = \widehat{S}.$$

The following lemma is trivial but useful Douglas Jr., Santos, and Sheen (2000).

Lemma 3.1. *The degrees of freedom* (11) *and* (12) *determine, respectively,* $\varphi \in \widehat{Q}(\widehat{K})$ *and* $\psi \in \widehat{S}(\widehat{K})$ *uniquely.*

The following proposition states an immediate but fundamental property of $\widehat{Q}$ and $\widehat{S}$ that is important in obtaining effective nonconforming methods Douglas Jr., Santos, and Sheen (2000);)Douglas Jr., Santos, Sheen, and Ye (1999).

Proposition 3.1. *If an element of* $\pi_\tau \widehat{Q}$ *or* $\pi_\tau \widehat{S}$ *vanishes at the centroid of a face of* $\widehat{K}$, *it is orthogonal to constants on that face.*

For $0 < h < 1$, let Ω be decomposed into nonoverlapping three-dimensional rectangular hexahedra $\{\Omega_j : j = 1, \cdots, J\}$ with their edges bounded by h:

$$\overline{\Omega} = \cup_j \overline{\Omega}_j, \quad \Omega_j \cap \Omega_k = \emptyset, \quad j \neq k.$$

Then $Q(\Omega_j)$ and $S(\Omega_j)$ are defined by scaling and translating from $\widehat{Q}$ and $\widehat{S}$. Let

$$\Gamma_j = \partial\Omega_j \cap \Gamma, \quad \Gamma_{jk} = \partial\Omega_j \cap \partial\Omega_k = \Gamma_{kj}.$$

3.2 The nonconforming mixed finite element method

Set

$$\mathbf{V}^h = \{\varphi \in [L^2(\Omega)]^3 : \varphi_j := \varphi|_{\Omega_j} \in Q(\Omega_j),$$
$$\varphi_j(m_{jk}) = \varphi_k(m_{jk}),$$
$$m_{jk} \text{ being the centroid of } \Gamma_{jk} \forall \{j, k\} \in I^h\},$$

$$W^h = \{\psi \in [L^2(\Omega)]^3 : \psi|_{\Omega_j} \in S(\Omega_j)\},$$

where I^h denotes the index set of all internal interfaces Γ_{jk}.

The *nonconforming mixed finite element method* is then defined as follows: find $\{E^h, H^h\} \in \mathbf{V}^h \times W^h$ such that

$$\big((i\omega\varepsilon + \sigma)E^h, \varphi\big) - \sum_j \big[(H^h, \mathbf{\nabla} \times \varphi)_j$$
$$+ \sum_k \big\langle \mathbf{v}_j \times E_j^h, \pi_\tau\varphi \big\rangle_{\Gamma_{jk}} \big] + \big\langle \alpha\pi_\tau E^h, \pi_\tau\varphi \big\rangle_\Gamma$$
$$= (F, \varphi), \quad \varphi \in \mathbf{V}^h, \tag{13a}$$

$$i\omega(\mu H^h, \psi) + \sum_j (\mathbf{\nabla} \times E^h, \psi)_j = (G, \psi), \psi \in W^h. \tag{13b}$$

Denoting by $\|\cdot\|_{0,h}$ the broken $L^{(\Omega)}$-norm, we then have the following *a priori* error estimate:

Theorem 3.1. *Suppose that* $\sigma_* > 0$. *Let* $\{E, H\}$ *and* $\{E^h, H^h\}$, $0 < h < 1$, *be the solutions to* (10) *and* (13), *respectively. Then,*

$$\|E - E^h\|_0 + \|H - H^h\|_0 + \|\mathbf{\nabla} \times (E - E^h)\|_{0,h}$$
$$\leq Ch^{1/2}\left(\|E\|_2 + h^{1/2}\|H\|_1\right). \tag{14}$$

The proof of the above theorem is given in Douglas Jr., Santos, and Sheen (2000), where $\varepsilon^* = 0$ is assumed, but including the case $\varepsilon_* > 0$ does not change any argument given there.

3.3 Hybridization

Denote by $\langle\langle\cdot,\cdot\rangle\rangle_{\Gamma_{jk}}$ the approximation to $\langle\cdot,\cdot\rangle_{\Gamma_{jk}}$ obtained by using the mid-point rule on Γ_{jk} so that

$$\langle\langle u, v\rangle\rangle_{\Gamma_{jk}} = |\Gamma_{jk}|(u\overline{v})(m_{jk}),$$

where m_{jk} and $|\Gamma_{jk}|$ denote the centroid and the measure of Γ_{jk}, respectively.

Following Arnold and Brezzi (1985); Fraeijs de Veubeke (1965); Fraeijs de Veubeke (1975), the hybridization of the procedure (13) is given by introducing the space $\tilde{\Lambda}^h$ of Lagrange multipliers associating its elements with $\mathbf{v}_j \times E_j(m_{jk})$ on Γ_{jk}. Thus set

$$\mathcal{N}C_{-1}^h = \{\varphi \in [L^2(\Omega)]^3 : \varphi|_{\Omega_j} \in Q(\Omega_j)\}$$

$$\tilde{\Lambda}^h = \{\tilde{\lambda}^h : \tilde{\lambda}^h|_{\Gamma_{jk}} = \tilde{\lambda}_{jk} \in [P_0(\Gamma_{jk})]^2;$$
$$\tilde{\lambda}_{jk} + \tilde{\lambda}_{kj} = 0 \quad \text{for all } jk \in I^h\},$$

and $P_0(\Gamma_{jk})$ the set of constant functions defined on Γ_{jk}.

The *global hybridized nonconforming mixed finite element method* is then defined as follows: find $\{\tilde{E}^h, \tilde{H}^h, \tilde{\lambda}^h\}$ $\in \mathcal{N}C_{-1}^h \times W^h \times \tilde{\Lambda}^h$ such that

$$\begin{aligned}
&((i\omega\varepsilon + \sigma)\tilde{E}^h, \varphi) - \sum_j \big[(\tilde{H}^h, \nabla \times \varphi)_j \\
&\quad + \sum_k \langle\langle \tilde{\lambda}^h, \pi_\tau\varphi \rangle\rangle_{\Gamma_{jk}}\big] + \langle\langle \alpha\pi_\tau\tilde{E}^h, \pi_\tau\varphi \rangle\rangle_\Gamma \\
&\quad = (F, \varphi), \quad \varphi \in \mathcal{N}C_{-1}^h,
\end{aligned} \tag{15a}$$

$$i\omega(\mu\tilde{H}^h, \psi) + \sum_j (\nabla \times \tilde{E}^h, \psi)_j = (G, \psi), \quad \psi \in W^h, \tag{15b}$$

$$\sum_{j,k \in I^h} \langle\langle \theta, \pi_\tau\tilde{E}^h \rangle\rangle_{\Gamma_{jk}} = 0, \quad \theta \in \tilde{\Lambda}^h. \tag{15c}$$

Notice that (15c) is equivalent to imposing the condition $\pi_\tau\tilde{E}_j^h(m_{jk}) = \pi_\tau\tilde{E}_j^h(m_{jk})$ at the centroid m_{jk} of interfaces Γ_{jk}. Also, as in Douglas Jr., Santos, and Sheen (2000), we immediately have the following theorem on existence and uniqueness for Problem (15).

Theorem 3.2. *Let* $\sigma_* > 0$. *Then the problem* (15) *is uniquely solvable.*

4   A Seidel-type domain decomposition method

4.1   A Seidel-type domain decomposition iterative procedure

Set $f_j = f|_{\Omega_j}$ for any function f defined on Ω. The differential domain decomposition problem for solving (1) is to find $\{E_j, H_j\}$, for $j = 1, \cdots, J$, such that

$$(i\omega\varepsilon + \sigma)E_j - \nabla \times H_j = F_j \quad \text{in} \quad \Omega_j, \tag{16a}$$

$$i\omega\mu H_j + \nabla \times E_j = G_j \quad \text{in} \quad \Omega_j, \tag{16b}$$

$$\alpha\pi_\tau E_j + \nu \times H_j = 0 \quad \text{on} \quad \Gamma_j, \tag{16c}$$

with the interface consistency conditions

$$\nu_j \times H_j = -\nu_k \times H_k \text{ and } \pi_\tau E_j = \pi_\tau E_k \quad \text{on } \Gamma_{jk} \forall k. \tag{17}$$

Instead of (17), the Robin-type transmission condition

$$(\nu_j \times H_j + \beta_{jk}\pi_\tau E_j) = -\nu_k \times H_k + \beta_{jk}\pi_\tau E_k \quad \text{on} \quad \Gamma_{jk} \forall k \tag{18}$$

will be imposed. In (18), β_{jk} is a complex function defined on Γ_{jk} which will be specified later. The weak formulation of the Problem (16) and (18) is given as follows: find $\{E_j, H_j\} \in H^*(\text{curl}; \Omega_j) \times [L^2(\Omega_j)]^3$, $j = 1, \cdots, J$, such that

$$\begin{aligned}
&((i\omega\varepsilon + \sigma)E_j, \varphi)_j - (H_j, \nabla \times \varphi)_j \\
&\quad + \sum_k \langle \beta_{jk}(\pi_\tau E_j - \pi_\tau E_k) + \nu_k \times H_k, \pi_\tau\varphi \rangle_{\Gamma_{jk}}, \\
&\quad + \langle \alpha\pi_\tau E_j, \pi_\tau\varphi \rangle_{\Gamma_j} = (F_j, \varphi)_j, \quad \varphi \in H^*(\text{curl}; \Omega_j),
\end{aligned}$$

$$i\omega(\mu H_j, \psi)_j + (\nabla \times E_j, \psi)_j = (G_j, \psi)_j, \psi \in [L^2(\Omega_j)]^3.$$

Set $\mathbf{V}_j^h = \mathcal{N}C_{-1}^h|_{\Omega_j}$ and $W_j^h = W^h|_{\Omega_j}$. Also let Λ^h be the set of Lagrange multipliers associated with $(\nu_j \times H_j)(m_{jk})$ on Γ_{jk} given in by

$$\Lambda^h = \Big\{ \lambda^h | \; \lambda^h|_{\Gamma_{jk}} = \lambda_{jk}^h \in \Lambda_{jk} \quad \forall\{j, k\} \in I^h \Big\},$$

where $\Lambda_{jk} = [P_0(\Gamma_{jk})]^2$, and set

$$\Lambda_j^h = \Big\{ \lambda_j^h | \; \lambda_j^h|_{\Gamma_{jk}} = \lambda_{jk}^h \in \Lambda_{jk} \quad \forall k \Big\}.$$

From now on associate $(\nu_j \times H_j)(m_{jk})$ on Γ_{jk} with λ_{jk}.

We are now in a position to define a *Seidel-type domain decomposition iterative procedure*. Choose an initial guess $\{E_j^{h,0}, H_j^{h,0}, \lambda_j^{h,0}\} \in \mathbf{V}_j^h \times W_j^h \times \Lambda_j$ for all j. Then, iteratively for $n \geq 1$, find $\{E_j^{h,n}, H_j^{h,n}, \lambda_j^{h,n}\} \in \mathbf{V}_j^h \times W_j^h \times \Lambda_j^h$, for $j = 1, 2, \cdots$, fulfilling

$$\begin{aligned}
&((i\omega\varepsilon + \sigma)E_j^{h,n}, \varphi)_j - (H_j^{h,n}, \nabla \times \varphi)_j \\
&\quad - \sum_k \langle\langle \lambda_{jk}^{h,n}, \pi_\tau\varphi \rangle\rangle_{\Gamma_{jk}} + \langle\langle \alpha\pi_\tau E_j^{h,n}, \pi_\tau\varphi \rangle\rangle_{\Gamma_j} \\
&\quad = (F_j, \varphi)_j, \quad \varphi \in \mathbf{V}_j^h,
\end{aligned} \tag{19a}$$

$$\begin{aligned}
&i\omega(\mu H_j^{h,n}, \psi)_j + (\nabla \times E_j^{h,n}, \psi)_j = (G_j, \psi)_j, \\
&\quad \psi \in W_j^h,
\end{aligned} \tag{19b}$$

$$\begin{aligned}
&\lambda_{jk}^{h,n} = -\lambda_{kj}^{h,n^*} + \beta_{jk}(\pi_\tau E_k^{h,n^*} - \pi_\tau E_j^{h,n})(m_{jk}), \\
&\quad \text{for all } k,
\end{aligned} \tag{19c}$$

where n^* is defined by

$$n^* = \begin{cases} n-1, & j < k \\ n, & j > k. \end{cases}$$

Remark 4.1. *It should be remarked that in the updating procedure* (19c) *data in use passed from neighboring subdomains are most up-to-date; in this sense* (19) *is said to be a Seidel-type scheme. Instead of the above, if n^* were defined as $n-1$ for all j,k, the procedure* (19) *should be regarded as a Jacobi-type scheme in the sense that the updating procedure* (19c) *would use informations from the past step; this procedure has been studied in Douglas Jr., Santos, and Sheen (2000).*

4.2 Convergence of the iterative procedure

The localized Problem (16)-(18) is weakly formulated as follows: find $\{\tilde{E}_j^h, \tilde{H}_j^h, \tilde{\lambda}_j^h\} \in \mathbf{V}_j^h \times W_j^h \times \Lambda_j^h$ such that

$$((i\omega\varepsilon + \sigma)\tilde{E}_j^h, \varphi)_j - (\tilde{H}_j^h, \nabla \times \varphi)_j - \sum_k \langle\langle\tilde{\lambda}_{jk}^h, \pi_\tau\varphi\rangle\rangle_{\Gamma_{jk}}$$
$$+ \langle\langle\alpha\pi_\tau\tilde{E}_j^h, \pi_\tau\varphi\rangle\rangle_{\Gamma_j} = (F_j, \varphi)_j, \quad \varphi \in \mathbf{V}_j^h, \qquad (20a)$$

$$i\omega(\mu\tilde{H}_j^h, \psi)_j + (\nabla \times \tilde{E}_j^h, \psi)_j = (G_j, \psi)_j, \psi \in W_j^h, \qquad (20b)$$

$$\tilde{\lambda}_{jk}^h = -\tilde{\lambda}_{kj}^h + \beta_{jk}(\pi_\tau\tilde{E}_k^h - \pi_\tau\tilde{E}_j^h)(m_{jk}), \text{for all } k. \qquad (20c)$$

We will show the convergence of the solution $\{E_j^{h,n}, H_j^{h,n}, \lambda_j^{h,n}\}$ of Problem (19) to the solution $\{\tilde{E}_j^h, \tilde{H}_j^h, \tilde{\lambda}_j^h\}$ of Problem (20). We will restrict the convergence proof to the case $\beta_{jk} = \beta$. for all j,k. Set, for all j,

$$u_j^n = E_j^{h,n} - \tilde{E}_j^h, \quad v_j^h = H_j^{h,n} - \tilde{H}_j^h, \quad \theta_j^n = \lambda_j^{h,n} - \tilde{\lambda}_j^h. (21)$$

Then a subtraction of (20) from (19) gives the iteration error equations:

$$((i\omega\varepsilon + \sigma)u_j^n, \varphi)_j - (v_j^n, \nabla \times \varphi)_j - \sum_k \langle\langle\theta_{jk}^n, \pi_\tau\varphi\rangle\rangle_{\Gamma_{jk}}$$
$$+ \langle\langle\alpha\pi_\tau u_j^n, \pi_\tau\varphi\rangle\rangle_{\Gamma_j} = 0, \varphi \in \mathbf{V}_j^h, \qquad (22a)$$

$$i\omega(\mu v_j^n, \psi)_j + (\nabla \times u_j^n, \psi)_j = 0, \quad \psi \in W_j^h, \qquad (22b)$$

$$\theta_{jk}^n = -\theta_{kj}^{n^*} + \beta(\pi_\tau u_k^{n^*} - \pi_\tau u_j^n)(m_{jk}) \forall k. \qquad (22c)$$

Choose $\varphi = u_j^n$ and $\psi = v_j^n$ in (22a) and (22b), respectively. Then we have

$$\sum_k \langle\langle\theta_{jk}^n, \pi_\tau u_j^n\rangle\rangle_{\Gamma_{jk}} = ((i\omega\varepsilon + \sigma)u_j^n, u_j^n)_j$$
$$- i\omega(\mu v_j^n, v_j^n)_j + \langle\langle\alpha\pi_\tau u_j^n, \pi_\tau u_j^n\rangle\rangle_{\Gamma_j}. \qquad (23)$$

Taking the real and imaginary parts in the above equation, we get

$$\text{Re}\sum_k \langle\langle\theta_{jk}^n, \pi_\tau u_j^n\rangle\rangle_{\Gamma_{jk}} = (\sigma u_j^n, u_j^n)_j$$
$$+ \langle\langle\alpha_R\pi_\tau u_j^n, \pi_\tau u_j^n\rangle\rangle_{\Gamma_j} \qquad (24)$$

and

$$\text{Im}\sum_k \langle\langle\theta_{jk}^n, \pi_\tau u_j^n\rangle\rangle_{\Gamma_{jk}} = \omega(\varepsilon u_j^n, u_j^n)_j$$
$$- \omega(\mu v_j^n, v_j^n)_j - \langle\langle\alpha_I\pi_\tau u_j^n, \pi_\tau u_j^n\rangle\rangle_{\Gamma_j}. \qquad (25)$$

We rearrange (22c) so that

$$\theta_{jk}^n = -\theta_{kj}^{n-1} + \beta(\pi_\tau u_k^{n-1} - \pi_\tau u_j^n)(m_{jk}),$$
$$j < k, \qquad (26a)$$

$$\theta_{jk}^n = -\theta_{kj}^n + \beta(\pi_\tau u_k^n - \pi_\tau u_j^n)(m_{jk}), \quad j > k, \qquad (26b)$$

$$= \theta_{jk}^{n-1} - \beta(\pi_\tau u_j^{n-1} - \pi_\tau u_k^n)(m_{jk})$$
$$+ \beta(\pi_\tau u_k^n - \pi_\tau u_j^n)(m_{jk}) \qquad (26c)$$

$$= -\theta_{kj}^{n-1} + \beta(\pi_\tau u_k^{n-1} - 2\pi_\tau u_j^{n-1})(m_{jk})$$
$$- \beta(\pi_\tau u_j^n - 2\pi_\tau u_k^n)(m_{jk}). \qquad (26d)$$

Here, (26c) follows from (26b) by applying (26a) and (26d) follows from (26c) by applying (26b) for $n-1$. Motivated by (26a) and (26d), we define the pseudo-energy for the iterative procedure (19):

$$R^n(\{u^n, v^n, \theta^n\}) = \sum_{j<k} \left|\theta_{jk}^n + \beta\pi_\tau u_j^n(m_{jk})\right|_{0,\Gamma_{jk}}^2$$
$$+ \sum_{j>k} \left|\theta_{kj}^n + \beta\pi_\tau u_k^n(m_{jk})\right|_{0,\Gamma_{jk}}^2. \qquad (27)$$

Denote $|||f||| = \sum_j \langle\langle f, f\rangle\rangle_{\Gamma_j}$ for a function defined on Γ. Then the following recurrence relation for the decay in pseudo-energy holds.

Theorem 4.2. *With the pseudo-energy* (27)*, we have the following general recurrence relation which is independent of differential or weak problems:*

$$R^n = R^{n-1} - 8\text{Re}\sum_{j,k} \left\langle\theta_{jk}^{n-1}, \beta\pi_\tau u_j^{n-1}\right\rangle_{\Gamma_{jk}} \qquad (28)$$

Proof. Since by (26a), we have

$$R^n = 2 \sum_{j<k} \left| \theta_{jk}^n + \beta \pi_\tau u_j^n \right|_{0,\Gamma_{jk}}$$

$$= 2 \sum_{j<k} \left| \theta_{kj}^{n-1} - \beta \pi_\tau u_k^{n-1} \right|_{0,\Gamma_{jk}} \qquad \text{by (26a)}$$

$$= 2 \sum_{j>k} \left| \theta_{jk}^{n-1} - \beta \pi_\tau u_j^{n-1} \right|_{0,\Gamma_{jk}}$$

$$= 2 \sum_{j>k} \left| -\theta_{kj}^{n-1} + \beta(\pi_\tau u_k^{n-1} - 2\pi_\tau u_j^{n-1}) \right|_{0,\Gamma_{jk}} \text{ by (26a)}$$

$$= 2 \sum_{j<k} \left| \theta_{jk}^{n-1} - \beta(\pi_\tau u_j^{n-1} - 2\pi_\tau u_k^{n-1}) \right|_{0,\Gamma_{jk}}$$

$$= 2 \sum_{j<k} \left| \theta_{jk}^{n-1} + \beta \pi_\tau u_j^{n-1} \right.$$

$$\left. -2\beta(\pi_\tau u_j^{n-1} - \pi_\tau u_k^{n-1}) \right|_{0,\Gamma_{jk}}$$

$$= R^{n-1} - 8\,\mathrm{Re} \sum_{j<k} \left\langle \theta_{jk}^{n-1} + \beta \pi_\tau u_j^{n-1}, \right.$$

$$\left. \beta(\pi_\tau u_j^{n-1} - \pi_\tau u_k^{n-1}) \right\rangle_{\Gamma_{jk}}$$

$$+ 8 \sum_{j<k} \left| \beta(\pi_\tau u_j^{n-1} - \pi_\tau u_k^{n-1}) \right|_{0,\Gamma_{jk}}$$

$$= R^{n-1} - 8\,\mathrm{Re} \sum_{j<k} \left\langle \theta_{jk}^{n-1} + \beta \pi_\tau u_k^{n-1}, \right.$$

$$\left. \beta(\pi_\tau u_j^{n-1} - \pi_\tau u_k^{n-1}) \right\rangle_{\Gamma_{jk}}$$

$$= R^{n-1} - 8\,\mathrm{Re} \sum_{j,k} \left\langle \theta_{jk}^{n-1}, \beta \pi_\tau u_j^{n-1} \right\rangle_{\Gamma_{jk}}$$

since

$$\mathrm{Re} \left[\sum_{j<k} \left\langle -\beta \pi_\tau u_k^{n-1}, \beta(\pi_\tau u_j^{n-1} - \pi_\tau u_k^{n-1}) \right\rangle_{\Gamma_{jk}} \right.$$

$$\left. + \sum_{j<k} \left\langle \theta_{jk}^{n-1}, \beta \pi_\tau u_k^{n-1} \right\rangle_{\Gamma_{jk}} \right]$$

$$= \mathrm{Re} \sum_{j<k} \left\langle \beta \pi_\tau u_k^{n-1}, -\beta(\pi_\tau u_j^{n-1} - \pi_\tau u_k^{n-1}) + \theta_{jk}^{n-1} \right\rangle_{\Gamma_{jk}}$$

$$= \mathrm{Re} \sum_{j>k} \left\langle \beta \pi_\tau u_j^{n-1}, \beta(\pi_\tau u_j^{n-1} - \pi_\tau u_k^{n-1}) + \theta_{kj}^{n-1} \right\rangle_{\Gamma_{jk}}$$

$$= -\mathrm{Re} \sum_{j>k} \left\langle \beta \pi_\tau u_j^{n-1}, \theta_{jk}^{n-1} \right\rangle_{\Gamma_{jk}} \qquad \text{by (26a)}$$

$$= -\mathrm{Re} \sum_{j>k} \left\langle \theta_{jk}^{n-1}, \beta \pi_\tau u_j^{n-1} \right\rangle_{\Gamma_{jk}}.$$

This proves (28). ∎

From Theorem 4.2 and (23) it is immediate to have the following theorem.

Theorem 4.3. *Let* $\beta = \beta_R + i\beta_I$. *Then, with the pseudo-energy (27), the following recurrence relation holds:*

$$R^n = R^{n-1} - 8\beta_R \left[\left\| \sigma^{1/2} u^{n-1} \right\|^2 + \left\| \left\| \alpha_R^{1/2} \pi_\tau u^{n-1} \right\| \right\|^2 \right]$$

$$+ 8\beta_I \left[\omega \left\| \mu^{1/2} v^{n-1} \right\|^2 - \omega \left\| \varepsilon^{1/2} u^{n-1} \right\|^2 \right]$$

$$+ \left\| \left\| \alpha_I^{1/2} \pi_\tau u^{n-1} \right\| \right\|^2 \right]. \tag{29}$$

Remark 4.4. *In the Jacobi-type iterative procedure given in Douglas Jr., Santos, and Sheen (2000) the pseudo-energy is given in the form*

$$R_J^n(\{u^n, v^n, \theta^n\}) = \sum_{j,k} \left| \theta_{jk}^n + \beta \pi_\tau u_j^n(m_{jk}) \right|_{0,\Gamma_{jk}}^2$$

instead of (27), the recurrence relation is given in the form:

$$R_J^n = R_J^{n-1} - 4\beta_R \left[\left\| \sigma^{1/2} u^{n-1} \right\|^2 + \left\| \left\| \alpha_R^{1/2} \pi_\tau u^{n-1} \right\| \right\|^2 \right]$$

$$+ 4\beta_I \left[\omega \left\| \mu^{1/2} v^{n-1} \right\|^2 - \omega \left\| \varepsilon^{1/2} u^{n-1} \right\|^2 \right]$$

$$+ \left\| \left\| \alpha_I^{1/2} \pi_\tau u^{n-1} \right\| \right\|^2 \right].$$

Thus the energy decays in the Seidel-type procedure is roughly as twice fast as that in the Jacobi-type one.

According to Theorem 4.3, we have the following convergence result.

Theorem 4.5. *Assume that* β *satisfies the following conditions:*

$$\beta_R \sigma_* + \beta_I \omega \varepsilon^* > 0, \ \beta_R \alpha_{R_*} - \beta_I \alpha_{I_*} \geq 0,$$

$$\beta_R > 0, \ \beta_I \leq 0; \tag{30}$$

Then, the iteration error $\{u^n, v^n, \theta^n\}$ *satisfying (22) converges to zero as n tends to* ∞, *and* R^n *tends to zero as well.*

Proof. Due to (29), we have

$$R^n = R^0 - 8 \sum_{k=1}^{n-1} \sum_j \Big[(\beta_R \sigma + \beta_I \omega \varepsilon)(\sigma u_j^k, u_j^k)_j$$

$$- \beta_I \omega \mu (v_j^k, v_j^k)_j$$

$$+ (\beta_R \alpha_R - \beta_I \alpha_I) \langle\langle \pi_\tau u_j^k, \pi_\tau u_j^k \rangle\rangle_{\Gamma_j} \Big]$$

Thus, under Condition (30), we see that R^n is a nonnegative nonincreasing sequence, and therefore,

$$\sum_{k=1}^{\infty} \Big[\beta_R \sum_j \big\{ (\sigma u_j^k, u_j^k)_j + \langle\langle \alpha_R \pi_\tau u_j^k, \pi_\tau u_j^k \rangle\rangle_{\Gamma_j} \big\}$$

$$- \beta_I \sum_j \big\{ \omega(\mu v_j^k, v_j^k)_j - \omega(\varepsilon u_j^k, u_j^k)_j$$

$$+ \langle\langle \alpha_I \pi_\tau u_j^k, \pi_\tau u_j^k \rangle\rangle_{\Gamma_j} \big\} \Big] < \infty.$$

Suppose that (30) holds. Then we have that $u_j^n \to 0$ in $[L^2(\Omega_j)]^3$ as n tends to ∞, which implies that $\boldsymbol{\nabla} \times u_j^n \to 0$ in $[H^{-1}(\Omega_j)]^3$ as n tends to ∞. Due to finite dimensionality of $\mathbf{V}_j^h$, $\boldsymbol{\nabla} \times u_j^n \to 0$ in $[L^2(\Omega_j)]^3$ as n tends to ∞. Hence, (22b) leads to that $v_j^n \to 0$ in $[L^2(\Omega_j)]^3$ as n tends to ∞. Next, as n tends to ∞, (22a) tends to $-\sum_k \langle\langle \theta_{jk}^n, \pi_\tau \varphi \rangle\rangle_{\Gamma_{jk}} = 0$ for all $\varphi \in \mathbf{V}_j^h$, in particular for $\varphi = \tilde{\varphi}_{jk}$ such that $\pi_\tau \tilde{\varphi}(m_{jk}) = \theta_{jk}^n$ and vanishes at the other five centroids of $\partial \Omega_j$. This implies that θ_{jk}^n converges to zero as n tends to ∞ for all $\{j, k\} \in I^h$. The convergence of R^n to zero follows from the formula (27). The proof is complete. $\qquad\square$

4.3 An estimate for the spectral radius of the iterative procedure

Let $T_{F,G} = T_{F,G}(\beta, \cdot) =: \mathcal{NC}_{-1}^h \times W^h \times \Lambda^h \to \mathcal{NC}_{-1}^h \times W^h \times \Lambda^h$ be the affine map such that, for any $(U, V, \theta) \in \mathcal{NC}_{-1}^h \times W^h \times \Lambda^h$, $\{E, H, \lambda\} = T_{F,G}\{U, V, \theta\} = T_{F,G}(\beta; \{U, V, \theta\})$ is the solution of the following problem: for $j = 1, 2, \cdots$,

$$((i\omega\varepsilon + \sigma)E_j, \varphi)_j - (H_j, \boldsymbol{\nabla} \times \varphi)_j - \sum_j \langle\langle \lambda_{jk}, \pi_\tau \varphi \rangle\rangle_{\Gamma_{jk}}$$

$$+ \langle\langle \alpha \pi_\tau E_j, \pi_\tau \varphi \rangle\rangle_{\Gamma_j} = (F_j, \varphi)_j, \quad \varphi \in \mathbf{V}_j^h, \qquad (31a)$$

$$i\omega(\mu H_j, \psi)_j + (\boldsymbol{\nabla} \times E_j, \psi)_j = (G_j, \psi)_j, \psi \in W_j^h, \qquad (31b)$$

$$\lambda_{jk} = \begin{cases} -\theta_{kj} + \beta_{jk}(\pi_\tau U_k - \pi_\tau E_j)(m_{jk}), & j < k, \\ -\lambda_{kj} + \beta_{jk}(\pi_\tau E_k - \pi_\tau E_j)(m_{jk}), & j > k. \end{cases}$$

Immediately, the argument given in Douglas Jr., Santos, and Sheen (2000) gives the following result.

Lemma 4.6. *If $\{E, H, \lambda\}$ is a fixed point of $T_{F,G}$, then $\lambda_{jk} = -\lambda_{kj}$ for all $\{j, k\}$. Moreover, the pair $\{E, H, \lambda\}$ is a solution of (20) if and only if it is a fixed point of $T_{F,G}$.*

The operator $T_{F,G}$ can be decomposed as the sum of $T_{0,0}\{U, V, \theta\}$ and $T_{F,G}\{0, 0, 0\}$, and hence $\{U, V, \theta\}$ is a fixed point of $T_{F,G}$ if and only if

$$T_{F,G}\{U, V, \theta\} = \{U, V, \theta\} = T_{0,0}\{U, V, \theta\} + T_{F,G}\{0, 0, 0\}.$$

Observe that solving (19) for $j = 1, 2, \cdots$ is equivalent to applying the operator $T_{F,G}$ to $\{E^{h,n-1}, H^{h,n-1}, \lambda^{h,n-1}\}$:

$$\{E^{h,n}, H^{h,n}, \lambda^{h,n}\} = T_{F,G}\{E^{h,n-1}, H^{h,n-1}, \lambda^{h,n-1}\}$$

$$= T_{0,0}\{E^{h,n-1}, H^{h,n-1}, \lambda^{h,n-1}\} + T_{F,G}\{0, 0, 0\}. \qquad (32)$$

Also, due to Lemma 4.6, $\{\tilde{E}^h, \tilde{H}^h, \tilde{\lambda}^h\}$, which is the solution of (20), satisfies

$$\{\tilde{E}^h, \tilde{H}^h, \tilde{\lambda}^h\} = T_{F,G}\{\tilde{E}^h, \tilde{H}^h, \tilde{\lambda}^h\}$$

$$= T_{0,0}\{\tilde{E}^h, \tilde{H}^h, \tilde{\lambda}^h\} + T_{F,G}\{0, 0, 0\}. \qquad (33)$$

Then the subtraction of (33) from (32) gives

$$\{u^n, v^n, \theta^n\} = T_{0,0}\{u^{n-1}, v^{n-1}, \theta^{n-1}\}.$$

Thus solving (22) for all j is exactly the application of the operator $T_{0,0}$ to $\{u^{n-1}, v^{n-1}, \theta^{n-1}\}$, and therefore the error reduction at each iteration is dominated by the spectral radius of $T_{0,0}$, which depends on the choice of β. We wish to find a way of optimal choice β.

Let γ be an eigenvalue of $T_{0,0}$ and $\{E, H, \lambda\}$ the associated eigenvector, so that

$$T_{0,0}\{E, H, \lambda\} = \gamma\{E, H, \lambda\}.$$

Immediately, (27) leads to

$$R(T_{0,0}\{E, H, \lambda\}) = |\gamma|^2 R(\{E, H, \lambda\}). \qquad (34)$$

On the other hand, Theorem 4.3, incorporated with (24) and (25), gives the equality:

$$R(T_{0,0}\{E, H, \lambda\}) = R(\{E, H, \lambda\})$$

$$- 8 \Bigg[\beta_R \operatorname{Re} \sum_{j,k \in I^h} \langle\langle \lambda_{jk}, \pi_\tau E_j \rangle\rangle_{\Gamma_{jk}}$$

$$+ \beta_I \operatorname{Im} \sum_{j,k \in I^h} \langle\langle \lambda_{jk}, \pi_\tau E_j \rangle\rangle_{\Gamma_{jk}} \Bigg]. \qquad (35)$$

Thus, a combination of (34) and (35) gives

$$|\gamma|^2 = 1 - \frac{8}{R(\{E,H,\lambda\})}\left[\beta_R \operatorname{Re}\sum_{j,k\in I^h}\langle\langle\lambda_{jk},\pi_\tau E_j\rangle\rangle_{\Gamma_{jk}}\right.$$

$$\left.+\beta_I \operatorname{Im}\sum_{j,k\in I^h}\langle\langle\lambda_{jk},\pi_\tau E_j\rangle\rangle_{\Gamma_{jk}}\right].\tag{36}$$

Obviously, $|\gamma| < 1$ if β satisfies

$$\beta_R \operatorname{Re}\sum_{j,k\in I^h}\langle\langle\lambda_{jk},\pi_\tau E_j\rangle\rangle_{\Gamma_{jk}}$$

$$+\beta_I \operatorname{Im}\sum_{j,k\in I^h}\langle\langle\lambda_{jk},\pi_\tau E_j\rangle\rangle_{\Gamma_{jk}} > 0.\tag{37}$$

The next step is to derive a bound for $R(\{E,H,\lambda\})$, hopefully, in certain norms of E, H, and λ. For this, we observe that, since $\{E,H,\lambda\}$ is an eigenvector of $T_{0,0}$ with the associate eigenvalue γ, (31) with $F = G = 0$ leads to

$$((i\omega\varepsilon+\sigma)E_j,\varphi)_j - (H_j,\nabla\times\varphi)_j - \sum_k\langle\langle\lambda_{jk},\pi_\tau\varphi\rangle\rangle_{\Gamma_{jk}},$$

$$+\langle\langle\alpha\pi_\tau E_j,\pi_\tau\varphi\rangle\rangle_{\Gamma_j} = 0,\quad \varphi\in\mathbf{V}_j^h,\tag{38a}$$

$$i\omega(\mu H_j,\psi)_j + (\nabla\times E_j,\psi)_j = 0,\quad \psi\in W_j^h,\tag{38b}$$

$$\gamma\lambda_{jk} = -\lambda_{kj} + \beta(\pi_\tau E_k - \gamma\pi_\tau E_j)(m_{jk}),\quad j < k,\tag{38c}$$

$$\lambda_{jk} = -\lambda_{kj} + \beta(\pi_\tau E_k - \pi_\tau E_j)(m_{jk}),\quad j > k.\tag{38d}$$

Due to (27) and (38),

$$R(\{E,H,\lambda\}) = \sum_{j<k}\left|\frac{1}{\gamma}(\lambda_{kj}-\beta\pi_\tau E_k(m_{jk}))\right|^2_{0,\Gamma_{jk}}$$

$$+\sum_{j>k}\left|\lambda_{kj}+\beta\pi_\tau E_k(m_{jk})\right|^2_{0,\Gamma_{jk}}$$

$$=\sum_{j<k}\left|\lambda_{jk}+\beta\pi_\tau E_j(m_{jk})\right|^2_{0,\Gamma_{jk}}$$

$$+\sum_{j>k}\left|\frac{1}{\gamma}(\lambda_{jk}-\beta\pi_\tau E_j(m_{jk}))\right|^2_{0,\Gamma_{jk}}$$

$$\leq\frac{1}{|\gamma|^2}\Big\{\sum_{j<k}\left|\lambda_{jk}+\beta\pi_\tau E_j(m_{jk})\right|^2_{0,\Gamma_{jk}}$$

$$+\sum_{j>k}\left|\lambda_{jk}-\beta\pi_\tau E_j(m_{jk})\right|^2_{0,\Gamma_{jk}}\Big\}$$

$$\leq\frac{2}{|\gamma|^2}\sum_{j,k\in I^h}\Big\{\left|\lambda_{jk}\right|^2_{0,\Gamma_{jk}}$$

$$+|\beta|^2\left|\pi_\tau E_j(m_{jk})\right|^2_{0,\Gamma_{jk}}\Big\}\tag{39}$$

as desired.

The final step begins with combining (36) and (39) to have

$$|\gamma|^2 \leq \frac{1}{1+4M(\beta)},\tag{40}$$

where

$$M(\beta) = \left[\beta_R \operatorname{Re}\sum_{j,k\in I^h}\langle\langle\lambda_{jk},\pi_\tau E_j\rangle\rangle_{\Gamma_{jk}}\right.$$

$$\left.+\beta_I \operatorname{Im}\sum_{j,k\in I^h}\langle\langle\lambda_{jk},\pi_\tau E_j\rangle\rangle_{\Gamma_{jk}}\right]$$

$$\Big/\left[\sum_{j,k\in I^h}\Big\{\left|\lambda_{jk}\right|^2_{0,\Gamma_{jk}}+|\beta|^2\left|\pi_\tau E_j(m_{jk})\right|^2_{0,\Gamma_{jk}}\Big\}\right].\tag{41}$$

Observe that $\operatorname{Re}\sum_{j,k\in I^h}\langle\langle\lambda_{jk},\pi_\tau E_j\rangle\rangle_{\Gamma_{jk}} > 0$ and that (37) is equivalent to $M(\beta) > 0$.

The maximum M^* of $M(\beta)$ is taken where β is chosen optimally such that

$$|\beta|^2 = \frac{\sum_{j,k\in I^h}\left|\lambda_{jk}\right|^2_{0,\Gamma_{jk}}}{\sum_{j,k\in I^h}\left|\pi_\tau E_j(m_{jk})\right|^2_{0,\Gamma_{jk}}}\tag{42a}$$

and

$$\beta_I \operatorname{Re}\sum_{j,k\in I^h}\langle\langle\lambda_{jk},\pi_\tau E_j\rangle\rangle_{\Gamma_{jk}} = \beta_R \operatorname{Im}\sum_{j,k\in I^h}\langle\langle\lambda_{jk},\pi_\tau E_j\rangle\rangle_{\Gamma_{jk}}.\tag{42b}$$

In this case,

$$M^* = \frac{\left|\sum_{j,k\in I^h}\langle\langle\lambda_{jk},\pi_\tau E_j\rangle\rangle_{\Gamma_{jk}}\right|}{2\left[\sum_{j,k\in I^h}\left|\lambda_{jk}\right|^2_{0,\Gamma_{jk}}\right]^{1/2}\left[\sum_{j,k\in I^h}\left|\pi_\tau E_j(m_{jk})\right|^2_{0,\Gamma_{jk}}\right]^{1/2}},$$

and therefore, $|\gamma|$ satisfies

$$|\gamma|^2 \leq \frac{1}{1+4M^*}.$$

We summarize the above result in the following theorem, which suggests a method of the choice of the optimal β.

Theorem 4.7. *Let* $\{E,H,\lambda\}$ *be an eigenvector of* $T_{0,0}$ *with the associate eigenvalue* γ. *Assume that* (37) *holds.*

Then $|\gamma| < 1$. Moreover, β, with positive real part, is optimal if it satisfies (42)*; in this case, the following bound for $|\gamma|$ holds:*

$$|\gamma|^2 \le 1 \Big/ \left\{ 1 + 2 \left| \sum_{j,k\in I^h} \langle\langle \lambda_{jk}, \pi_\tau E_j \rangle\rangle_{\Gamma_{jk}} \right| \right.$$
$$\left. \left[\sum_{j,k\in I^h} |\lambda_{jk}|^2_{0,\Gamma_{jk}} \sum_{j,k\in I^h} |\pi_\tau E_j(m_{jk})|^2_{0,\Gamma_{jk}} \right]^{-1/2} \right\}. \quad (43)$$

Remark 4.8. *Notice that Conditions* (37) *and* (42b) *are equivalent to*

$$\beta_R \left\{ \left\|\sigma^{1/2}E\right\|^2 + \left\|\left|\alpha_R^{1/2}\pi_\tau E\right\|\right\|^2 \right\}$$
$$- \beta_I \left\{ \omega\left\|\mu^{1/2}H\right\|^2 - \omega\left\|\varepsilon^{1/2}E\right\|^2 + \left\|\left|\alpha_I^{1/2}\pi_\tau E\right\|\right\|^2 \right\} > 0$$

and

$$\beta_I \left\{ \left\|\sigma^{1/2}E\right\|^2 + \left\|\left|\alpha_R^{1/2}\pi_\tau E\right\|\right\|^2 \right\} = -\beta_R \left\{ \omega\left\|\mu^{1/2}H\right\|^2 \right.$$
$$\left. - \omega\left\|\varepsilon^{1/2}E\right\|^2 + \left\|\left|\alpha_I^{1/2}\pi_\tau E\right\|\right\|^2 \right\},$$

respectively.

Remark 4.9. *We remark that as $\left|\sum_{j,k\in I^h} \langle\langle \lambda_{jk}, \pi_\tau E_j \rangle\rangle_{\Gamma_{jk}}\right|$ vanishes, so do E and H, and $|\gamma|$ tends to 1.*

Remark 4.10. *With minor modifications in the above proof of Theorem 4.7, β's can be chosen such that they vary according to j and k. Also, practically it suffices to update β only once after certain number of iterations, instead of updating β at every iteration. However, the true solution $\{\tilde{E}^h, \tilde{H}^h, \tilde{\lambda}^h\}$ is not known in the actual computations, and hence the use of optimal choice of β as in Theorem 4.7, or its approximation, still needs further investigations.*

Remark 4.11. *It should be observed that the optimal β in Theorem 4.7 depends on the values of $\pi_\tau E$ and λ on the interfaces Γ_{jk}'s. Theorem 4.12 establishes the spectral radius estimate and choice of β to guarantee an optimal order of convergence. In actual computations, however, Theorem 4.7 suggests a much more efficient choice of β than this, although it is heuristic: for this, see the next section on numerical results.*

Instead of varying β as in Theorem 4.7, consider the case in which β is independent of solutions. For the following theorem, set

$$\zeta = \max_j \left\{ \frac{h_{\max(\Omega_j)}}{h_{\min(\Omega_j)}} \right\},$$

$$h_{\max} = \max_j \left\{ h_{\max}(\Omega_j) \right\}, \quad h_{\min} = \min_j \left\{ h_{\min}(\Omega_j) \right\},$$

where $h_{\min}(\Omega_j)$ and $h_{\max}(\Omega_j)$ are the minimum and maximum edge length of Ω_j, respectively. The following theorem gives an asymptotic estimate for the spectral radius of $T_{0,0}$.

Theorem 4.12. *Let $\rho(T_{0,0})$ be the spectral radius of $T_{0,0}$. Suppose $\mu_* > 0$. If β is chosen such that $\beta_R\alpha_{R_*} - \beta_I\alpha_{I_*} > 0$ and $\beta_R\sigma_* + \beta_I\omega\varepsilon^* > 0$ with $\beta_R > 0, \beta_I \le 0$, then*

$$|\rho(T_{0,0})| \approx 1 - Ch_{\min}. \quad (44)$$

Proof. We begin with finding a lower bound for $M(\beta)$ given by (41), which will give an upper bound for $|\gamma|$ by (40). For this, let $\tilde{\Omega}_j$ be an arbitrary element among Ω_j's and $\tilde{\Gamma}_{jk}$ an interior face of $\tilde{\Omega}_j$ common with another element Ω_k. Choose $\tilde{\varphi}$ in (38) such that

$$\pi_\tau\tilde{\varphi}(m_{jk}) = \begin{cases} \tilde{\lambda}_{jk} & \text{on} \quad \tilde{\Gamma}_{jk}, \\ 0 & \text{on} \quad \partial\Omega_j \setminus \tilde{\Gamma}_{jk}. \end{cases}$$

Recall the estimates given in Douglas Jr., Santos, and Sheen (2000):

$$\|\tilde{\varphi}\|^2_{0,\Omega_j} \le C_1 \langle\langle\tilde{\lambda}_{jk}, \tilde{\lambda}_{jk}\rangle\rangle_{\Gamma_{jk}}, \quad (45a)$$

$$\|\mathbf{\nabla}\times\tilde{\varphi}\|^2_{0,\Omega_j} \le C_2 \langle\langle\tilde{\lambda}_{jk}, \tilde{\lambda}_{jk}\rangle\rangle_{\Gamma_{jk}}, \quad (45b)$$

$$\langle\langle\pi_\tau E_j, \pi_\tau E_j\rangle\rangle_{\Gamma_{jk}} \le C_3 \|E_j\|^2_{0,\Omega_j}, \quad (45c)$$

where

$$C_1 = C\max_j \frac{h_{\max}(\Omega_j)^2}{h_{\min}(\Omega_j)}, \quad C_2 = C\max_j \frac{h_{\max}(\Omega_j)^2}{h_{\min}(\Omega_j)^3}, \quad (46)$$

$$C_3 = C\max_j \frac{h_{\max}(\Omega_j)}{h_{\min}(\Omega_j)^2}.$$

It follows from (38a) and (45) that

$$\langle\langle\tilde{\lambda}_{jk}, \tilde{\lambda}_{jk}\rangle\rangle_{\Gamma_{jk}} = \big((i\omega\varepsilon + \sigma)E_j, \tilde{\varphi}\big)_j - (H_j, \mathbf{\nabla}\times\tilde{\varphi})_j$$
$$\le \big[\sqrt{C_1}\big(\omega\|\varepsilon E_j\|_{0,\Omega_j} + \|\sigma E_j\|_{0,\Omega_j}\big)$$
$$+ \sqrt{C_2}\|H_j\|_{0,\Omega_j}\big] \langle\langle\tilde{\lambda}_{jk}, \tilde{\lambda}_{jk}\rangle\rangle^{1/2}_{\Gamma_{jk}},$$

from which it is immediate that

$$\langle\langle\lambda_{jk},\lambda_{jk}\rangle\rangle_{\Gamma_{jk}} \leq C_1\left(\omega^2\left\|\varepsilon E_j\right\|_{0,\Omega_j}^2 + \left\|\sigma E_j\right\|_{0,\Omega_j}^2\right)$$
$$+ C_2\left\|H_j\right\|_{0,\Omega_j}^2.$$

Assume that $\beta_R\sigma_* + \beta_I\omega\varepsilon^* > 0$ and $\mu_* > 0$. Choose $\beta_R > 0$ and $\beta_I \leq 0$. Then, a combination of (41), (24), (25), (45c), (47), and (46) results in

$$M(\beta) = \left[\beta_R\operatorname{Re}\sum_{j,k\in I^h}\langle\langle\lambda_{jk},\pi_\tau E_j\rangle\rangle_{\Gamma_{jk}}\right.$$

$$\left.+\beta_I\operatorname{Im}\sum_{j,k\in I^h}\langle\langle\lambda_{jk},\pi_\tau E_j\rangle\rangle_{\Gamma_{jk}}\right]$$

$$/\left[\sum_{j,k\in I^h}\left\{\left|\lambda_{jk}\right|_{0,\Gamma_{jk}}^2 + |\beta|^2\left|\pi_\tau E_j(m_{jk})\right|_{0,\Gamma_{jk}}^2\right\}\right]$$

$$= \left[\beta_R\left(\sigma\|E\|^2 + \alpha_R|||\pi_\tau E|||^2\right)\right.$$

$$\left.+\beta_I\left(\omega\varepsilon\|E\|^2 - \omega\mu\|H\|^2 - \alpha_I|||\pi_\tau E|||^2\right)\right]$$

$$/\left[\sum_{j,k\in I^h}\left\{\left|\lambda_{jk}\right|_{0,\Gamma_{jk}}^2 + |\beta|^2\left|\pi_\tau E_j(m_{jk})\right|_{0,\Gamma_{jk}}^2\right\}\right]$$

$$= \left[(\beta_R\sigma + \beta_I\omega\varepsilon)\|E\|^2 - \beta_I\omega\mu\|H\|^2\right.$$

$$\left.+(\beta_R\alpha_R + \beta_I\alpha_I)|||\pi_\tau E|||^2\right]$$

$$/\left[\sum_{j,k\in I^h}\left\{\left|\lambda_{jk}\right|_{0,\Gamma_{jk}}^2 + |\beta|^2\left|\pi_\tau E_j(m_{jk})\right|_{0,\Gamma_{jk}}^2\right\}\right]$$

$$\geq \left[(\beta_R\sigma_* + \beta_I\omega\varepsilon^*)\|E\|^2 - \beta_I\omega\mu_*\|H\|^2\right.$$

$$\left.+(\beta_R\alpha_R - \beta_I\alpha_I)\frac{h_{\min}}{Ch_{\max}^2}\|E\|^2\right]$$

$$/\left[C_1(\omega^2\varepsilon^{*2} + \sigma^{*2})\|E\|^2 + C_2\|H\|^2\right.$$

$$\left.+|\beta|^2C_3\|E\|^2\right]$$

$$= \left[\left(\beta_R\sigma_* + \beta_I\omega\varepsilon^* + (\beta_R\alpha_R - \beta_I\alpha_I)\frac{h_{\min}}{Ch_{\max}^2}\right)\times\right.$$

$$\left.\times\|E\|^2 - \beta_I\omega\mu_*\|H\|^2\right]$$

$$/\left[C_1(\omega^2\varepsilon^{*2} + \sigma^{*2})\|E\|^2 + C_2\|H\|^2\right.$$

$$\left.+|\beta|^2C_3\|E\|^2\right]$$

$$\geq \frac{1}{C_3}\left[\min\left\{\beta_R\sigma_* + \beta_I\omega\varepsilon^* + (\beta_R\alpha_R - \beta_I\alpha_I)\times\right.\right.$$

$$\times\frac{h_{\min}}{Ch_{\max}^2} - \beta_I\omega\mu_*\bigg\}\left\{\|E\|^2 + \|H\|^2\right\}\bigg]$$

$$/\left[\left[C_1C_3^{-1}(\omega^2\varepsilon^{*2} + \sigma^{*2}) + C_2C_3^{-1} + |\beta|^2\right]\right.$$

$$\left\{\|E\|^2 + \|H\|^2\right\}\bigg]$$

$$\geq \frac{h_{\min}}{C\zeta}\left[\min\left\{\beta_R\sigma_* + \beta_I\omega\varepsilon^* + (\beta_R\alpha_R - \beta_I\alpha_I)\times\right.\right.$$

$$\times\frac{h_{\min}}{Ch_{\max}^2} - \beta_I\omega\mu_*\bigg\}\bigg]$$

$$/\left[\left[C_1C_3^{-1}(\omega^2\varepsilon^{*2} + \sigma^{*2}) + C_2C_3^{-1} + |\beta|^2\right]\right]$$

The maximum of the last quantity is taken if we choose β such that

$$|\beta| = \sqrt{h_{\max}h_{\min}(\omega^2\varepsilon^{*2} + \sigma^{*2}) + \zeta}, \qquad (47)$$

$$\beta_R\left(\sigma_* + \alpha_R\frac{h_{\min}}{Ch_{\max}^2}\right) = -\beta_I\left(\omega\varepsilon^* + \omega\mu_* - \alpha_I\frac{h_{\min}}{Ch_{\max}^2}\right).$$

With this β, we have that $M(\beta) \geq \frac{h_{\min}}{C}$, and thus $|\gamma| \approx 1 - Ch_{\min}$.

This completes the proof.

$\square$

Remark 4.13. *Also observe that the results corresponding to Theorem 4.7 and Theorem 4.12 for the Jacobi-type procedure are immediate. Replacing only* (40) *by*

$$|\gamma|^2 \leq \frac{1}{1 + 4M(\beta)},$$

instead of (43), *one has*

$$|\gamma|^2 \leq 1\bigg/\left[1 + \left|\sum_{j,k\in I^h}\langle\langle\lambda_{jk},\pi_\tau E_j\rangle\rangle_{\Gamma_{jk}}\right|\right.$$

$$\left.\left\{\sum_{j,k\in I^h}\left|\lambda_{jk}\right|_{0,\Gamma_{jk}}^2\sum_{j,k\in I^h}\left|\pi_\tau E_j(m_{jk})\right|_{0,\Gamma_{jk}}^2\right\}^{-1/2}\right],$$

and then the estimate for the spectral radius will be accordingly affected. Indeed instead of (44),

$$|\gamma| \approx 1 - Ch_{\min}$$

with the same constant C as in (44).

4.4 The red-black Seidel-type procedure

In order to supplement the lack in parallelism of *Seidel-type iterative procedure*, *a red-black Seidel-type procedure* is proposed in what follows. Let the set of subdomain indices be divided into the two parts I_R and I_B, so that

$$\overline{\Omega} = \left[\cup_{j\in I_R}\overline{\Omega}_j\right] \bigcup \left[\cup_{j\in I_B}\overline{\Omega}_j\right], \quad \Omega_j \cap_{j\neq k} \Omega_k = \emptyset,$$

and each element $\Omega_j, j \in I_R$, is not adjacent to any element $\Omega_k, k \in I_R$. The *red-black Seidel-type iterative procedure* corresponding to Problem (19) is based on the the following updating scheme for $\lambda_{jk}^{h,n} \forall k \forall j$:

$$\lambda_{jk}^{h,n} = \begin{cases} -\lambda_{kj}^{h,n-1} + \beta_{jk}(\pi_\tau E_k^{h,n-1} - \pi_\tau E_j^{h,n})(m_{jk}) \\ \qquad\qquad \forall k, \forall j \in I_R, \\ -\lambda_{kj}^{h,n} + \beta_{jk}(\pi_\tau E_k^{h,n} - \pi_\tau E_j^{h,n})(m_{jk}) \\ \qquad\qquad \forall k, \forall j \in I_B. \end{cases} \quad (48a)$$

$$(48b)$$

Then the *red-black Seidel-type iterative procedure* consists of the alteration of the two substeps: the first substep of updating (48a) and solving (19a) and (19b) for all $j \in I_R$, and the second substep of updating (48b) and solving (19a) and (19b) for all $j \in I_B$.

Introduce the pseudo-energy for *red-black Seidel-type iterative procedure* in the form

$$R^n(\{u^n, v^n, \theta^n\}) = \sum_{j\in I_R}\sum_k \left|\theta_{jk}^n + \beta\pi_\tau u_j^n(m_{jk})\right|^2$$
$$+ \sum_{j\in I_B}\sum_k \left|\theta_{kj}^n + \beta\pi_\tau u_k^n(m_{jk})\right|^2.$$

With this pseudo-energy, all the results in the previous subsections for *Seidel-type* iterative procedure remain valid for *red-black Seidel-type iterative procedure*.

5 Numerical Results

Let the computational domain Ω be given as $[0, 4 \times 10^3m]^3$. For the physical parameters in our numerical simulation we use the data following Coggon (1971). Let the frequency ω be $10Hz$, and the electric permittivity and magnetic permeability contrasts $\varepsilon/\varepsilon_0, \mu/\mu_0$ be 10 and 1, where ε_0 is the electric permittivity of the vacuum and μ_0 the magnetic permeability of the free space. Let the
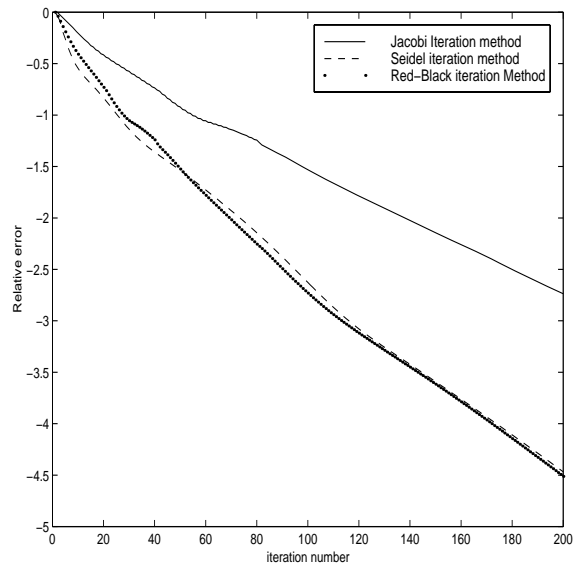

Figure 1 : The logarithmic relative L^2-error with the fixed β value

conductivity σ be $0.01\,S/m(Siemens/meter)$. The coefficient matrix α in the absorbing boundary condition (3c) is given by the diagonal matrix $(1-i)\sqrt{\frac{\sigma}{2\omega\mu}}I$. In the rest of the section some numerical results are illustrated to compare the classical *Gauss-Jacobi type algorithm* with the presented *Gauss-Seidel type* and *red-black type ones* with various updating procedures.

5.1 The case of homogeneous problems

Setting $F = G = 0$, we investigate in the convergence of the solutions of (3) to zeros. Denote by $\{u^{h,n}, v^{h,n}, \theta^{h,n}\}$ the numerical solution at the nth step error equations (22), and we calculate

$$\mathbf{Log}_{10}\left[\frac{||u^{h,n}||_0 + ||v^{h,n}||_0 + ||\nabla \times u^{h,n}||_{0,h}}{||u^{h,1}||_0 + ||v^{h,1}||_0 + ||\nabla \times u^{h,1}||_{0,h}}\right]$$

as logarithmic relative L^2-errors. Here, $u^{h,n}, v^{h,n}$ are the solutions at the nth iterative step. For the numerical computations in this subsection, the domain Ω is divided into the $80 \times 80 \times 80$ congruent subdomains.

We consider the case of updating procedure with a fixed β satisfying (47). Figure 1 shows the comparison of the Gauss-Jacobi, Gauss-Seidel, and red-black type iterative procedures. For all cases the speed of convergence of the Gauss-Seidel and Red-Black type procedures are faster (approximately by a factor of two) than that of the Gauss-Jacobi type one.

5.2 The case of nonhomogeneous problems

Let us take F and G such that the exact solutions of (3) are

$$
\begin{aligned}
E_{ex} &= (\phi(x,y,z),0,0) \\
H_{ex} &= 2000 \times \alpha \left(0, \frac{\partial\phi(x,y,z)}{\partial z}, -\frac{\partial\phi(x,y,z)}{\partial y}\right)
\end{aligned}
$$

where

$$
\phi(x,y,z) = \left(\frac{x}{2000}+1\right)\left(\frac{x}{2000}-1\right)\frac{y}{2000}\left(\frac{z}{2000}-1\right).
$$

As a synthetic model for nonhomogeneous case, we test the model in COMMEMI project Zhdanov, Varentsov, Weaver, Golubev, and Krylov (1997). It consists of a conductive block of 2 S/m embedded in a homogeneous body with conductivity $0.01\,S/m$. The anomaly size is $1\,km \times 1\,km \times 2\,km$ in the computational domain $4\,km \times 4\,km \times 4\,km$.
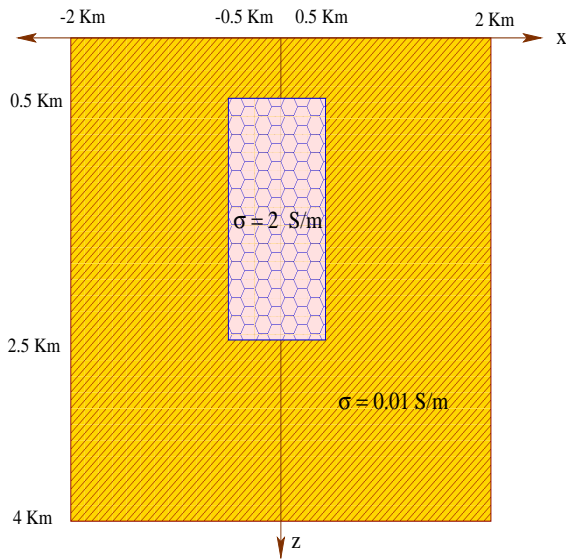

Figure 2 : Two-dimensional slice of synthetic anomaly model at $y = 0$

We define the relative L^2-error $\mathcal{E}_{rel}^{h,n}$ by the quotient

$$
\frac{\|E_{ex}-E^{h,n}\|_0 + \|H_{ex}-H^{h,n}\|_0 + \|\nabla\times(E_{ex}-E^{h,n})\|_{0,h}}{\|E_{ex}\|_0 + \|H_{ex}\|_0 + \|\nabla\times E_{ex}\|_{0,h}},
$$

where we used red-black type procedure with the stopping criterion such that if the relative error change from

the $n-1$st step to the current nth step is less than 0.001; that is,

$$
1 - \frac{\mathcal{E}_{rel}^{h,n}}{\mathcal{E}_{rel}^{h,n-1}} \leq 0.001.
$$

Although the theoretical convergence given in Theorem 3.1 is suboptimal, we remark that an optimal convergence rate of $O(h)$ can be observed in Table 1.

Table 1 : Relative L^2 errors and reduction ratio r

h (km)	Num. of Grids	Rel. L^2-Error	Red. Ratio r
1/4	$16\times 16\times16$	0.084063	
1/8	$32\times 32\times32$	0.042073	0.998
1/16	$64\times 64\times64$	0.021045	0.999
1/32	$128\times 128\times128$	0.010546	0.996

Figures 3 and 4 plot horizontal slices of the real part of the electric field $E_1^{h,N}$ across the source location for the homogeneous and nonhomogeneous cases, respectively. For these figures the stopping was made when

$$
\begin{aligned}
&\left[\|E^{h,n+1}-E^{h,n}\|_0 + \|H^{h,n+1}-H^{h,n}\|_0 \right. \\
&\quad \left. + \|\nabla\times(E^{h,n+1}-E^{h,n})\|_{0,h}\right] \\
&\Big/ \left[\|E^{h,n}\|_0 + \|H^{h,n}\|_0 + \|\nabla\times E^{h,n}\|_{0,h}\right]
\end{aligned}
$$

was 0.00017. From the snapshots in Figures 3 and 4 one may observe that the absorbing boundary conditions (3c) perform well.

5.3 Calculus of the spectral radius

In this subsection we will compute and confirm numerically the actual rate of convergence of the domain decomposition iteration that is estimated by the spectral radius given in (44) in §4.3. For this, let E, $E^{h,n}$ and $E^{h,\infty}$ be the exact, n-th iterative, and the limiting iterative solutions, respectively. The n-th error $e^{h,n} = \|E-E^{h,n}\|_{L^2(\Omega)}$ is then estimated by

$$
\begin{aligned}
e^{h,n} &\leq \|E-E^{h,\infty}\|_{L^2(\Omega)} + \|E^{h,\infty}-E^{h,n}\|_{L^2(\Omega)} \\
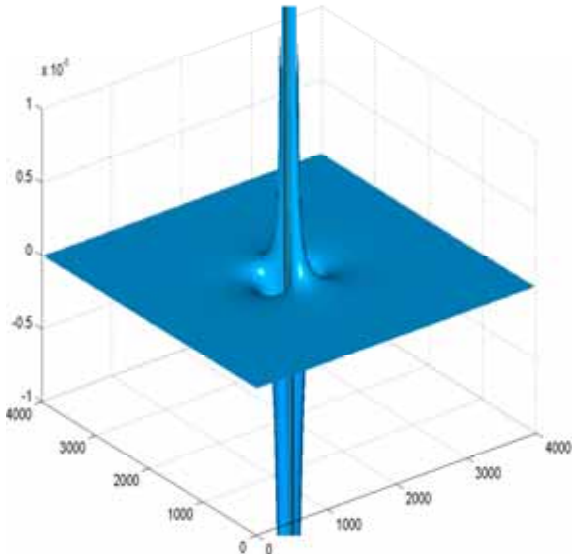&\leq C_1\|E\|_{L^2(\Omega)}h^{\alpha_1} + C_2(1-C_3 h^{\alpha_2})^n.
\end{aligned}
$$

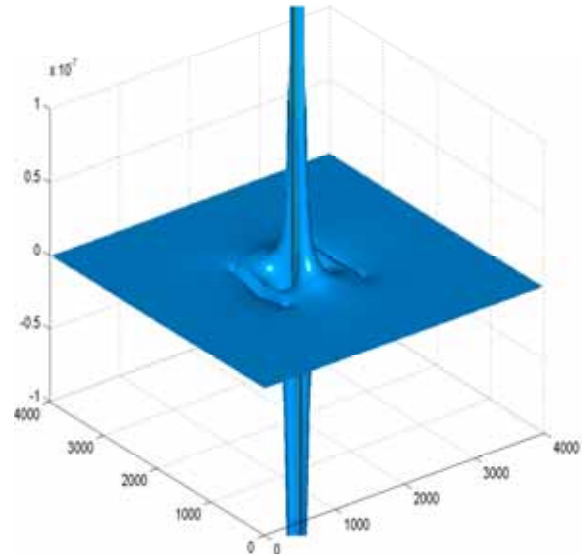**Figure 3** : The real part of $e_1^{h,N}$ for homogeneous model

Figure 4 : The real part of $e_1^{h,N}$ for nonhomogeneous model

Since α_1 is at least 0.5 by Theorem 3.1 but nearly 1 by the numerical experiment given in §5.2, we will concentrate on calculating only C_3 and α_2 which relate with the iterative procedure. By using the following relations

$$e^{h,n} - e^{h,n+1} \approx C_2(1 - C_3 h^{\alpha_2})^n C_3 h^{\alpha_2}$$

$$e^{h,n+1} - e^{h,n+2} \approx C_2(1 - C_3 h^{\alpha_2})^{n+1} C_3 h^{\alpha_2},$$

we have

$$\frac{e^{h,n+1} - e^{h,n+2}}{e^{h,n} - e^{h,n+1}} \approx 1 - C_3 h^{\alpha_2}$$

$$\frac{e^{2h,n+1} - e^{2h,n+2}}{e^{2h,n} - e^{2h,n+1}} \approx 1 - C_3 2^{\alpha_2} h^{\alpha_2}.$$

It seems to be improper to take a fixed number n, and hence we obtain the values by taking averages through $n = 1$ to a certain number (=50). In Table 2, we present numerical calculus with β satisfying (47) for the spectral radius coefficients C_3 and α_2:

$$\rho(T_{0,0}) \leq 1 - C_3 h^{\alpha_2}.$$

Table 2 : Numerical calculus of coefficients in the spectral radius estimation

	Jacobi-type	Red-Black
C_3	0.537675	1.152814
α_2	1.078837	1.156271

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