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# A Local Strictly Nondecreasing Material Law for Modeling Softening and Size-Effect: a Discrete Approach

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**Abstract:** In this study nonlocality is discussed with regard to the differential and discrete formulations. Here, nonlocality is found to be a concept attaining not to the description of the material, but to the governing equations. This has made it possible to discuss the opportunity of introducing nonlocality in the constitutive equations, in order to give respectability to strain-softening damage models. When using the differential formulation, a length scale must be introduced into the material description of a strain-softening modeling, particularly when the size-effect is involved. In the opinion of the Author, this need lies in the basics themselves of the differential formulation, performing the limit process. Actually, with the reduction of global variables to point (and instant) variables, we loose metrics. Consequently, metrics must be reintroduced a-posteriori, by means of a length scale, if we want to describe more than 0-dimensional (nonlocal) effects. Here it is shown how a length scale is intrinsic in Physics. Avoiding the limit process, that is, using a discrete formulation, we preserve the length scale of Physics and do not need to recover it. In this sense, it may be asserted that the discrete formulation is nonlocal in itself and does not require nonlocal constitutive relationships for modeling nonlocal effects. Obtaining a nonlocal formulation by using local constitutive laws and discrete operators seems to be possible and physically appealing. Numerical results are provided here, showing how a formulation using discrete operators and a local constitutive law is able to model softening and size-effect, which is impossible for differential local approaches. The mathematical and physical well-posedness and the existence itself of strain-softening are also discussed.

**keyword:** Strain-softening, size-effect, effective law, nonlocality, Cell Method, meshfree

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## 1 Introduction

Recently, a new experimental procedure for identifying the constitutive law in uniaxial compression has been proposed for brittle heterogeneous materials: the procedure of the effective law [Ferretti (2001); Ferretti and Di Leo (2003); Ferretti (2004b)]. This procedure produces evidence against the existence of strain-softening (i.e. the decline of stress at increasing strain) and identifies a monotone strictly nondecreasing material law for concrete specimens in uniaxial compression (Fig. 2), whose average stress versus average strain diagrams, $\overline{\sigma} - \overline{\varepsilon}$, are softening (Fig. 1).

The procedure is based on the idea that strain-softening is not a real material property, such as argued in several papers of last century, particularly of the '80s [Hadamard (1903); Hudson et al. (1971); Dresher and Vardoulakis (1982); Bergan (1983); Hegemier and Read (1983); Sandler and Wright (1983); Wu and Freud (1983)], and in a recent discussion on the existence of strain-softening [Ferretti (2004a)], which analyses the problem from the analytical and physical points of view. The effective law is then identified experimentally, being careful to properly treat the acquired data when the material law is derived from the experimental load-displacement curves, $N - u$. Actually, it is a common practice to define the stress and strain at a point as the average stress $\overline{\sigma}$ on the nominal area, $A_n$, and the average strain $\overline{\varepsilon}$ on the specimen height, $L$. This corresponds to assuming that the cross-sectional area does not change until crushing, which occurs suddenly, with propagation of sub-vertical macro-cracks (Fig. 3a). However, the actual failure mechanism of concrete specimens develops internally, with internal macro-cracks propagating through the specimens from the very beginning of the compression test. In cylindrical specimens, these cracks isolate a resistant inner core of bi-conic shape (Fig. 3b), while the outer part is expelled along the radial direction and splits into several portions (scheme in Fig. 3a). At the end of the

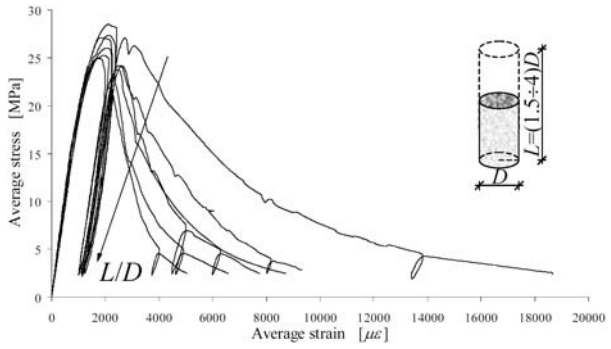


**Figure 1** : Size effect for the $\overline{\sigma}-\overline{\varepsilon}$ diagrams

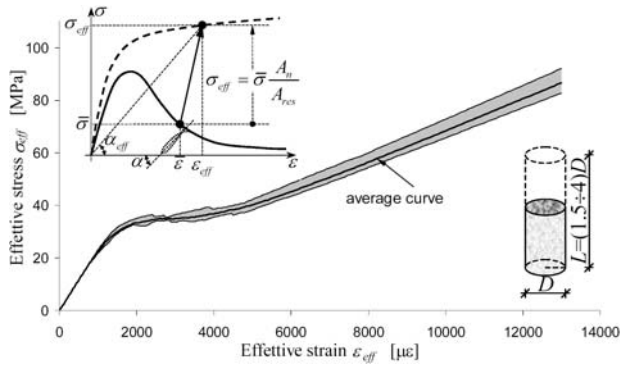


**Figure 2** : Dispersion range of the effective law for variable slenderness and average curve

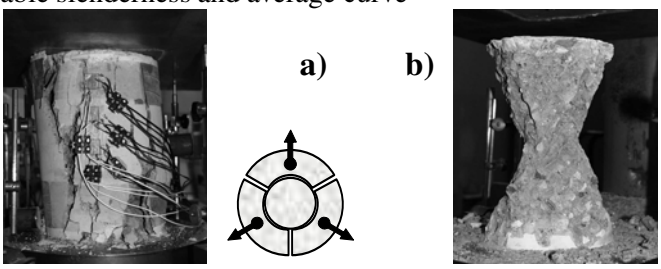


**Figure 3** : Concrete specimen at the end of the test, with scheme of splitting on the middle cross-section **(a)**, and concrete specimen at the end of the test, after removal of the outer part **(b)**

test, no evident crack propagation seems to afflict the biconic inner core (Fig. 3b). The sub-vertical cracks on the external surface are a secondary effect of the actual failure mechanism, which gradually modifies the resistant structure and, consequently, the effective cross-sectional area, or resistant area $A_{res}$, of the specimen.

In Ferretti (2004a, b), the effective stress, $\sigma_{eff}$, and the effective strain, $\varepsilon_{eff}$, have been introduced as constitutive parameters describing the actual material behavior. Reductions of $A_{res}$ are evaluated experimentally, by means of microseismic and energetic damage parameters, and used for deriving $\sigma_{eff} = N/A_{res}$ from the experimental $N-u$ curves. $\varepsilon_{eff}$ is derived from the knowledge of $\sigma_{eff}$ and the behavior of unloading-reloading (Fig. 2). In conclusion, the identification procedure of the effective law does not consist of a mere scale factor: the material is separated from the structure scale and the constitutive behavior is no more the mirror image of a structural problem at a lower scale [Ferretti (2004e)]. Thus, the two curves, $N-u$ ($\overline{\sigma}-\overline{\varepsilon}$) and $\sigma_{eff}-\varepsilon_{eff}$, are not identical in shape. In particular, as previously said, $\sigma_{eff}-\varepsilon_{eff}$ turns out not to be softening even if $N-u$ is.

As can be appreciated in Fig. 2, the effective law is size-effect insensitive, since the dispersion range for specimens of different slenderness is very narrow. Other interesting results related to the identification procedure of the effective law concern the Poisson's ratio, the volumetric strain, and the viscosity. By means of strain acquisitions into the presumed resistant structure, it was found that the Poisson's ratio is almost independent of the longitudinal strain [Ferretti (2004c)], concrete never exhibits dilatancy [Ferretti (2004c)], and viscosity has no substantial effects on the macroscopic behavior of concrete solids [Ferretti (2004d)]. What we know as concrete dilatancy [Brace et al. (1966); Di Leo et al. (1979)] is an apparent effect, due to a technique of data acquisition which inadequately evaluates the influence on the experimental data of a failure mechanism with splitting and consequent crack openings [Ferretti (2004c)]. Moreover, since the displacement consists of two parts, one constitutive and one related to crack openings, the time-dependent behavior of displacements cannot be related solely to a material property. Also crack openings play a role in time-dependent effects. Therefore, displacement time-dependence is, partly, a structural (and not material) effect. The identification procedure of the effective law shows that displacement time-dependence is determined, mostly, not by material viscosity, but by crack propagation.

The question we want answer in this paper is whether or not the effective law is suitable for numerical analysis. Actually, the effective law is a local material law, with the stress at a given point uniquely depending on the current values, and also the previous history, of deformation

at that point only, and many years of research have shown that the classical local continuum concept, leading to constitutive models falling within the category of simple nonpolar materials [Noll (1972)], does not seem to be adequate for modeling heterogeneous materials. Local constitutive relationships between stress and strain tensors are not adequate for describing the mechanical behavior of solids in the classical differential formulation, since no material is an ideal continuum, decomposable into a set of infinitesimal material volumes, each of which can be described independently. All materials, natural and manmade, are characterized by microstructural details whose size ranges over many order of magnitude [Bažant and Jirásek (2002)]. In constructing a material model, one must select a certain resolution level below which the microstructural details are not explicitly visible. Instead of refining the explicit resolution level, it is often more effective to use various forms of generalized continuum formulation, dealing with material that are nonsimple or polar, or both. A list of enriched continuum models is provided in Bažant and Jirásek (2002). Among these, a great variety of nonlocal models was developed.

In the following, attention will be focused on nonlocal models. Some preliminary ideas on nonlocal elasticity can be traced back to the late $19^{th}$ century [Duhem (1893)]. Beginning with Krumhansl (1965), Rogula (1965), Eringen (1966), Kunin (1966), and Kröner (1968), the idea was promulgated that heterogeneous materials should properly be modeled by some type of nonlocal continuum. Nonlocal continua are continua in which the stress at a certain point is not a function of the strain at the same point, but a function of the strain distribution over a certain representative volume of the material centered at that point [Bažant and Chang (1984)]. Thus, nonlocality is tantamount to an abandonment of the principle of the local action of classical continuum mechanics [Bažant and Jirásek (2002)].

It will be shown here (Section 2.2) that nonlocal constitutive laws between stress and strain tensors are not strictly needed to construct a material model. They are required only if a differential formulation is used, since differential operators are local. In other words, the effective law is suitable for modeling nonlocal effects if used with a formulation which is nonlocal in itself (Section 4.3). A formulation involving nonlocality in itself, the Cell Method, will be presented in Section 3.

## 2 The differential formulation: merits and limits

About three centuries ago, Newton made an epoch with his Philosophiae Naturalis Principia Mathematica, by introducing the notion of limit process, a wonderful conquest for mankind. Since then, the elegance of the limit notion led to every experimental law being provided with a differential formulation.

With the advent of computers, about fifty years ago, the differential equations were discretized by means of one of various discretization methods, since the numerical solution cannot be achieved in the most general case if a discrete description of physical laws is not provided (Figs. 4, 5). The custom of operating in the context of a differential formulation, from the introduction of calculus forth, both when pursuing an analytical, approximated or numerical solution, did not leave any room to the possibility of attaining a direct discrete formulation of field equations. Nevertheless, the very need to discretize the differential equations in order to achieve a numerical solution gives rise to the question of whether it is possible to formulate the physical laws directly in an algebraic manner, through a direct finite formulation.

It was precisely the applicability to a wide range of real cases, allowed by the numerical analysis, which highlighted the limits of the differential formulation, i.e., the impossibility for the numerical solution to converge to the actual result for some peculiar cases. The unlimited trust in the powerful instrument of the limit process led the researchers not to search the reason for this in the differential formulation, but somewhere else. Here, we want to discuss whether some numerical problems may derive just from the consolidated custom of formulating the problem in differential form. In particular, our attention will be focused on modeling the size-effect, the impossibility of which in the context of the classical local continuum concept is one of the main reasons leading to the development of nonlocal approaches.

### *2.1 Nonlocal approaches*

Nonlocal models take into account possible interactions between the given point and other material points. Theoretically, the stress at a point can depend on the strain history in the entire body, but the long range interactions certainly diminish with increasing distance, and can be neglected when the distance exceeds a certain limit called the interaction radius $R$ [Jirásek and Bažant (2001)]. The

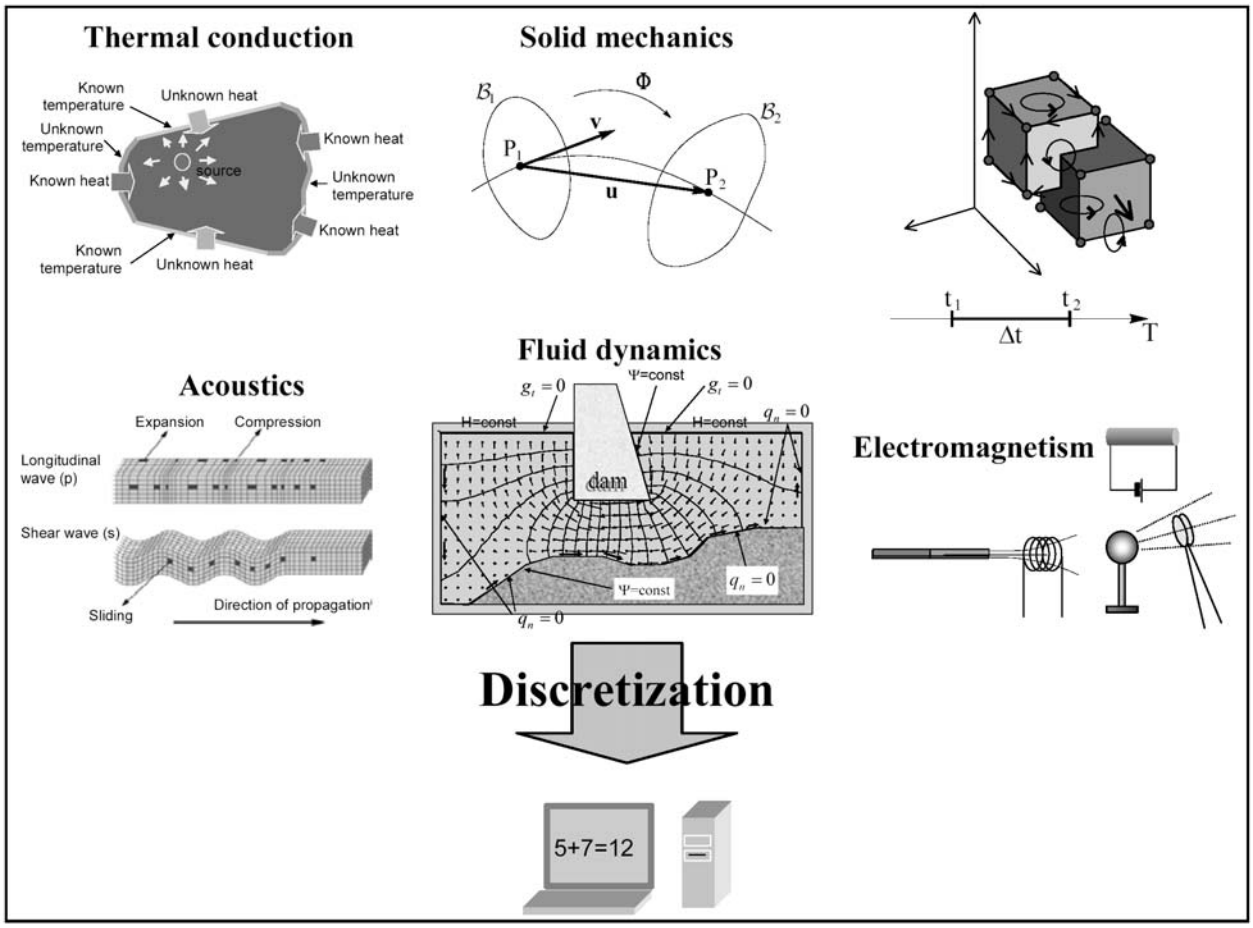


**Figure 4** : Discretization as an obligatory step in passing from Physics to its numerical analysis

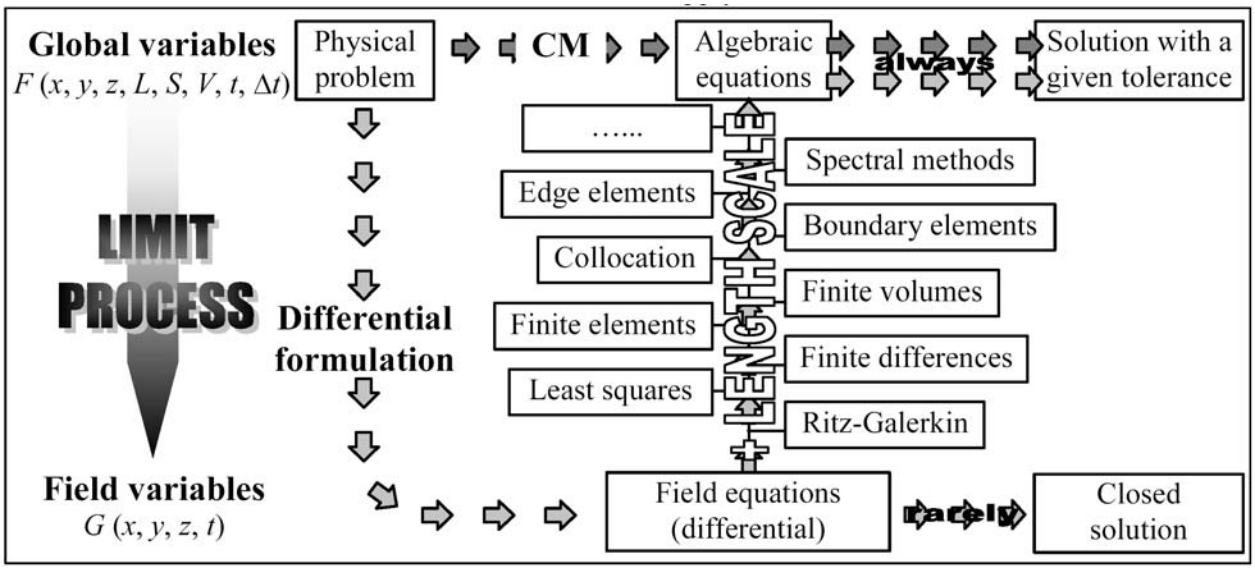


**Figure 5** : How to achieve the solution thought the Cell Method and the differential formulation

interval, circle or sphere, of radius $R$ is called the domain of influence.

As discussed in Bažant and Jirásek (2002), physical justifications of the nonlocality well-posedness may be summarized as follows:

1. Homogenization of the heterogeneous microstructure on a scale sufficiently small for it to be impossible to consider the smoothed strain field as uniform.
2. Homogenization of regular or statistically regular lattices or frames.
3. Need to capture the size-effects observed in experiments and in discrete simulations.
4. Impossibility to simulate numerically the observed distributed cracking with local continuum models.
5. Dependence of the microcrack growth on the average deformation of a finite volume surrounding the whole microcrack, and not on the local stress or strain tensor at the point corresponding to the microcrack center [Bažant (1991)].
6. Microcrack interaction, leading to either amplification of the stress intensity factor or crack shielding depending on the orientations of the microcracks, the orientation of the vectors joining the centers, and the size of the microcracks [Bažant (1994)].
7. Density of geometrically necessary dislocations in metals, whose effect, after continuum smoothing, naturally leads to a first-gradient model (metal plasticity).
8. Paradoxical situations or incorrect predictions arising from a Wiebull-type weakest link theory of quasibrittle structural failure on the assumption that the failure probability at a point of a material depends on the continuum stress at the point, rather than on the average strain from a finite neighborhood of that point.

Some studies have also been made to justify the characteristic length in the nonlocal approach by microstructure [Bažant and Pijaudier-Cabot (1989)]. A sophisticated explanation of the need for nonlocal terms in homogenized elastic models of random composites has been given by Drugan and Willis (1996) and Luciano and Willis (2001).

Nonlocal approaches were employed in various branches of physical sciences, e.g. in optimization of slider bearings [Rayleigh (1918)], or in modeling of liquid crystals [Oseen (1933)], radiative transfer [Chandrasekhar (1950)], electric wave phenomena in the cortex [Hodgkin (1964)], and solid mechanics [Rogula (1965)]. As far as the last branch is concerned, solid mechanics, there exist two types of problem motivated by the need to improve the classical continuum description with the incorporation of an internal length parameter: those with strain-softening and those with no strain-softening at all. They all share the common need of modeling the size-effect, which is impossible in the context of the classical plasticity. Only discrete numerical simulations, such as the random particle and lattice models [Bažant et al. (1990); Schlangen and van Mier (1992); Schlangen (1993); Jirásek and Bažant (1995); van Mier (1997)], have succeeded in bringing to light the existence of a nonstatistical size-effect.

Earlier studies on nonlocal elasticity [Duhem (1893); Krumhansl (1965); Rogula (1965); Eringen (1966); Kunin (1966); Kröner (1968)] were addressed to problems in which the size-effect is not caused by material softening. For the most part, these studies were motivated by homogenization of the atomic theory of Bravais. They aimed at a better description of phenomena taking place in crystals on a scale comparable to the range of interatomic forces [Jirásek and Rolshoven (2002)]. They showed that nonlocal continuum models approximately reproduce the dispersion of short elastic waves and improve the description of interactions between crystal defects such as vacancies, interstitial atoms and dislocations.

The term "nonlocal" has in the past been used with two meanings, one narrow and one broad. In the narrow sense, it refers strictly to the models with an averaging integral. In the broad sense, it refers to all the constitutive models that involve a characteristic length (material length), which also include the gradient models [Bažant and Jirásek (2002)]. Generally speaking, integral-type nonlocal models replace one or more state variables by their nonlocal counterparts, obtained by weighted averaging over a spatial neighborhood of each point under consideration [Jirásek and Bažant (2001)]. This leads to an abandonment of the principle of local action, as previously stated. In gradient-type nonlocal models, the principle of local action is preserved and the field in the

immediate vicinity of the point is taken into account by enriching the local constitutive relations with the first or higher gradients of some state variables or thermodynamic forces. A length scale is incorporated into the material description of both the integral- and gradient-type nonlocal models.

### 2.2 Where to include the length scale

According to Rogula's mathematical definition of nonlocality in the narrow sense, an operator $A$ is called local if, whenever $u(\mathbf{x}) = v(\mathbf{x})$ for all $\mathbf{x}$ in a neighborhood of the point $x_0$, then $Au(\mathbf{x}_0) = Av(\mathbf{x}_0)$ [Rogula (1982)]. As pointed out in Bažant and Jirásek (2002), the differential operators satisfy this condition, because the derivatives of an arbitrary order do not change if the differentiated functions change only outside a small neighborhood of the point at which the derivatives are taken. On the basis of this statement, the opportunity of including a length scale in the constitutive relationships is discussed here: in the opinion of the Author of the present paper, the locality of the differential operators is the main reason why nonlocal material models must be introduced in solid mechanics, in order to satisfy the nonlocality of physical phenomena. In other words, if the differential operators are local, any formulation using differential operators is local. Thus, the differential formulation is not adequate in itself for describing nonlocal effects.

The reason for this non-adequateness lies in the basics themselves of the differential formulation: performing the limit process. Actually, the differential formulation requires field functions (Fig. 5), which have to depend on point position, $x$, $y$, $z$, and instants, $t$. Only on this condition is it possible to find the derivatives and, then, to apply the differential formulation.

If the field functions are not directly described in terms of point position and instants, they are obtained from global variables by performing densities and rates (Fig. 5). Global variables are domain variables, depending on point position, $x$, $y$, $z$, and instants, $t$, but also on line extensions, $L$, areas, $S$, volumes, $V$, and time intervals, $\Delta t$. By performing a limit analysis of the mean global variables, we obtain a set of non-metric equations, with no reference to measures of length, area, and volume. Also the information on time-intervals dependence is lost. Thus, the multi-dimensional geometrical content of the global variables vanishes and we loose the possibility of describing more than 0-dimensional effects, that is, the nonlocal effects. In other words, performing the limit process acts as a projection from 3D Physics into 0D Physics. A description of phenomena living in more-than-zero-dimensional Physics is not possible in 0D Physics if a length scale is not supplied.

It is now a commonly accepted fact that the solution of a problem can be governed by the ratio of the physical dimensions of a structure to an intrinsic material length. Since this dependence on the size-effect cannot be resolved by a differential formulation, due to having lost the geometrical information, metrics must be reintroduced a-posteriori, in the discretization stage (Fig. 5), if we want to model the nonlocal effects. We are thus faced with two processes, one the inverse of the other, with no guarantee of them being perfectly dual as stands the geometrical content of the variables we are dealing with. The material lengths that are present in various forms of generalized continuum theories arise from the homogenization procedure and have their origin in the characteristics of the heterogeneous microstructure that are not explicitly resolved by the differential formulation. Nonlocal models have, as a matter of fact, two parameters with the dimension of length, one of them characterizing the length and the other the width of the process zone [Bažant and Jirásek (2002)].

One may ask, now, where the length scale must be reintroduced. In nonlocal approaches, a length scale is incorporated in constitutive laws, but there is no evidence that this choice is the only, or the physically most appealing one. In order to answer the question about where to reintroduce the length scale, let us examine the physical variables and their classification. In accordance with Hallen (1962), Penfield and Haus (1967), and Tonti (1972), all physical variables belong to one of the following three classes:

1. *Configuration variables*, describing the field configuration (displacements for solid mechanics, spatial velocity for fluidodynamics, electric potential for electrostatics, temperature for thermal conduction, etc.). All variables linked to configuration variables by operations of sum, integration, difference, division by a length, an area, a volume, and an interval, limit process, time and space derivatives are configuration variables.

2. *Source variables*, describing the field sources (forces for solid mechanics and fluidodynamics,

masses for geodesy, electric charges for electrostatics, electric currents for magnetostatics, heat sources for thermal conduction, etc.). All variables linked to source variables by operations of sum, integration, difference, product and division by a length, an area, a volume, and an interval, time and space derivatives are source variables.

3. *Energetic variables*, resulting from the product of a configuration variable for a source variable.

The equations relating configuration variables to each other and source variables to each other are structure equations, while those relating configuration to source variables are constitutive equations.

Now, each physical phenomenon occurs in space, and space has a multi-dimensional geometrical structure. If the physical variables are able to describe phenomena in space, such as they actually are, then the physical variables themselves have a multi-dimensional geometrical structure. Consequently, the variables are implicitly associated with geometrical objects provided with an extension (points, but also lines, areas and volumes). As an example, a flux is associated with a surface, a voltage with a line, and a mass content with a volume (Fig. 6). Displacements in solid mechanics, the kinetic potential in flow mechanics, the gauge function of electromagnetism, and the iconal function in optics are examples of variables associated with points in space (and time), without being densities or rates. In order to attain a proper description of Physics, this strict relationship between variables and geometrical structure of space must be preserved in the solving equations. As discussed previously, this preservation is not guaranteed by performing the limit process. Thus, reduction of global variables to point and instant variables is not physically appealing.

As far as Solid Mechanics is concerned, volume forces, which are source variables, are associated with a length scale in dimension 3, since their geometrical referent is a volume. Analogously, surface forces, which are source variables, have a two-dimensional geometrical referent (the surface); strains, which are configuration variables, have a one-dimensional geometrical referent (the line); and displacements, which are configuration variables, have a zero-dimensional geometrical referent (the point). In conclusion, it seems that dimensional scales and nonlocal effects are associated with the variables directly, and not with the equations relating variables by each other. Nonlocality seems to be a property of the governing equations, and not of the constitutive laws. In other words, Physics has an intrinsic length scale, in the sense that nonlocality attains to the physical phenomenon in its complex, and not necessarily to some type of material model. Consequently, if the problem is studied in the context of a formulation not preserving nonlocality, the differential formulation, nonlocality must be recovered by means of some type of enriched continuum models. Otherwise, if nonlocality were preserved by the formulation, there would be no longer any need to employ nonlocal material models.

Since performing the limit process is the main cause of nonlocality non-preservation in the differential formulation, such as previously discussed, we can then expect that a discrete formulation, which avoids the limit process, could be nonlocal in itself. Actually, if the differential formulation is abandoned in favor of a discrete one, the limit process is no longer performed and we can directly operate in 3D Physics. In this case, the length scale is naturally associated with global variables and nonlocal effects are intrinsically taken into account. As a proof of what has been asserted, one should consider that the theories of nonlocal elasticity advanced by Eringen and Edelen in the early 1970s [Edelen et al. (1971); Eringen (1972); Eringen and Edelen (1972)] attributed a nonlocal character to body forces, mass, entropy, and internal energy. These are all global variables whose geometrical referent is a volume. It is thus clear that they, like all variables whose geometrical referent is more than zero-dimensional, cannot be properly described in a context in which all variables are related to points.

Moreover, it is worth noting that, in the theories of nonlocal elasticity developed by Eringen and Edelen, nonlocality was a property of the elastic problem in its complex, and not only of the constitutive relationships. Thus, in these theories of nonlocality we can already find the idea that nonlocality is a property of the governing equations. Nevertheless, this idea was not developed any more, since the theories of nonlocal elasticity were too complicated to be calibrated and experimentally verified, let alone to be applied to any real problems [Bažant and Jirásek (2002)]. Treating only the stress-strain relationships as nonlocal, while the equilibrium and kinematic equations and the corresponding boundary conditions retain their standard form, was needed later [Eringen and Kim (1974); Eringen et al. (1977)], in order to provide

| *Physical theory* | **Global variable** | **Referent geometrical object** |
|---|---|---|
| *Thermal conduction* | **Temperature** | ***P*** |
| *Thermal conduction* | **Electrical potential** | ***P*** |
| *Solid mechanics* | **Displacement** | ***P*** |
| *Fluid mechanics* | **Velocity** | ***P*** |
| *Electromagnetism* | **Voltage** | ***L*** |
| *Solid mechanics* | **Stretching** | ***L*** |
| *Acoustics* | **Velocity circulation** | ***L*** |
| *Electromagnetism* | **Charge flow** | ***S*** |
| *Fluid dynamics* | **Discharge** | ***S*** |
| *Thermal conduction* | **Heat** | ***S*** |
| *Solid mechanics* | **Surface force** | ***S*** |
| *Mechanics* | **Mass content** | ***V*** |

**Figure 6** : Association between physical variables and points (P), lines (L), surfaces (S), and volumes (V) for several physical theories

a practical formulation of these early theories. Consequently, incorporating the length scale in the constitutive relationships only is a practical simplification of a more general theory and has no evident justifications from the physical point of view.

In conclusion, reintroducing or preserving nonlocality in governing equations seems to be physically more correct than reintroducing nonlocality in constitutive equations. When speaking of reintroduction, we deal with a differential formulation, while, when speaking of preservation, we deal with a discrete formulation. The difference is not negligible, since, in order to reintroduce a length scale, it is necessary to develop an adequate approach, the nonlocal approach, while, in order to preserve the length scale, it is sufficient to avoid the limit process, using discrete approaches, and a nonlocal formulation is automatically obtained. Obtaining a nonlocal formulation by using local constitutive laws and discrete operators seems thus to be possible, as well as physically appealing. If this were the case, the new nonlocal formulation would be advantageous from the numerical point of view, since the numerical solution is achieved faster by using discrete rather than differential operators.

The use of a discrete formulation instead of a differential one also has a justification based on the microstructure of matter. When performing densities and rates, the intention is to formulate the field laws in an exact form. Nevertheless, the density finding process is carried out without considering whether a physical significance exists for the limit one is performing. In fact, since matter is discrete on a molecular scale, performing the limit process of the mean densities with the extent of the geometrical object going to zero makes no sense. Moreover, the differential formulation can only be solved for very simple geometries and particular boundary conditions. To obtain a solution in the general case, the differential equations must be expressed in a discrete form (for any differential method, Fig. 5). Consequently, the final solution is an approximation in all cases. It therefore seems unnecessary to use exact equations if, to solve them, we must introduce some kind of approximation.

## 3 The discrete formulation: the Cell Method

### *3.1 Theoretical basics of the Cell Method*

In a discrete nonlocal formulation, all operators must be discrete and the limit process must be avoided at each level of the formulation. The direct or physical approach [Huebner (1975); Livesley (1983); Fenner (1996)], used at the beginning of the finite element method (FEM), is not suitable to this aim, since it starts from point-wise conservation equations and the discrete formulation is induced by the differential formulation. Even the Finite Volume Method (FVM) and the Finite Differences Method (FDM) are based on a differential formulation. The Cell Method (CM) is a new numerical method for solving field equations [Tonti (2001a)], whose aim is to provide a direct finite formulation of field equations, without requiring a differential formulation (Fig. 5).

The physical laws are formulated directly in an algebraic manner, through a direct finite formulation which is very simple, strictly related to experimental facts, and easy to employ in numerical solutions of field problems, as many years of research in this field have shown [Tonti (1995); Mattiussi (1997); Nappi et al. (1997); Cosmi (2000); Mattiussi (2000); Nappi et al. (2000); Tonti (2000); Cosmi (2001); Cosmi and Di Marino (2001); Ferretti (2001); Marrone (2001a); Marrone (2001b); Mattiussi (2001); Nappi and Tin-Loi (2001); Tonti (2001a); Tonti (2001b); Tonti (2001c); Tonti (2001d); Zovatto (2001); Bellina et al. (2002); Ferretti et al. (2002a); Ferretti et al. (2002b); Marrone et al. (2002); Tonti (2002); Ferretti (2003a); Ferretti and Di Leo (2003); Ferretti et al. (2003); Ferretti (2004f); Ferretti (2004g)]. Some basic concepts of the CM are very similar to those of the direct or physical approach. The CM is also similar to the vertex-based scheme of the FVM and can be considered as a generalization of the FDM [Ferretti (2003)]. Nevertheless, the similarity is apparent, since the CM is not based on a differential formulation and can be used for building a discrete nonlocal formulation.

The theoretical basics of the CM lead to a unified description of Physics, by highlighting the geometrical, algebraic and analytical structure which is common to different physical theories.

The CM uses cell complexes (Fig. 7), which are not simply the result of a domain discretization, needed by the numerical analysis. They substitute the coordinate systems when we need to describe not only points, but also line, surfaces, and volumes. 0- and more than 0-dimensional quantities are described directly, avoiding the limit process and the subsequent discretization, by associating them with nodes, edges, surfaces, and volumes of the cell complexes. Consequently, the governing equations are expressed in the discrete form directly. In conclusion, we could say that cell complexes represent the generalization of the coordinate systems, when the geometrical counterpart of physical variables is taken into account.

The geometrical structure of space is very rich with the CM. Choose a set of points in space, said the set of primal nodes $\boldsymbol{P}$ (black points in Fig. 7). The lines connecting the primal nodes (black lines in Fig. 7) define a spatial mesh, said the primal cell complex. Edges areas and volumes of the primal cell complex are, respectively, the primal sides $\boldsymbol{L}$, surfaces $\boldsymbol{S}$, and volumes $\boldsymbol{V}$. These geometrical

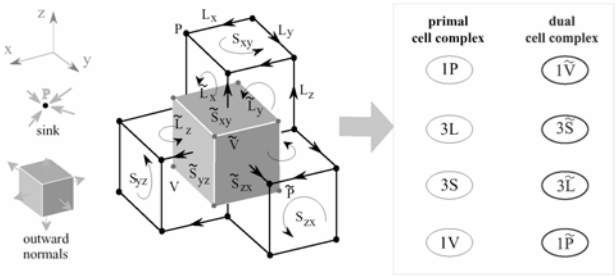


**Figure 7** : Correspondence between objects of the primal and dual cell complexes in 3D space

objects can be endowed with orientation. Whenever the orientation of a space element lies on the element itself, an *inner orientation* is established, while, whenever the orientation of a space element depends on the space in which the element is embedded, an *outer orientation* is established (Fig. 8). In Fig. 7, an inner orientation has been defined for the elements in dimension 0, 1, 2 and 3 of the primal cell complex.

Now, consider the surfaces, locus of the points which are equidistant from each pair of primal nodes (gray surfaces in Fig. 7). These further surfaces can be used for building a second spatial mesh (Fig. 7), said the dual cell complex. Points, edges, areas, and volumes of the dual cell complex are, respectively, the dual nodes $\tilde{\mathbf{P}}$, sides $\tilde{\mathbf{L}}$, surfaces $\tilde{\mathbf{S}}$, and volumes $\tilde{\mathbf{V}}$. Once defined an inner orientation for the primal cell complex, the dual cell complex turns out to be provided with an outer orientation (Fig. 7). The relationship between cell complexes, primal and dual, and type of orientation, inner and outer, was discovered by Veblen and Whitehead (1932), and introduced in Physics by Schouten (1951) and van Dantzing (1956).

Each geometrical object of the dual cell complex can be put in dual correspondence with one geometric object of the primal cell complex, staggered with respect to the former one (Fig. 7):

- each node of the dual complex is contained in one cell of the primal complex;
- each edge of the dual complex intersects a face of the primal complex;
- each face of the dual complex is intersected by one edge of the primal complex;
- each volume of the dual complex contains one node of the primal complex.

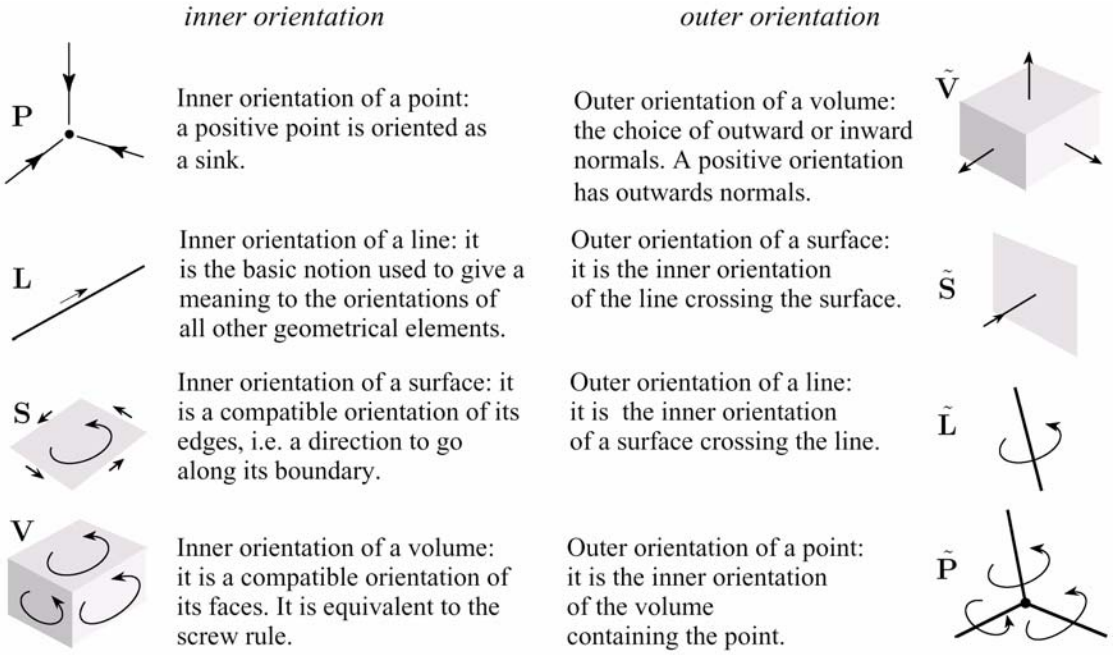


**Figure 8** : Inner and outer orientations in three-dimensional space [Tonti (2001a)]

Following the practice of algebraic topology, the CM considers the nodes, edges, faces, and volumes as cells of dimension zero, one, two, and three, respectively. They are also denoted as 0-cells, 1-cells, 2-cells, and 3-cells. With this position, in a $n$-dimensional space the dual correspondence is established between primal (dual) cells of dimension $p$ and dual (primal) cells of dimension $n-p$.

The elements of both cell complexes turn out to be strictly associated with variables (Fig. 9). In effect, the variables of each physical theory are not related to the physical objects of a single complex. In particular, source variables are always associated with the elements of the dual complex only and configuration variables are always associated with the elements of the primal complex only. Incidentally, it is worth noting that not only variables, but also their classification can be put in relationship with geometry: since the objects of the primal cell complex are natural geometrical referents of configuration variables, while the objects of the dual cell complex are natural geometrical referents of source variables, by providing the primal cell complex with an inner orientation, the configuration variables of any field theory are associated with cells endowed with an inner orientation, while the source variables are associated with cells endowed with an outer orientation (Fig. 9). This further states the strong relationship existing between geometry and Physics.

In conclusion, a strict correspondence between variables and geometry exists, never highlighted before the CM. Due to this correspondence, a proper mathematical description of any physical phenomenon cannot leave out of consideration the geometrical structure of the phenomenon itself. The coordinate systems of the differential formulation are not sufficient for describing the properties of orientation and dual correspondence between cells of dimension 1, 2, $\ldots n$ in a $n$-dimensional space, since they are only adequate to describe points (in space and time). On the contrary, a system of dual cells seems to be quite adequate to preserve the geometric structure of all involved variables. The association of physical variables to elements of a cell complex and its dual was introduced by Okada and Onodera (1951) and Branin (1966). In the CM, the strong coupling between physical variables and oriented space elements becomes the key to give a direct discrete formulation to physical laws of fields. In the opinion of the Author, this coupling can be considered as a further justification of the nonlocality physical well-posedness, since speaking of geometric content and nonlocality is substantially the same thing.

In a two-dimensional domain with triangular-shaped primal cells, the dual cells can be obtained by joining the circumcenters, the barycenters, or the incenters of primal cells. Inner and outer orientations for primal and dual

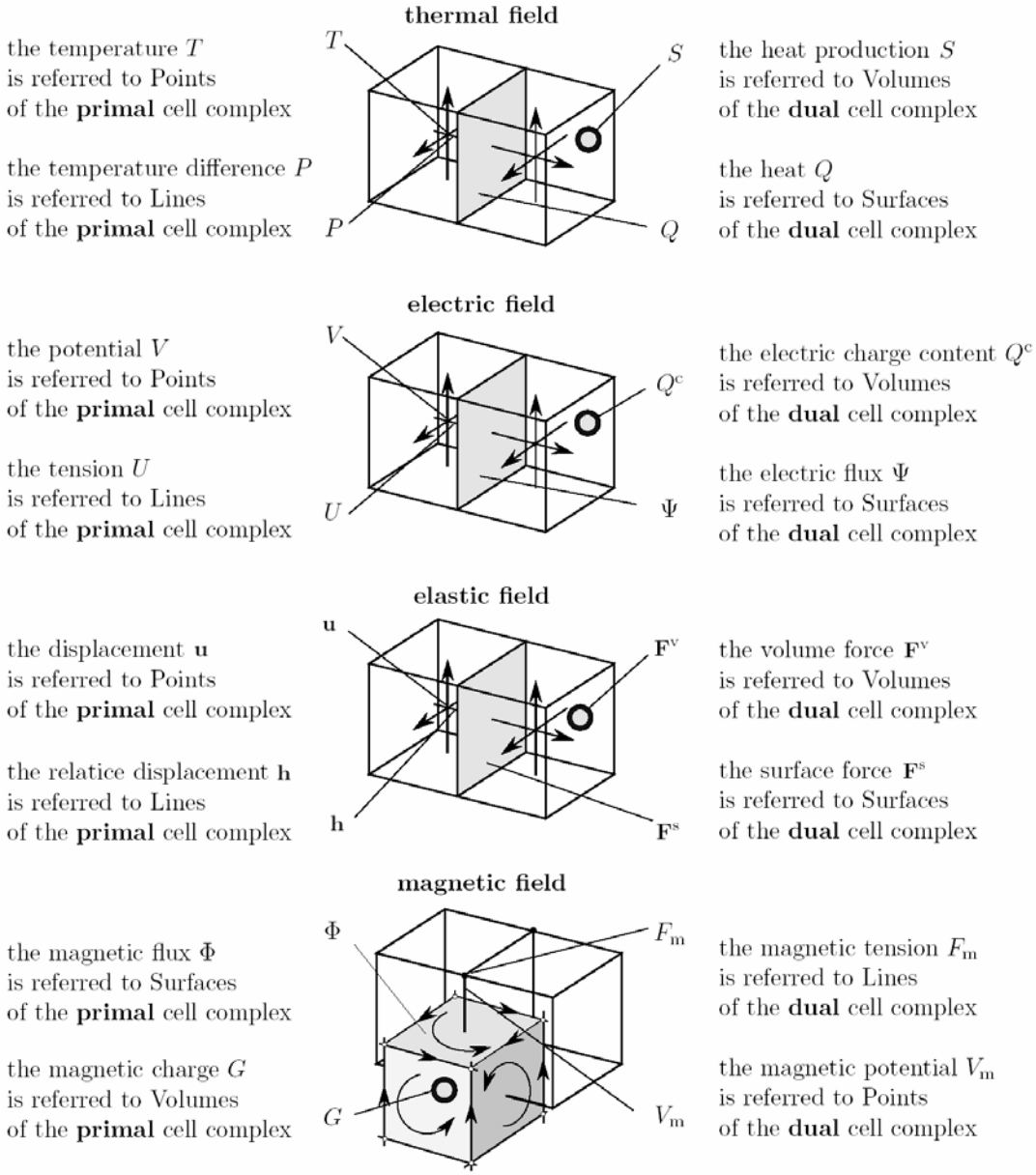


**Figure 9** : Association between physical variables and cell complexes [Tonti (2001a)]

cell complexes in a two-dimensional domain are shown in Fig. 10, for the case of triangular-shaped primal cells and circumcentric dual cells. An example of barycentric dual cells for triangular-shaped primal cells is provided in Fig. 13. Time elements and their duals can be built analogously to the primal and dual cell complexes (Fig. 11).

In order to preserve the geometrical content of cells, the primal mesh is provided with a thickness also in two-dimensional domains, which is a unit thickness (Fig. 12). Moreover, since each dual volume must contain one node of the primal complex, in two-dimensional domains primal and dual cell complexes turn out to be shifted along the body thickness (Fig. 12). The use of two cell complexes shifted in thickness somewhat resembles the use of imbricate elements in FEM analysis. Imbricate elements represent the first application of nonlocal continuum to regularization of material instability problems for strain-softening materials [Bažant et al. (1984); Belytschko et al. (1986)]. They overlap like tiles on a roof (Fig. 13). In the FEM, imbrication is a way to provide the continuum with nonlocality properties. In the CM,

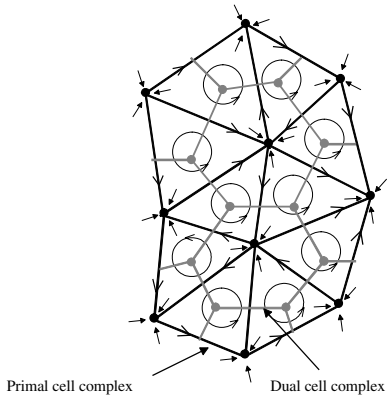


**Figure 10** : Inner and outer orientation in 2D cell complexes with circocentric dual cells

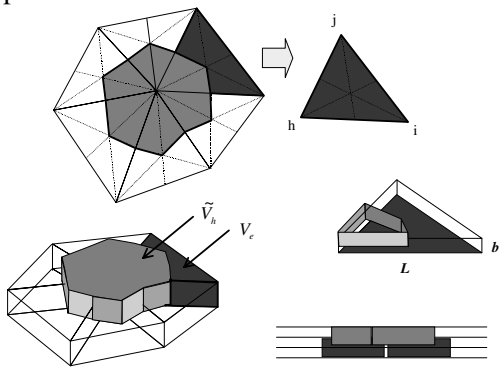


**Figure 12** : Staggering in 2D cell complexes with barycentric dual cells

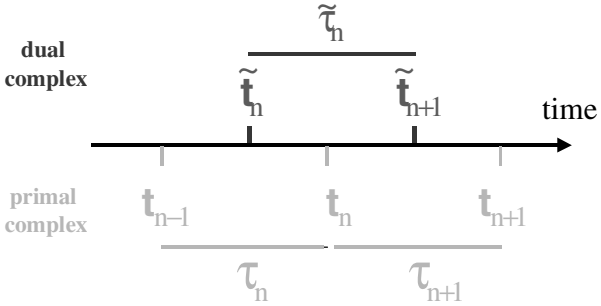


**Figure 11** : Duality between temporal elements in time space

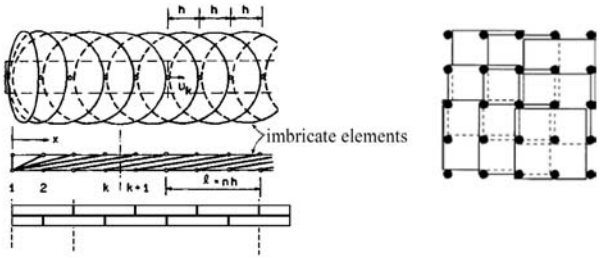


**Figure 13** : Two examples of imbrication (overlapping) of finite elements [Bažant et al. (1984)]

shifting and overlapping only have a meaning linked to the classification of physical variables into configuration and source variables. The equilibrium is then enforced on dual polygons, such as in the vertex-based scheme of the FVM [Mavripilis (1995)], and they never overlap in the classical CM.

It is worth noting that the cumbersome double transformation from finite to differential and then back to finite, typical of the differential formulation (Fig. 5, Section 2.2), is avoided by starting directly from physical laws in finite form, such as the CM does. The advantage, besides being numerical, is also of physical well-posedness, since the correspondence between variables, which are global variables, and relative geometrical referent is never lost with the CM. In conclusion, the discrete formulation is more appealing than the differential formulation from the physical point of view.

The CM is also more appealing as far as the aforementioned discussion on the discrete nature of matter is concerned. Actually, since the use of point functions is no longer necessary because of leaving the differential formulation, the CM deals with (discrete) equations that are not in conflict with the discrete nature of matter.

### *3.2 Linear interpolation in two-dimensional domains*

The linear interpolation of the CM has been shown in Ferretti (2003a). Here, only the basics of the interpolation are provided.

As discussed previously, in the CM the displacement vectors $\mathbf{u}_i$, $\mathbf{u}_j$, $\mathbf{u}_k$ are associated with the nodes $\mathbf{P}_i$, $\mathbf{P}_j$, $\mathbf{P}_k$ of the primal cell $\mathbf{A}_{ijk}$, while the interface forces, $\mathbf{Q}_{nm}$ $(n = i, j, k; m = j, k, i)$, and the volume forces, $\mathbf{F}_n$, are associated with the dual edges intersecting the primal edge $\mathbf{P}_n\mathbf{P}_m$ and the dual cell surrounding the primal node $\mathbf{P}_k$, respectively (Fig. 15).

The linear interpolation of the CM is obtained by employing affine interpolation functions of the displacement field in every primal cell. The dimensions of primal cells are chosen so that the assumption of uniform strain is accurate. A homogeneous state of strain is associated with

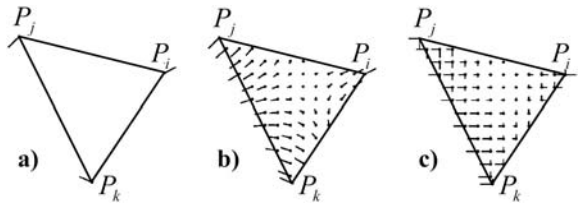


**Figure 14** : Primal nodes **(a)**, displacement field **(b)**, and displacement components (c) assuming an affine field

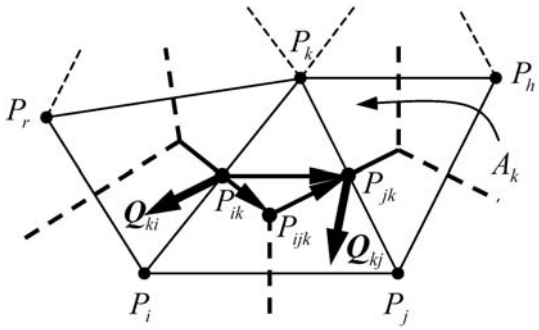


**Figure 15** : Property of the stress field

the complex of primal cells, represented by the affine transformation related to the displacement components of the primal nodes (Fig. 14).

The affine transformation can be expressed in the form:

$$\mathbf{h}_{nm} = \mathbf{H}_{ijk}\mathbf{L}_{nm}, \quad (n = i, j, k; m = j, k, i). \tag{1}$$

In Eq. 1, $\mathbf{H}_{ijk}$, displacement gradient of $\mathbf{A}_{ijk}$, is a double tensor which, in general, is not symmetric, and $\mathbf{h}_{nm}$ is the relative displacement associated with the oriented side $\mathbf{L}_{nm} = (\mathbf{P}_m - \mathbf{P}_n) = [\Delta x_{nm} \Delta y_{nm}]^T$. The determinant of the transformation is twice the magnitude of the oriented area of $\mathbf{A}_{ijk}$, with its sign (corkscrew rule):

$$\Delta_{ijk} = \det \mathbf{H}_{ijk} = \begin{vmatrix} x_i & y_i & 1 \\ x_j & y_j & 1 \\ x_k & y_k & 1 \end{vmatrix} = 2\mathbf{A}_{ijk}. \tag{2}$$

The components of the infinitesimal strain tensor $\varepsilon = [\varepsilon_x \quad \varepsilon_y \quad 2\varepsilon_{xy}]^T$ can be evaluated as:

$$\boldsymbol{\varepsilon} = \mathbf{B}\mathbf{u}, \tag{3}$$

where,

$$\mathbf{u} = \begin{bmatrix} w_k & v_k & w_i & v_i & w_j & v_j \end{bmatrix}^T, \tag{4}$$

$$\mathbf{B} = \frac{1}{\Delta_{ijk}} \begin{bmatrix} \Delta y_{ji} & 0 & \Delta y_{kj} & 0 & \Delta y_{ik} & 0 \\ 0 & \Delta x_{ij} & 0 & \Delta x_{jk} & 0 & \Delta x_{ki} \\ \Delta x_{ij} & \Delta y_{ji} & \Delta x_{jk} & \Delta y_{kj} & \Delta x_{ki} & \Delta y_{ik} \end{bmatrix}. \tag{5}$$

The equilibrium equations, written for the dual cells, take the form:

$$\mathbf{Q}_{kj} = -\mathbf{Q}_{jk}, \tag{6}$$

$$\sum_j \mathbf{Q}_{kj} + \mathbf{F}_k = 0, \tag{7}$$

$$\sum_j (\mathbf{P}_{jk} - \mathbf{O}) \times \mathbf{Q}_{kj} + (\mathbf{G}_k - \mathbf{O}) \times \mathbf{F}_k = 0. \tag{8}$$

It is then possible to express each interface force as a function of the two stress tensors associated with the junctions of the corresponding interface:

$$\mathbf{Q}_{kj} = \boldsymbol{\sigma}_{jhk}\mathbf{R}\left(\mathbf{P}_{jk} - \mathbf{P}_{jhk}\right) - \boldsymbol{\sigma}_{ijk}\mathbf{R}\left(\mathbf{P}_{jk} - \mathbf{P}_{ijk}\right). \tag{9}$$

$\mathbf{R}$ represents a counter-clockwise rotation by 90 ° :

$$\mathbf{R} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{10}$$

For each $\mathbf{A}_k$ dual cell, the following property follows from assuming a uniform stress field (Fig. 15):

$$\begin{aligned} &\boldsymbol{\sigma}_{ijk}\mathbf{R}\left(\mathbf{P}_{jk} - \mathbf{P}_{ijk}\right) - \boldsymbol{\sigma}_{ijk}\mathbf{R}\left(\mathbf{P}_{ik} - \mathbf{P}_{ijk}\right) \\ &\quad = \boldsymbol{\sigma}_{ijk}\mathbf{R}\left(\mathbf{P}_{jk} - \mathbf{P}_{ik}\right) \\ &\quad = \frac{1}{2}\boldsymbol{\sigma}_{ijk}\mathbf{R}\left(\mathbf{P}_j - \mathbf{P}_i\right) \end{aligned} \tag{11}$$

By means of Eq. 11, Eq. 7 can then be written as:

$$\frac{1}{2}\sum_j \boldsymbol{\sigma}_{ijk}\mathbf{R}\left(\mathbf{P}_j - \mathbf{P}_i\right) = \mathbf{F}_k. \tag{12}$$

For cells located on the boundary, Eq. 7 takes the form:

$$\sum_j \mathbf{Q}_{kj} + \mathbf{Q}_k + \mathbf{F}_k = 0. \tag{13}$$

$\mathbf{Q}_k$ is the interface force to the boundary.

These equations are implemented in the same manner as for FEM:

- a compatibility equation: Eq. 3;

- the constitutive law:

$$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon}. \tag{14}$$

- an indefinite equilibrium equation: by means of Eqs. 9, 14, and 3, the interface force can be expressed as a function of displacement:

$$\mathbf{S} = \mathbf{RL}, \tag{15}$$

$$\mathbf{N} = \begin{bmatrix} \mathbf{S}_x & 0 & \mathbf{S}_y \\ 0 & \mathbf{S}_y & \mathbf{S}_x \end{bmatrix}, \tag{16}$$

$$\mathbf{Q} = \mathbf{N}\boldsymbol{\sigma} = \mathbf{ND}\boldsymbol{\varepsilon} = (\mathbf{NDB})\,\mathbf{u}. \tag{17}$$

By using Eq. 7, a linear system of equations can be written for each dual cell, in the form:

$$\mathbf{F} = \mathbf{KU}. \tag{18}$$

**F** and **U** are the force and displacement vectors respectively, and **K** is analogous to the stiffness matrix in the FEM, which is symmetric, and defined as positive for properly constrained systems. To assemble **K**, the contribution of each primal cell is computed, taking into account the relationship between the local and global node numbering schemes (as in the FEM):

$$\mathbf{u} = \mathbf{CU}. \tag{19}$$

**C** is a Boolean matrix describing the location of the vertices.

### 3.3 *Quadratic interpolation in two-dimensional domains*

One can obtain a quadratic interpolation of displacements inside each primal triangle ***c*** of nodes ***h***, ***i***, and ***j***, by choosing the vertex ***h*** as the origin of affine coordinates. Dispose the axes of the affine coordinates, $\xi$ and $\eta$, along the sides ***hi*** and ***hj***, respectively (Fig. 16). Then, assume the segments ***hi*** and ***hj*** as unit along the corresponding axis.

Now, denote the three midpoints of sides ***hi***, ***ij***, and ***hj***, as ***p***, ***q***, and ***r***, respectively (Fig. 16). On the assumption of knowing the displacements at nodes ***h***, ***i***, ***j***, ***p***, ***q***, and ***r***, a quadratic interpolation of the displacements at each node

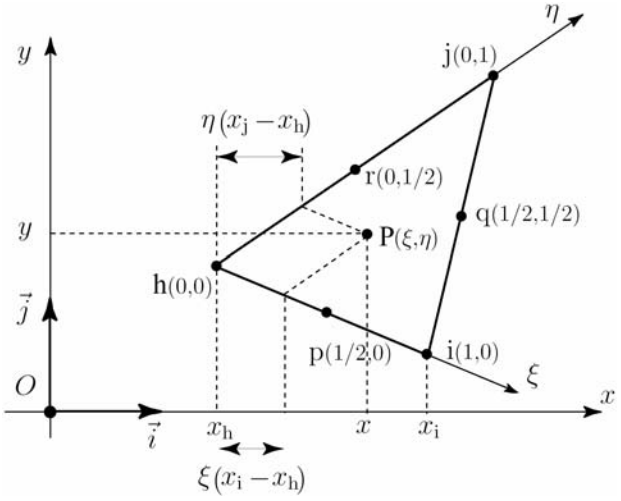


**Figure 16** : Relationships between global Cartesian and local affine coordinates [Tonti (2001a)]

of the triangle is given by (affine coordinates):

$$\begin{pmatrix} u^c(\xi,\eta) & v^c(\xi,\eta) \end{pmatrix} = \begin{pmatrix} 1 & \xi & \eta & \xi^2 & \xi\eta & \eta^2 \end{pmatrix} \begin{bmatrix} a_h & b_h \\ a_i & b_i \\ a_j & b_j \\ a_p & b_p \\ a_q & b_q \\ a_r & b_r \end{bmatrix}_c. \tag{20}$$

The coefficients $a_k$ and $b_k$ must be determined by imposing to Eq. 20 to be satisfied at nodes ***h***, ***i***, ***j***, ***p***, ***q***, and ***r***. By considering the affine coordinates of nodes ***h***, ***i***, ***j***, ***p***, ***q***, and ***r***, we obtain:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1/2 & 0 & 1/4 & 0 & 0 \\ 1 & 1/2 & 1/2 & 1/4 & 1/4 & 1/4 \\ 1 & 0 & 1/2 & 0 & 0 & 1/4 \end{bmatrix} \begin{bmatrix} a_h & b_h \\ a_i & b_i \\ a_j & b_j \\ a_p & b_p \\ a_q & b_q \\ a_r & b_r \end{bmatrix}_c = \begin{bmatrix} u_h & v_h \\ u_i & v_i \\ u_j & v_j \\ u_p & v_p \\ u_q & v_q \\ u_r & v_r \end{bmatrix}_c. \tag{21}$$

Said $[C]$ the reciprocal of the matrix of the nodal affine coordinates, it is possible to express the unknown coeffi-

cients in function of the nodal displacements:

$$\left[\begin{array}{cc} a_k & b_k \end{array}\right]_c = [C] \left[\begin{array}{cc} u_k & v_k \end{array}\right]_c . \tag{22}$$

With the use of affine coordinates, the matrix $[C]$ is the same in each triangle. Finally, the displacement at each point of the triangle in function of its nodal values is given by Eq. 20 with the use of Eq. 22:

$$\begin{array}{l} \left(\begin{array}{cc} u^c(\xi,\eta) & v^c(\xi,\eta) \end{array}\right) \\ \quad = \left(\begin{array}{cccccc} 1 & \xi & \eta & \xi^2 & \xi\eta & \eta^2 \end{array}\right) [C] \left[\begin{array}{cc} u_k & v_k \end{array}\right]_c . \end{array} \tag{23}$$

The operations of making gradients, expressing the constitutive equation, and deriving the fundamental equation follow as shown for the linear interpolation. Contrarily to what happens with the FEM, the stiffness matrix with quadratic interpolation does not turn out to be symmetric.

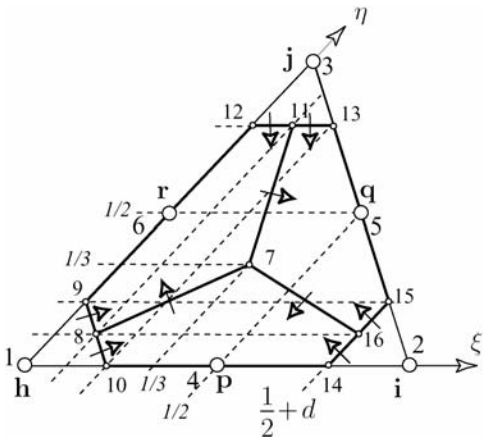


**Figure 17** : Construction of the influence polygons and orientation of their edges [Tonti (2001a)]

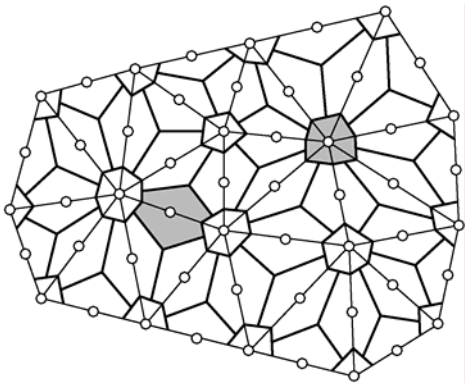

**Figure 18** : Primal (thin line) and dual (thick line) mesh for a plane domain [Tonti (2001a)]

As far as the convergence rate is concerned, it has been shown [Tonti (2001a)] that it depends on the choice of dual polygons. The most convenient choice, giving a convergence rate equal to four [Cosmi (2000)], is the use of Gauss points for building the dual polygons around nodes ***h***, ***i***, and ***j*** (Fig. 17). Said $L$ the length of the given triangle side, the distance of the two Gauss points from the ends of that side is equal to:

$$g = \frac{1}{2}\left(1 - \frac{1}{\sqrt{3}}\right) L. \tag{24}$$

As far as the dual polygons for nodes p, q, and r are concerned, they are built using Gauss points, side midpoints of the dual polygon around nodes h, i, and j, and the triangle barycentre, as shown in Fig. 17. The arising mesh for a plane domain is shown in Fig. 18.

Attaining a fourth-order convergence with the quadratic interpolation of the CM is all the more relevant as it was not possible to attain convergence greater than second order for any of the methods which are similar to the CM, such as the direct or physical approach of the FEM, the vertex-based scheme of the FVM, and the FDM.

### *3.4 A meshfree approach for the CM*

A meshfree approach of the CM is due to Zovatto (2001). Meshfree and meshless approaches are very useful in those problems of fracture mechanics in which the crack is simulated as a discontinuity of the displacement field. Actually, crack geometry updating and remeshing on the whole domain are very expensive from the computational point of view. A critical analysis of the state of the art on meshfree and meshless techniques is provided in Belytschko et al. (1996).

When applied to the CM, the meshfree technique leads to the generation of one dual polygon on each node of the primal complex, independently from the shape of other dual polygons. At the generic node ***i*** of the primal complex, the local generation of the dual polygon directly follows from the local generation of the primal mesh around ***i***. Primal nodes defining the local primal mesh around ***i*** are called boundary nodes. The number of boundary nodes, $n$, is the same for each primal node internal to the domain. A different number of boundary nodes can be chosen for the nodes internal to the domain and the nodes on the contour (Fig. 19). For geometrical reasons of mesh distortion, $n$ can vary from four to eight.

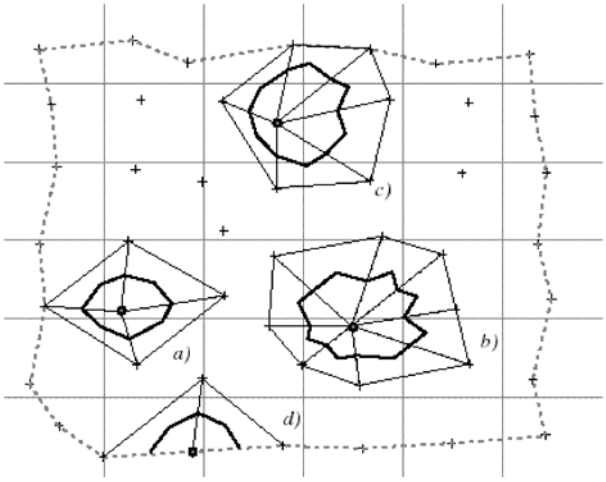


**Figure 19** : Generation of local dual polygons for nodes on the domain contour (d), and for nodes internal to the domain and primal local meshes with four (a), eight (b), and six (c) boundary nodes [Zovatto (2001)]

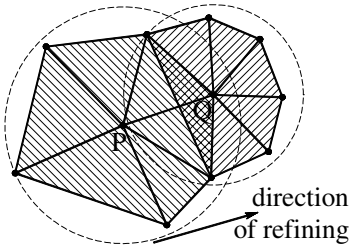


**Figure 20** : The nearest n nodes for nodes P and Q in a mesh with local refining ($n = 6$)

Once fixed $n$, the algorithm of local mesh generation at the generic node $\boldsymbol{i}$ proceeds as follows:

1. the $n$ nodes lying nearest to $\boldsymbol{i}$ are chosen as boundary nodes;

2. boundary nodes are ordered in anticlockwise sense, starting from the node for which the segment collecting $\boldsymbol{i}$ with the node itself forms the minor angle with the axis of the abscises;

3. the local primal mesh is generated by the segments joining $\boldsymbol{i}$ with the $n$ boundary nodes, and the boundary nodes to each other in anticlockwise sense.

With this algorithm, the local primal mesh turn out to be triangular shaped (Fig. 19). The local dual polygon is then generated by joining the barycenters of the local primal mesh in an anticlockwise sense (Fig. 19).

The analysis presented in the following of this paragraph belongs to a study on the meshfree CM, recently developed by the Author of the present paper. It was seen that attention must be paid in assuring that the union of all dual polygons is equal to the domain. Actually, portions of the domain not covered by dual polygons would be treated by the CM as local imperfections. Moreover, local dual polygons may overlap somewhere. Due to the automatic generation of dual polygons when the number of boundary nodes is fixed a-priori, the degree of overlapping depends on the distribution of primal nodes. If the distance between primal nodes is almost constant, the degree of overlapping is the same for each primal node, while, in case of a primal mesh with local refining or coarsening, the degree of overlapping may change from node to node. Since overlapping increases the nonlocality degree at the given node $\boldsymbol{i}$, this second case results in changing the nonlocality degree from node to node. In order to make sure that the nonlocality degree is the same at each node, $n$ must be changed node by node.

In the case of $n$ fixed and primal mesh with local refining or coarsening, a further case may occur: if **P** belongs to the $n$ nodes nearest to **Q**, **Q** does not necessarily belong to the $n$ nodes nearest to **P** (Fig. 20). Thus, the stiffness matrix is not symmetric if primal nodes are not distributed in a sufficiently regular manner. This results in a non-biunique interaction between **P** and **Q**.

The fact that the interaction between two material nodes may not be biunique raises some doubts from the theoretical point of view. Even in this further case, the problem may be avoided by adjusting $n$ at each node. For example, $n$ may represent the number of nodes internal to a circle of given radius, centered on the node $\boldsymbol{i}$. An example of $n$ adjusting is provided in Fig. 21, for a plate with internal crack. A high degree of refining is imposed both on the crack and on an internal contour. In Fig. 21, all primal and dual meshes are plotted, in order to show the biuniqueness of the relationship between nodes. Due to the biuniqueness, with this modified algorithm the stiffness matrix is symmetric.

The problem of nonsymmetric stiffness matrix is shared by many nonlocal models, such as Eringen's nonlocal theories of plasticity in the strain [Eringen (1981)] and stress [Eringen (1983)] space, the nonlocal plasticity model of Bažant and Lin [Bažant and Lin (1988b)],

the plasticity with nonlocal softening variable [Planas et al. (1993), Hu and Wittmann (2000)], and the combination of nonlocal and local softening variables [Vermeer and Brinkgreve (1994); Planas et al. (1996); Strömberg and Ristinmaa (1996)]. The causes of nonsymmetry in nonlocal damage models were explained by Bažant and Pijaudier-Cabot (1988). They lie in the nonsymmetric character of interaction weights. The studies on the structure of the nonlocal matrix [Huerta and Pijaudier-Cabot (1994)] and on numerical implementation and convergence rates [Jirásek and Patzák (2002)] did not show any convergence problem attributed to nonsymmetry. The same can be asserted for the nonsymmetric matrix of the meshfree CM [Zovatto (2001)].

Finally, it is worth noting that no problem arises when the boundary conditions of Dirichlet or Neuman are imposed in the implementation of the meshfree CM [Zovatto (2001)], while this is not the case in the implementation of the meshfree FEM. A proposal for solving the problem of the boundary conditions in the meshfree FEM can be found in Zhu and Atluri (1998).

As pointed out by Chen et al. (2000), a meshfree approximation in FEM possess intrinsic nonlocal properties, since the approximation functions are not locally constructed. In FEM simulations, nonlocal properties of meshfree approximations are exploited to incorporate an intrinsic length scale, regularizing problems with material instabilities. In the CM, a meshfree approach does not directly involve increasing the CM intrinsic degree of nonlocality. Actually, only the procedure of mesh building has changed, and not the approximations used to achieve the solution. This happens since the CM is very ductile and can be easily adapted to a meshfree formulation without having to change the structure of the method. Nevertheless, an increase in the degree of nonlocality can occur if the local mesh building leads to local primal meshes which overlap.

In order to clarify last statement, let us classify primal nodes in function of their degree of relationship with the given node (Fig. 22). Nodes belonging to the local primal mesh at the given point are nodes of first level (Fig. 22). The strain at these nodes directly influences the stress at the given point. Some of these nodes may be shared by two or more local primal meshes (Fig. 21).

We can then define a set of primal nodes whose local primal mesh shares some node with the local primal mesh at the given point. This set of primal nodes is said the set of nodes of second level (Fig. 22). The strain at a node of second level directly influences the stress at the corresponding node of first level and, thus, indirectly influences the stress at the given point. That is, nodes of second level are not directly connected to the given node. They are connected to $\boldsymbol{i}$ through the nodes of first level (Fig. 22). Analogously, we can then define nodes of third level, fourth level, and so on (Fig. 22). This establishes a sort of chain of interaction between nodes (Fig. 22), with the strain at each node influencing the stress at the given node by an amount which is proportional to its position into the chain. The chain is also established in the classical formulation of the CM, since the need to cover the whole domain is sufficient to impose the node sharing. In conclusion, in the CM the stress at a point actually depends on the strain history in the entire body, and interactions between nodes diminish with increasing distance. In the CM meshless approach, overlapping between local primal meshes increases the number of connections at each level. This results in an increased nonlocality degree at each level. Due to the nonlocal nature of Physics, we can expect an increased degree of nonlocality to lead to a better approximated solution. We can thus expect that the approximation degree of the solution would decrease with the increase of the degree of overlapping. Since the higher degree of overlapping is obtained with circular tributary areas, we can finally expect that the highest accuracy of the solution would be reached with circular dual polygons. Numerical analyses confirming this statement are in progress, at the moment.

## 4 Strain-softening modeling

The enrichment of the classical continuum by incorporating nonlocal effects into the constitutive equations is often used in order to avoid the ill-posedness of boundary value problems with strain-softening constitutive models. When the material tangent stiffness matrix ceases to be positive definite, the governing differential equations may lose ellipticity. Finite element solutions of such problems exhibit a pathological sensitivity to the element size and do not converge to physically meaningful solutions as the mesh is refined [Jirásek (1999)]. Actually, the boundary value problem does not have a unique solution with continuous dependence on the given data [Jirásek and Bažant (2001)]. To remedy the loss of ellipticity, a length scale must be incorporated, implicitly or explicitly, into the material description or the formula-

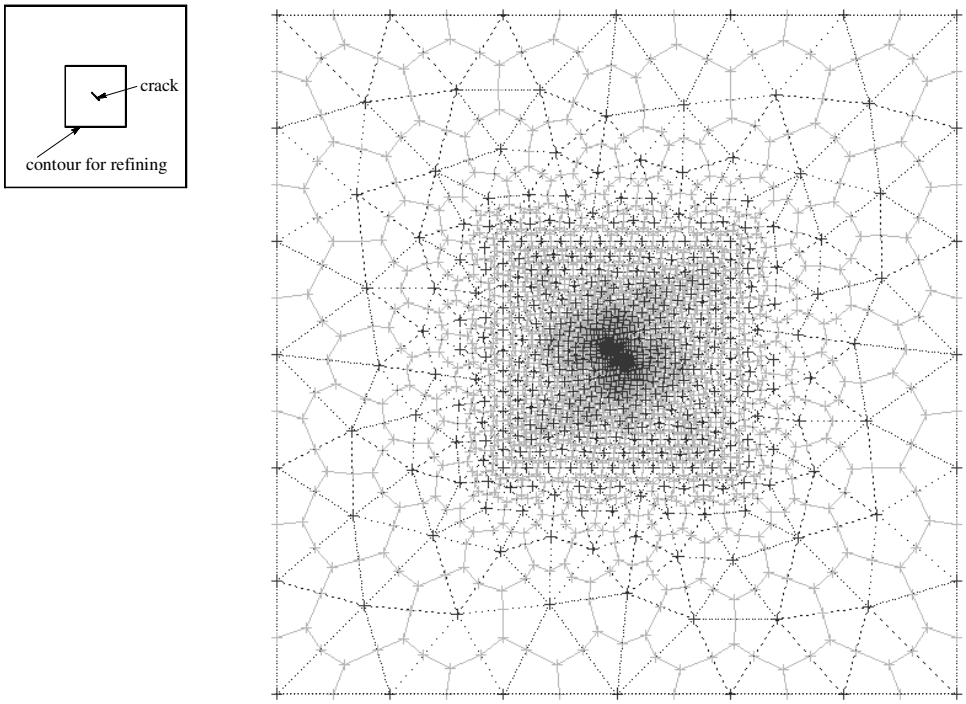


**Figure 21** : Local mesh generation for a cracked plate with refining on an internal contour and on the crack, when the interaction between n

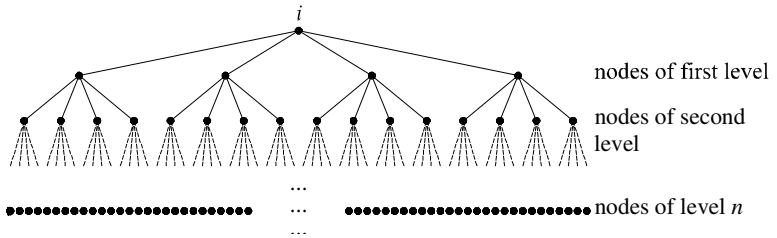


**Figure 22** : Connections between several levels of node to node interaction

tion of the boundary value problem [Chen et al. (2000)]. From experimental tests on heterogeneous brittle materials with traditional identification process, it appears that the strain-softening zone is of finite size and dissipates a finite amount of energy. However, when strain-softening is applied in conjunction with the classical local continuum concept and the differential formulation, the strain-softening zone is found to localize, in those simple cases for which exact solutions have been found, into a zone of zero thickness [Bažant and Chang (1984)]. Thus, the numerical solution by finite element converges with mesh refinement to a strain-softening zone of zero thickness and to zero energy dissipated by failure. Strain localizes into a narrow band whose width depends on the element size and tends to zero as the mesh is refined [Jirásek (1998b)]. This is not a realistic result. The corresponding load-displacement diagram always exhibits snapback for a sufficiently fine mesh, independent of the size of the structure and of the ductility of the material.

Incorporating a length scale remedies the loss of ellipticity, since the actual width of the zone of localized plastic strain is related to the heterogeneous material microstructure and can be correctly predicted only by models having a parameter with the dimension of length [Jirásek and Rolshoven (2002)]. As previously said, the length scale is absent from standard theories of elasticity or plasticity, and must be introduced by an appropriate enhancement. A properly formulated enhancement has a regularizing effect in differential formulations, since it acts as a local-

ization limiter restoring the well-posedness of the boundary value problem. As far as the reasons for the absence of a length scale from standard theories are concerned, it has been shown above (Section 2.2) how they follow directly from performing the limit process. The lack of a length scale is thus directly bonded to the use of the differential formulation.

Early extensions of the nonlocal concept to strain-softening material, leading to the so-called imbricate continuum, are due to Bažant in 1984. They were later improved by the nonlocal damage theory [Pijaudier-Cabot and Bažant (1987); Bažant and Pijaudier-Cabot (1988)] and adapted for concrete [Saouridis (1988); Saouridis and Mazars (1992)]. Nonlocal formulations were elaborated for a wide spectrum of models [Jirásek and Bažant (2001)], including softening plasticity [Bažant and Lin (1988b); Vermeer and Brinkgreve (1994); Nilsson (1994); Planas et al. (1996); Strömberg and Ristinmaa (1996); Nilsson (1997); Borino et al. (1999)], hardening crystal plasticity [Gao and Huang (2001)], progressively cavitating porous plastic solids [Leblond et al. (1994); Tvergaard and Needleman (1995); Needleman and Tvergaard (1998)], smeared crack models [Bažant and Lin (1988a); Jirásek and Zimmermann (1998)], and microplane models [Bažant and Ožbolt (1990); Ožbolt and Bažant (1996)].

Form the purely phenomenological point of view, the choice of the variable to be averaged remains to some extent arbitrary. This leads to a great number of possible nonlocal formulations. Nevertheless, one must be careful when selecting a certain formulation from the literature, because almost all of them capture the onset of localization properly, but some fail to give physically reasonable results at later stages of the localization process [Jirásek (1998b)]. The basic model with damage evolution driven by the damage energy release rate is not suitable for quasibrittle materials, since it gives the same response in tension and in compression. A number of nonlocal damage formulations of the simple isotropic damage model with one scalar damage variable appeared during the last decade, aimed at emphasizing the effect of tension on the propagation of cracks. A unified nonlocal formulation applicable to any inelastic constitutive model with softening as a reliable localization limiter is not available at present.

On the basis of the discussion on where nonlocality must be introduced (Section 2.2), the question we want to answer in the following is whether or not is it possible to model softening by means of a discrete approach, the Cell Method, and a local material law, the effective law.

### *4.1 How to formulate the constitutive law in a discrete nonlocal approach*

The last point to consider in building a discrete nonlocal formulation using local constitutive laws is how to formulate the constitutive law in order for it to be actually local, with the nonlocality of governing equations not automatically extending to the constitutive laws by scale change. In accordance with the identification procedure of the effective law (Section 1), we can, and actually must, separate the material from the structure scale [Ferretti (2004a,b)], since the constitutive behavior is not the mirror image of a structural problem at a lower scale. The separation between the two scales is motivated by experimental observation [Ferretti (2001)], stating that we cannot study the behavior of compressed concrete in the context of continuum mechanics, since macro-cracks isolating the inner resistant core develop from the very beginning of the compression test forth.

Only on the assumption of scales separation is it possible to associate the nonlocality with the governing equations only, without automatic extension to the material laws. We can thus state that the discrete nonlocal formulation with the CM and local constitutive laws strongly needs to be used with the effective law, since it is the only local material law satisfying the qualification of scales separation. On the other hand, the effective law strongly needs to be used with the Cell Method, if we intend to model nonlocal effects, since the CM is the only truly discrete formulation, and only a truly discrete formulation affords the possibility of modeling nonlocal effects by using a local material law.

It is worth noting that the governing equations of the discrete nonlocal formulation never lose ellipticity, since the effective law is monotone strictly nondecreasing. Consequently, by introducing the scales separation the ill-posedness of boundary value problems never poses. There remains to prove, however, the ability of a CM code with effective law of modeling the softening behavior of load-displacements curves. We also need to prove the ability of the code to model the size-effect, in order this new approach could actually be called nonlocal. Numerical results on the structural softening and size-effect modeling will be provided in Section 4.3.

### 4.2 Comparison between discrete and differential nonlocal approaches

As an example of nonlocal damage formulation, consider the nonlocal version of the simple isotropic damage model [Pijaudier-Cabot and Bažant (1987)]. Said $\boldsymbol{\sigma}$ the stress tensor, $\boldsymbol{\varepsilon}$ the strain tensor, $\mathbf{D}^e$ the elastic material stiffness matrix, and $\boldsymbol{\omega}$ the scalar damage parameter, this model postulates the stress-strain relationship for the given integration point $\mathbf{x}_p$ in the form:

$$\boldsymbol{\sigma}(\mathbf{x}_p) = (1-\boldsymbol{\omega}(\mathbf{x}_p))\,\mathbf{D}^e(\mathbf{x}_p)\boldsymbol{\varepsilon}(\mathbf{x}_p). \tag{25}$$

In Eq. 25, the evolution of the damage parameter is controlled by the weighted average of the damage energy release $Y$ over a certain neighborhood of the given integration point $\mathbf{x}_p$:

$$\overline{Y}(\mathbf{x}_p) = \int_V \alpha(\mathbf{x}_p,\xi)\,Y(\xi)\,d\xi. \tag{26}$$

$\alpha(\mathbf{x}_p,\xi)$ in Eq. 26 is a given nonlocal weight function.

Initially, the damage parameter is equal to zero, and the response of the material is linear elastic. When the stored energy reaches a certain critical value, the damage parameter starts growing, reflecting the gradual loss of integrity of the material. During unloading, the damage parameter remains constant even if $\overline{Y}$ decreases. That is, $\omega$ depends on the maximum previously reached value of $\overline{Y}$:

$$\omega(\mathbf{x}_p) = f\left(\max\left(\overline{Y}\right)\right). \quad \omega(\mathbf{x}_p,t) = f\left(\max_{\tau\le t}\overline{Y}(\tau)\right) \tag{27}$$

Applying the nonlocal averaging to a variable that can never decrease is motivated by instability modes arising when the averaging operator is applied to the total strain tensor $\boldsymbol{\varepsilon}$, such as happens in the original nonlocal model for strain softening with imbricate finite elements [Bažant et al. (1984)]. The function $f$ in Eq. 27 can be identified from the uniaxial stress-strain curve.

Averaging of different variables gives rather different responses [Jirásek (1998b)]. The complete fracture is correctly reproduced by models that average the equivalent strain, the energy release rate, or the compliance variable. In a uniaxially stretched softening bar, these variables lead to residual strength approaching zero as the applied elongation is increased. On the contrary, the evaluation of the inelastic stress from the nonlocal strain, the same as the averaging of the damage variable, inelastic stress, or inelastic stress increment, leads to spurious residual stresses and to an expansion of the softening zone across the entire bar [Bažant and Jirásek (2002)]. From these last formulations, locking effects and, sometimes, loss of convergence follow. Once the number of integration points, $\mathbf{x}_p$, has been chosen, the stiffness matrix is found by replacing the integrals of the standard formulae with a sum of contributions from the finite number of integration points [Jirásek (1999)]. Said $p$ the summation index running from 1 to the total number of integration points, $w(\mathbf{x}_p)$ the integration weight, and $\mathbf{B}(\mathbf{x}_p)$ the strain-displacement matrix defined through the standard approximation between strain and vector of nodal displacements, $\boldsymbol{\varepsilon}(\mathbf{x}_p) = \mathbf{B}(\mathbf{x}_p)\mathbf{d}$, the vector of internal forces takes the following form:

$$\mathbf{f} = \sum_p w(\mathbf{x}_p)\,\mathbf{B}^T(\mathbf{x}_p)\,\boldsymbol{\sigma}(\mathbf{x}_p). \tag{28}$$

The nonlocal averaging integral in Eq. 26 is also approximated by a finite sum, in which $\alpha_{pq}$ are interaction coefficients depending on the weight function $\alpha$ and the volume in the neighborhood of the given integration point:

$$\overline{Y}(\mathbf{x}_p) = \sum_q \alpha_{pq} Y(\mathbf{x}_p). \tag{29}$$

By substituting Eq. 25 into Eq. 28, using the strain-displacement matrix, and introducing the shorthand notation, $\omega(\mathbf{x}_p) = \omega_p$, $\mathbf{B}(\mathbf{x}_p) = \mathbf{B}_p$, etc., we obtain:

$$\mathbf{f} = \sum_p w_p\,(1-\omega_p)\,\mathbf{B}_p^T\mathbf{D}_p^e\mathbf{B}_p\mathbf{d} = \sum_p w_p\,(1-\omega_p)\,\mathbf{K}_p^e\mathbf{d}. \tag{30}$$

Thus, the secant stiffness matrix, $\mathbf{K}^u$, turns out to assume the value:

$$\mathbf{K}^u = \sum_p w_p\,(1-\omega_p)\,\mathbf{K}_p^e. \tag{31}$$

By differentiating the internal forces with respect to the nodal displacements and introducing the vector $\mathbf{f}_p^e = \mathbf{K}_p^e\mathbf{d}$, it is also possible to obtain the tangent stiffness matrix, $\mathbf{K}$ [Jirásek (1999)]:

$$\mathbf{K} = \mathbf{K}^u - \sum_{p,q} w_p f_p' \alpha_{pq} \mathbf{f}_p^e \left(\mathbf{f}_q^e\right)^T. \tag{32}$$

$f_p' = f'\left(\overline{Y}_p\right)$ for loading and $f_p' = 0$ for unloading/reloading below the current damage threshold. The

term $w_p f'_p \alpha_{pq} \mathbf{f}^e_p \left(\mathbf{f}^e_q\right)^T$ represents the contribution of the nonlocal interaction between $\mathbf{x}_p$ and $\mathbf{x}_q$ to the overall stiffness. The sum is performed only over those pairs of integration points that are closer than the interaction radius. Reducing the interaction to points internal to the interaction radius leads to assembling matrices much smaller than usual ones. Actually, each pair of integration points, $\mathbf{x}_p$ and $\mathbf{x}_q$, contributes only to the $p-q$ element of the global stiffness matrix. On the other hand, the bandwidth increases due to the nonlocal interaction. Finally, the global stiffness matrix is in general not symmetric, since the interaction weights $\alpha_{pq}$ are not symmetric. The assembly procedure of the global stiffness matrix is similar to the usual one.

In the opinion of the Author, performing the sum only over those pairs of integration points that are closer than the interaction radius is equivalent to enforcing the equilibrium on those local dual polygons, in which the dual nodes are chosen as the nodes internal to the domain of influence. That is, the nonlocal approach is equivalent to a meshfree CM approach in which it is not the number of boundary nodes which has been fixed previously, but their distance from the corresponding primal node. Nevertheless, the interaction between the pairs of nodes would be biunique if dual nodes were chosen internal to the domain of influence. Thus, the global stiffness matrix would be symmetric with the meshfree CM. Finally, due to the intrinsic CM nonlocality and the equivalence between nonlocal approaches and meshfree CM, we can state that damage can be properly described by means of the local isotropic damage model if it is used in conjunction with the CM.

Most nonlocal damage formulations lead to a progressive shrinking of the zone in which local strains increase [Pijaudier-Cabot and Bažant (1987); Jirásek and Zimmermann (1998)]. The thickness of the zone of increasing damage can never be smaller than the support diameter of the nonlocal weight function. Numerical problems thus occur, when the residual stiffness of the material inside this zone becomes too small. These numerical problems are all the more severe if body forces are present, leading to divergence of the equilibrium iteration process. Transition from highly localized strains to displacement discontinuities embedded in the interior of finite elements (Fig. 23) can be used to remedy the loss of convergence when body forces are present [Jirásek (1998a)]. As pointed out in Jirásek (1999), this approach is appealing from the physical point of view, since in the final stage of degradation the material should no longer be considered as a continuum. However, the Author of the present paper argues that the transition technique corresponds to a description of the stress field in terms of displacements, and not of strains (Fig. 23). Thus, the stress field is not related to the microscopic behavior of the material, assumed to be strain-softening, but to the macroscopic behavior of the structure, which is not necessarily homothetic to the former (Section 1). The transition technique is thus equivalent to introducing a scale separation between the load-displacement and stress-strain relationships. In particular, the stress-strain relationships modeled by the transition technique may not be strain-softening. In other words, is not clear whether a case of strain-softening is actually modeled by the transition technique. In conclusion, the existence itself of the strain-softening behavior would not be ensured by this approach. Therefore, in the opinion of the Author, capturing the correct crack trajectory without any numerical instabilities through a transition technique can-

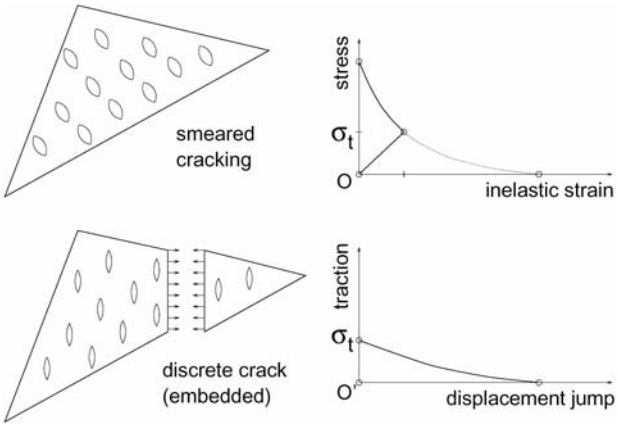


**Figure 23** : Transition from a continuum model to a discontinuity [Jirásek (1999)]

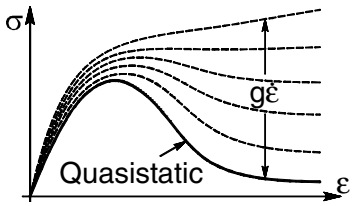


**Figure 24** : Behavior of the dynamic continuum rate-dependent model

not be considered a proof of the strain-softening mathematical well-posedness and, thus, of the existence itself of strain-softening. In effect, as better shown in Section 4.3, displacement discontinuity with opening of macroscopic cracks has shown itself to be per-se sufficient to model the softening branch (Fig. 25) and size-effect (Fig. 26) in load-displacement diagrams of compressed specimens, even if a monotone and local constitutive law is used [Ferretti (2003b); Ferretti and Di Leo (2003)].

It is worth noting that even the proposal of Sandler and Wright (1983) for dynamic continuum problems introduces a modified constitutive relationship, in order to avoid the problem of hyperbolic rather than elliptic governing equations. Actually, Sandler and Wright treat strain-softening in the rate-dependent manner, with a rate-dependent viscoelastic correction factor applied to the stress (Fig. 24). With this choice, a bounded solution exists for any instant, and the problem turns out to be dynamically well-posed. However, the correction of the value of $\sigma$ provided by the term $g\dot{\varepsilon}$ in Fig. 24 is equivalent to changing the shape of the quasistatic constitutive relationship. That is, it is equivalent to introducing a scale separation. Once more, it is not clear whether or not the modified constitutive relationship is of the softening type. Finally, it is possible to show [Ferretti (2004d)] that the difference $\sigma_{eff} - \overline{\sigma}$ in the identification procedure of the effective law (Fig. 2) plays the same role as the term $g\dot{\varepsilon}$ in Fig. 24, and the viscoelastic term is equal to a rate-independent factor which is related to the decrease of cross-sectional area. Consequently, the rate-dependent correction factor by Sandler and Wright and the correction factor of the effective law give the same value of stress. Since the effective law is monotone strictly nondecreasing, this result and the former discussion on the transition technique further prove how the problem of the existence of strain-softening is actually still an open issue. The apparent contradiction in establishing equality between a rate-dependent and a rate-independent factor has been extensively debated in Ferretti (2004d).

As far as the observation that in the final stage of the degradation process the material should no longer be considered as a continuum is concerned [Jirásek (1999)], it must be recalled that the identification procedure of the effective law (Section 1, Fig. 2) ceases to consider the concrete specimen as a continuum from the early beginning of the compression test forth. As previously said (Section 4.1), this assumption is motivated by the experimental evidence. It may be then asserted that the transition technique and the identification process of the effective law are two alternative ways for introducing a scale separation between load-displacement and stress-strain. This may be considered as a further indirect validation of the identifying procedure of the effective law, both as far as the monotone strictly nondecreasing behavior of the effective law and the results on Poisson's ratio, dilatancy, and viscosity are concerned. Actually, in the identification procedure of the effective law, both the non-softening behavior of the effective law, the constancy of the Poisson's ratio, the absence of real increase in the volume of a solid when placed under pressure, and the insignificant contribution of viscosity to the time-dependent behavior at constant load of concrete are direct results of having assumed that strain measurements acquired on the specimen surface are not real strains from the moment in which starts to propagate an inner crack [Ferretti (2004c); Ferretti (2004d)], quite similar to the discrete crack of Fig. 23. As previously recalled (Section 1), surface strains, and, particularly, the average circumferential strain usually employed for deriving Poisson's ratio and volumetric strain, are affected by the crack openings (Fig. 3a). Consequently, they cannot be directly used in the identification process, while the strain acquisitions into the presumed resistant core can, since we have assumed that macro-cracks do not occur in the inner core.

### *4.3 Results of the discrete nonlocal approach*

When modeling the propagation of a crack through a mesh, the geometry of the mesh must be modified as the crack propagates. Two different strategies are available to study fracture mechanics using the FEM. The first strategy describes fracture as a sharp drop in the normal stress, due to the evolution of damage to the material [Gurson (1977), Rousselier (1981)]. The second strategy describes the crack as a displacement discontinuity represented by the separation of its edges. Having to simulate the failure mechanism with propagation of dominant cracks assumed by the effective law, the code we have employed here uses the second strategy, together with a technique of intra-element propagation and automatic remeshing [Ferretti (2003a)]. The numerical constitutive law we have used, derived by the identified effective law, is shown in Fig. 27. The numerical analysis has been carried out on cylindrical specimens of concrete. On Fig.

28, one can see the final crack path provided by the code for the bottom left quarter of the longitudinal section.

The numerical curves of Figs. 25 and 26 refer to plain concrete, while those of Fig. 29 refer to concrete wrapped with sheets of carbon fiber composites (CFRP) [Ferretti and Di Leo (2003)]. These curves confirm that softening in plain concrete attains to the structure and does not necessarily correspond to material softening, whose existence is not guaranteed. Moreover, wrapped cylinders are modeled by means of a single material law, without any correction factor taking into account the triaxial state of stress induced by wrapping, as usually done. This is very appealing from the physical point of view and indirectly validates the effective law as a constitutive law. Actually, material models depending on the amount of wrapping are not strictly speaking constitutive. They are models of structural and not material behavior, which need to be calibrated on the single test. Also the size-effect is modeled without any parameter calibration: the different behaviors in Fig. 26 merely come from the faster decrease of $A_{res}$ at increasing $L/D$ ratios, well reproduced numerically on the basis of the different velocity of crack propagation

Finally, these results allow us to conclude that the monotone behavior of wrapped specimens is not induced by the composite stiffness, very high in comparison with the concrete one. It follows from the wrapping ability to oppose the modifications of resistant structure, leading the load-displacement and the effective law not to differ substantially in shape. The softening behavior of unwrapped specimens follows merely from the large modifications of resistant structure.

It is worth noting that, in order to simulate the modified interactions between material points, due to fracture propagation, nonlocal models must be able to continuously recompute the interaction weights for all interacting pairs of integration points. Recomputation is needed since long-range interaction between material points becomes more and more difficult, and finally impossible, as the fracture propagates. Thus, the interaction length must be decreased as the fracture propagates. This reflects in a high computational burden. Matters are different if a CM code with intra-element propagation is used [Ferretti (2003a)]. Actually, since the nonlocal approach is implicit into the CM, the modified nonlocal behavior is automatically taken into account as the geometry is updated. No further computational burden is required when an internal point becomes a boundary node due to the fracture propagation.

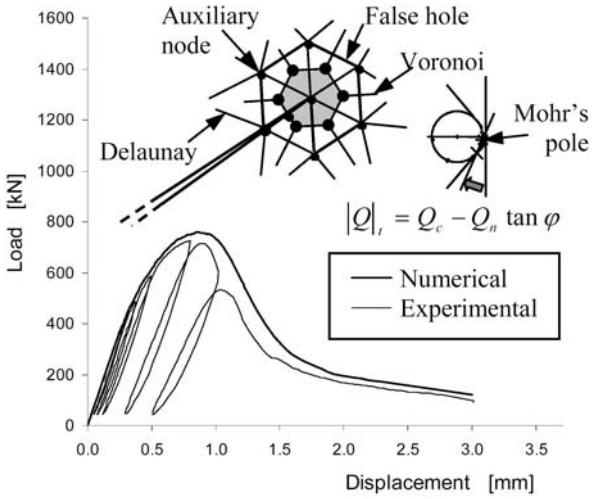


**Figure 25** : Comparison between numerical and experimental results for compressed concrete specimens

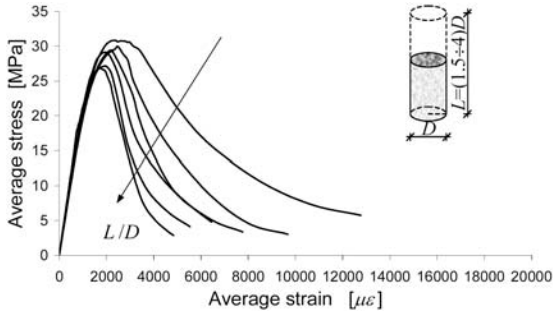


**Figure 26** : Numerically evaluated size-effect on average stress-average strain diagrams (see Fig. 1)

In Bažant and Chang (1984) and Jirásek and Rolshoven (2002), it was shown that numerical instabilities do not occur only if softening laws taking into account both the local and nonlocal effects are used. This means that the principle of the local action of the classical continuum mechanics must somehow be taken into account even in a nonlocal approach. This is exactly what happens in a CM code with a local constitutive model, nonlocality being ensured by the discrete formulation. In the opinion of the Author, the use of a local/nonlocal constitutive model in the FEM is thus equivalent to the use of a local constitutive model in the CM. This equivalence is also proved by

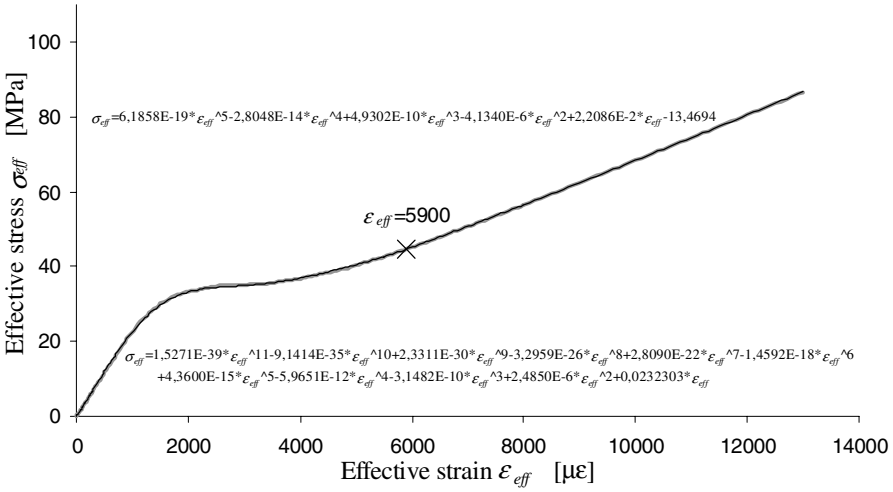


**Figure 27** : Monoaxial constitutive law adopted for concrete modeling

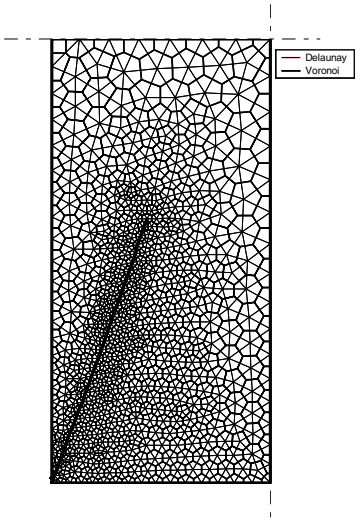


**Figure 28** : Crack path

the capability of the CM with local constitutive model to succeed where classical plasticity fails, requiring an improvement of the classical continuum description: modeling the size-effect (Fig. 26). Thus, one of the main historical reasons for improving the classical continuum description ceases to exist if the differential formulation is abandoned in favor of a discrete one. Numerical simulations of size-effect provided by a CM code with local constitutive model are collected in Ferretti (2003b).

It must be noticed that nonlocal theories aiming at regularizing the localization problem usually neglect the nonlocal elastic effects, and apply nonlocal averaging only to an internal variable (or thermodynamic force) linked to dissipative processes [Jirásek and Rolshoven (2002)]. This choice is justified on the basis of the smooth strain distribution characterizing the elastic regime, leading to a good approximation provided by the standard local theory. The implicit nonlocal approach of the CM allows us to take into account nonlocal effects in the elastic regime automatically too. This occurs since the CM nonlocality is derived from geometrical properties naturally linked to physical variables, and not from dissipative processes. Due to this implicit nonlocality, the transition between elastic regime and strain-localization regime is no longer critical for the accurateness of the numerical analysis. Thus, distinguishing between ante and after strain-localization regime is no longer necessary.

## 5 Conclusions

In this paper, two topics have been discussed, which usually are not put in direct relationship: nonlocal approaches and the existence of strain-softening. As far as the second topic is concerned, it has been shown how some of the expedients used for treating strain-softening by avoiding the numerical instability associated with a material tangent stiffness matrix that ceases to be positive definite are equivalent to introducing a scale separation between the load-displacement and stress-strain laws. That is, they are equivalent to employing a relationship between effective stress and effective strain that is, not necessarily, softening. On the other hand, the identification procedure of the effective law, assuming that strain-softening is not a real material property but the result of inhomogeneous deformation caused by the experimental technique, is based just on the assumption that we can and actually must separate the material (stress-strain) from the structure (load-displacement) scale. Such a procedure provides the first experimental evidence against the existence of strain-softening in concrete and identifies a monotone strictly nondecreasing effective law. However, we could argue that this law is not interesting from the numerical point of view, since, being a local law, it is not suitable for modeling problems in which the size-effect is involved.

As far as nonlocality is concerned, it has been observed that there is no evident physical justification for incorporating a length scale into the numerical formulation.

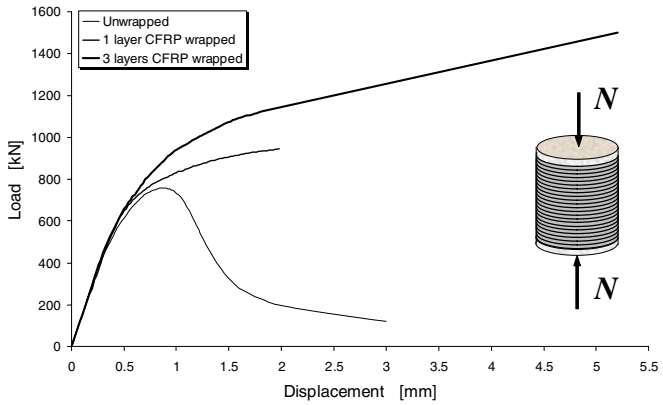


**Figure 29** : Numerical load-displacement curves for unwrapped and CFRP wrapped specimens

This need is only due to the use of a differential formulation, in which the length scale is absent since the metric notions have been lost in performing the limit process. The impossibility of treating some kind of problems in the context of a differential formulation with classical continuum description comes from the ill-posedness of describing Physics without taking into account the geometrical counterpart of variables. Moreover, incorporating a length scale into the constitutive relationships only, leaving unchanged the equilibrium and kinematic equations and the corresponding boundary conditions, is motivated by the need to provide a practical formulation and, once more, has no real physical justification. In effect, the discussion on where nonlocality must be taken into account (Section 2.2) concludes that it is physically more appealing to have governing rather than constitutive nonlocal laws. Thus, the incorporation of a length scale into the material description is not per-se necessary at all. What is really necessary is to describe the nonlocal nature of physical phenomena, that is, preserve the metrics of Physics. This may also be achieved by avoiding the limit process, with a discrete formulation. It has also been discussed here (Section 3.1) how the Cell Method (CM) is the only formulation that is truly discrete, at present. This method allows us to express the governing equations in discrete form directly, without loosing metrics. Several numerical similarities between the nonlocal approach and the CM, both in its classical and meshfree formulations, have also been highlighted in the present paper. In conclusion, if the goal is to perform a nonlocal analysis by means of the differential formulation, the constitutive equations must necessarily be modified in order to incorporate nonlocal effects. On the contrary, if the goal is to perform a nonlocal analysis by means of the CM, no need to modify the constitutive equations arises, since the CM is nonlocal in itself.

The question is not merely which type of continuum to associate with a differential or discrete formulation, nonlocal or local, respectively. The discussion takes on a deeper meaning and allows us to find an unexpected link between nonlocality and existence of strain-softening. If the discrete formulation is nonlocal in itself, any local law can be used in modeling size-effect, even the monotone effective law. The results provided here actually shows that a CM code using the effective law is able to model the size-effect on compressed concrete cylinders. This legitimates the effective law from the numerical point of view. Moreover, the use of the effective law in a CM code with nodal relaxation technique and automatic remeshing allows us to simulate the propagation of the dominant cracks during loading. Due to the scale separation involved by crack propagation, the load-displacement curve associated with the monotone effective law turns out to be softening, in good agreement with the experimental data. Consequently, the discrete nonlocal approach also provides proof of how the structural behavior may be softening even if the material behavior is not. This result together with the identification procedure of the effective law definitely reopens the question of strain-softening, whose existence and mathematical well-posedness seemed to be no longer under discussion after the numerical successes of nonlocal approaches. The existence of strain-softening must not to be taken as a dogma, even if the experiments seem to indicate so [Reinhardt and Cornelissen (1984)]. Interpretation of experimental data is a very delicate matter. If the current interpretation is too simplistic, such as both numerical and experimental results seem to indicate for strain-softening, it must be subjected to discussion. The identification procedure of the effective law moves in this direction, and provides an experimental local law that is suitable for numerical analyses with a CM code.

Finally, it is worth noting that, while a nonlocal formulation applicable to any inelastic constitutive model with softening is not available at present (Section 4), the ability which the CM code with effective law has of modeling both the size-effect and the wrapping using a single material law, without any parameter to calibrate on the specimen slenderness or load conditions, configures the

discrete nonlocal approach as very promising in view of a unified formulation for solid mechanics. The constitutive nature of the effective law is proved just by the absence of any parameter to calibrate on the single test.

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