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A Radial Basis Function Collocation Approach in Computational Fluid Dynamics

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Abstract: This paper explores the application of the mesh-free radial basis function collocation method for solution of heat transfer and fluid flow problems. The solution procedure is represented for a Poisson reformulated general transport equation in terms of a-symmetric, symmetric and modified (double consideration of the boundary nodes) collocation approaches. In continuation, specifics of a primitive variable solution procedure for the coupled mass, momentum, and energy transport representing the natural convection in an incompressible Newtonian Bussinesq fluid are elaborated. A comparison of different collocation strategies is performed based on the two dimensional De Vahl Davis steady natural convection benchmark with Prandtl number $\mathrm{Pr} = 0.71$, and Rayleigh numbers $\mathrm{Ra} = 10^3$, 10^4, 10^5, 10^6. Multiquadrics radial basis functions are used. The three methods are assessed in terms of streamfunction extreme, cavity Nusselt number, and mid-plane velocity components. Best performance is achieved with the modified approach.

keyword: radial basis function collocation method, heat transfer, fluid flow, natural convection.

1 Introduction

The development of efficient as well as simple algorithms for the numerical solution of partial differential equations (PDEs) is of major interest in applied sciences and engineering. The most popular discrete approximate methods for PDEs are nowadays the finite difference (FDM), finite volume (FVM), the finite element (FEM), the spectral (SM), and the boundary element methods (BEM). Despite the powerful features of these methods, there are often substantial difficulties in applying them to realistic, geometrically complex three dimensional transient situations. A common drawback of the mentioned methods is the need to create a polygonisation, either in the domain and/or on its boundary. This type of meshing is often the most time consuming part of the solution process and is far from being fully automated. In recent years, a new class of methods has been developed which do not require polygonisation but use a set of nodes to approximate the solution. The rapid development of these type of methods and their classification is elaborated in the very recent monographs [Atluri and Shen (2002a), Liu (2003), Atluri (2004a)]. A number of mesh reduction techiquies such as the dual reciprocity boundary element method [Šarler and Kuhn (1999)], meshfree techniques such as the dual reciprocity method of fundamental solutions [Šarler (2002)], meshfree local Petrov Galerkin methods (MLPG) [Lin and Atluri (2001a, 2001b), Atluri and Shen (2002b), Atluri, Han and Rajendran (2004b)] have been developed for transport phenomena and solution of the Navier-Stokes equations. A simplest class of mesh-free methods in development today are the Radial Basis Function [Buhmann (2000)] Collocation Methods (RBFCM) [Kansa (1990a), Kansa (1990b)] which represent focus in present paper.

2 Governing equations

For the present purposes, a transport phenomena problem can be briefly described in a general manner as the numerical solution of Eulerian transport equation, defined on a fixed domain Ω with boundary Γ, of the kind

$$\frac{\partial}{\partial t}\left[\rho f\left(\Phi\right)\right] + \nabla\cdot\left[\rho\vec{v} f\left(\Phi\right)\right] = \nabla\cdot\left(D\nabla\Phi\right) + S \tag{1}$$

with ρ, Φ, t, $\vec{v}$, S, and D standing for the density, transport variable, time, velocity, source, and diffusion matrix. The transport variable stands, for instance for the velocity component in each co-ordinate direction, or temperature, or the mass fraction of a chemical species. The function f denotes the relation between the transported and the diffused variable such as for example the relation between the enthalpy and the temperature. The solution of the governing equation for the dependent variable Φ

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at final time $t = t_0 + \Delta t$ is sought, where t_0 represents the initial time and Δt the positive time increment. The solution is constructed by the initial and boundary conditions that follow. The initial value of the transport variable Φ at a point with position vector $\vec{p}$ and time t_0 is defined through the known function Φ_0

$$\Phi(\vec{p}, t_0) = \Phi_0; \ p \in \Omega \cup \Gamma \tag{2}$$

The boundary Γ is divided into not necessarily connected parts Γ^D and Γ^N with Dirichlet and Neumann boundary conditions, respectively. These boundary conditions are at the boundary point $\vec{p}$ with normal $\vec{n}_\Gamma$ and time $t < t \leq t_0 + \Delta t$ defined through known functions Φ^D and Φ^N

$$\Phi = \Phi^D; \ \vec{p} \in \Gamma^D, \frac{\partial}{\partial \vec{n}_\Gamma}\Phi = \Phi^N; \ \vec{p} \in \Gamma^N \tag{3}$$

The involved parameters of the governing equation and boundary conditions are assumed to depend on the transport variable, space and time. The solution procedure thus inherently involves iterations. The governing equation is transformed into a Poisson form as follows. The diffusion matrix is split into constant isotropic part DI and the remaining an-isotropic part D'

$$\underline{D} = D\underline{I} + \underline{D}' \tag{4}$$

The transport equation is subsequently cast into Poisson form

$$\nabla^2\Phi = \theta + \nabla \cdot \vec{\Theta} \tag{5}$$

with

$$\theta = \left\{ \frac{\partial}{\partial t}\left[\rho f(\Phi)\right] - S \right\} / D \tag{6}$$

$$\vec{\Theta} = \left[\rho \vec{v} f(\Phi) - D' \nabla \Phi\right] / D \tag{7}$$

The partial time derivative can be approximated by a two-level fully implicit finite difference

$$\frac{\partial}{\partial t}\left[\rho f(\Phi)\right] \approx \frac{1}{\Delta t}\left[\rho f(\Phi) - \rho_0 f(\Phi_0)\right] \tag{8}$$

The inhomogenous terms are Taylor expanded as

$$\theta = \theta + \theta_{,\Phi}\left(\Phi - \overline{\Phi}\right), \quad \vec{\Theta} = \vec{\overline{\Theta}} + \vec{\overline{\Theta}}_{,\Phi}\left(\Phi - \overline{\Phi}\right) \tag{9}$$

with the 'over-bar' denoting value at previous iteration. The final form of the transformed equation, suitable for iterative solution than becomes

$$\nabla^2\Phi = Q(\Phi) \tag{10}$$

$$Q(\Phi) = \overline{\theta} + \overline{\theta}_{,\Phi}\left(\Phi - \overline{\Phi}\right) + \nabla \cdot \vec{\overline{\Theta}} + \nabla \cdot \vec{\overline{\Theta}}_{,\Phi}\left(\Phi - \overline{\Phi}\right) \tag{11}$$

The solution of equation (10) might sometimes require additional under-relaxation

$$\Phi = \varepsilon_{rel}\left(\overline{\Phi} - \overline{\overline{\Phi}}\right) + \overline{\overline{\Phi}} \tag{12}$$

with the two 'over-bars' denoting the value of the unknown two iterations ago and $0 < \varepsilon_{rel} < 1$ the relaxation factor.

3 Radial Basis Function Collocation Method

3.1 Description of fields and partial derivatives

The unknown filed Φ is calculated in points $\vec{p}_n$; $n = 1, ..., N = N_\Gamma + N_\Omega$. The first N_Γ belong to the boundary and the last N_Ω to the domain. The unknown field is approximated by M approximation functions ψ_m and their coefficients ς_m of the type

$$\Phi(\vec{p}) \approx \sum_{m=1}^{M} \varsigma_m \psi_m(\vec{p}) \tag{13}$$

Similarly, a partial derivative over coordinate ζ_m of point $\vec{p}(\zeta_1, \zeta_2, \zeta_3)$ can be approximated as

$$\frac{\partial}{\partial \zeta_i}\Phi(\vec{p}) \approx \sum_{n=1}^{M} \varsigma_m \frac{\partial}{\partial \zeta_i}\left[\psi_m(\vec{p})\right] \tag{14}$$

$$\frac{\partial^2}{\partial \zeta_i^2}\Phi(\vec{p}) \approx \sum_{n=1}^{M} \varsigma_m \frac{\partial^2}{\partial \zeta_i^2}\left[\psi_m(\vec{p})\right] \tag{15}$$

by assuming sufficient admissibility of functions ψ_m. A suitable function is the multiquadric

$$\omega_m(\vec{p}) = \left(r_m^2 + c^2\right)^{1/2} \tag{16}$$

with r_m standing for the Euclidean distance between the field point $\vec{p}$ and the reference point $\vec{p}_m$ and c for the free parameter. The coefficients ς_m of the solution can be determined in various ways. The following three collocation strategies have been put into context of solving transport phenomena in this paper.

3.1.1 A-symmetric collocation

The most simple is the a-symmetric collocation, originally proposed by Kansa [1990a,1990b]. The coefficients ς_m of the solution are determined from a system of collocation equations. The first N_Γ equations read

$$\sum_{m=1}^{M} \varsigma_m(\chi_i^D \psi_{mi} + \chi_i^N \frac{\partial}{\partial n_\Gamma}\psi_{mi}) = \chi_i^D \Phi_i^D + \chi_i^N \Phi_i^N; i = 1,...,N_\Gamma \tag{17}$$

and the last $M - N_\Gamma$ equations read

$$\sum_{m=N_\Gamma+1}^{M} \varsigma_m \nabla^2 \psi_{mi} = Q_i;\ i = N_\Gamma + 1,...,M; \tag{18}$$

by setting

$$\psi_{mi} = \omega_{mi}, \quad M = N \tag{19}$$

The first N_Γ equations (17) have been written for the boundary points, and the last N^Ω equations (18) for the domain points, and $\mathrm{F}_i \equiv \mathrm{F}(\vec{p}_i)$, where F stands for an arbitrary function. The introduced boundary condition indicators take form

$$\chi_i^D = \begin{cases} 1; \vec{p}_i \in \Gamma^D \\ 0; \vec{p}_i \notin \Gamma^D \end{cases}, \quad \chi_i^N = \begin{cases} 1; \vec{p}_i \in \Gamma^N \\ 0; \vec{p}_i \notin \Gamma^N \end{cases} \tag{20}$$

3.1.2 Symmetric collocation

The system of equations (16,17) can be made symmetric by setting the Fasshauer's [Fasshauer (1996)] ansatz

$$\psi_{mi} = \sum_{m=1}^{N_\Gamma} \left(\chi_m^D \omega_{mi} + \chi_m^N \frac{\partial \omega_{mi}}{\partial n_\Gamma} \right) + \sum_{m=N_\Gamma+1}^{M} \nabla^2 \omega_{mi},$$

$$M = N;\ \ i = 1,....,M \tag{21}$$

The right-hand-side vector remains the same as in equations (17) and (18).

The symmetric solution gives better results [Power and Barraco (2002)] as the a-symmetric one.

3.1.3 Modified collocation

If we attempt the solution only in the boundary nodes or only in the domain nodes the information either on the governing equation or on the boundary conditions is lost in these two extremes. In boundary points, the governing equation as well as boundary conditions have to be valid. Respectively, the system of equations (17,18) can be made symmetric and consistent by attempting the solution with double consideration of the boundary nodes, as deduced in [Chen (2002)], by setting

$$\psi_{mi} = \sum_{m=1}^{N_\Gamma} \left(\chi_m^D \omega_{mi} + \chi_m^N \frac{\partial \omega_{mi}}{\partial n_\Gamma} \right) + \sum_{m=N_\Gamma+1}^{M} \nabla^2 \omega_{(m-N_\Gamma)i},$$

$$M = N_\Gamma + N \tag{22}$$

Collocation is first made in the boundary nodes where the boundary conditions are valid and subsequently in the boundary and domain nodes where the governing equation is valid. By inserting the approximation (22) into framework (17,18) gives an $(N_\Gamma + N) \times (N_\Gamma + N)$ symmetric system of algebraic equations. The right-hand-side vector is equal to

$$\chi_i^D \Phi_i^D + \chi_i^N \Phi_i^N; i = 1,...,N_\Gamma \tag{23}$$

$$\chi_{i-N_\Gamma}^D \Phi_{i-N_\Gamma}^D + \chi_{i-N_\Gamma}^N \Phi_{i-N_\Gamma}^N; i = N_\Gamma + 1,...,2N_\Gamma \tag{24}$$

$$Q_{i-2N_\Gamma};\ \ i = 2N_\Gamma + 1,...,N \tag{25}$$

3.1.4 The convergence criterion

The timestep iterations are stopped when the condition

$$|\Phi^{j+1}(\vec{p}_i, t) - \Phi^j(\vec{p}_i, t)| < \varepsilon_{itr} \tag{26}$$

is satisfied in all gridpoints $\vec{p}_i$, where j counts iteration level.

3.1.5 The steady-state criterion

The steady-state is assumed to be reached when the condition

$$|\Phi(\vec{p}_i, t + \Delta t) - \Phi(\vec{p}_i, t)| < \varepsilon_{ste} \tag{27}$$

is satisfied in all gridpoints $\vec{p}_i$.

4 Natural Convection Problem

In order to demonstrate the application and suitability of the represented solution procedures for a typical computational fluid dynamics problem, the following system of coupled mass, momentum, and energy equations is used. It describes the natural convection of an incompressible Newtonian Bussinesq fluid

$$\nabla \cdot \vec{v} = 0, \tag{28}$$

$$\frac{\partial}{\partial t}(\rho\vec{v})+\nabla\cdot(\rho\vec{v}\vec{v}) = -\nabla P+\mu\nabla^2\vec{v}+\rho\vec{a}[1-\beta(T-T_{\text{ref}})] \quad (29)$$

$$\frac{\partial}{\partial t}(\rho c_P T)+\nabla\cdot(\rho\vec{v}c_P T)=k\nabla^2 T \quad (30)$$

with P standing for pressure, μ for viscosity, $\vec{a}$ for acceleration, β for the volumetric thermal expansion coefficient, T_0 for the Bussinesq reference temperature, c_P for specific heat, k for thermal conductivity, and T for temperature. All material properties are assumed constant.

5 Solution Strategy

The energy equation is coupled with the momentum equation through the velocity field and the momentum equation is coupled with the energy equation through the body force. Respectively, the solution inherently involves iterations. Let us assume the velocity, pressure and temperature fields are all known at iteration level j.What follows explains the solution at the iteration level $j+1$.

5.1 Solution of the momentum equation

The solution of the momentum equation at iteration level $j+1$ is obtained in the following way: The Pressure Poisson Equation (PPE) is constructed by taking the divergence of the momentum equation

$$\nabla^2 P^{j+1}=\nabla\cdot\left\{-\frac{1}{\Delta t}\left(\rho\vec{v}^{\,j}-\rho\vec{v}_0\right)-\nabla\cdot\left(\rho\vec{v}^{\,j}\vec{v}^{\,j}\right) +\mu\nabla^2\vec{v}^{\,j}+\rho\vec{a}\left[1-\beta\left(T^j-T_{\text{ref}}\right)\right]\right\} \quad (31)$$

The Neumann pressure boundary conditions can be defined along the whole boundary Γ by taking the scalar product of the momentum equation with the normal on the boundary. This gives

$$\frac{\partial}{\partial n_\Gamma}P^{j+1}=\left\{-\frac{1}{\Delta t}\left(\rho\vec{v}^{\,j}-\rho\vec{v}_0\right)-\nabla\cdot\left(\rho\vec{v}^{\,j}\vec{v}^{\,j}\right) +\mu\nabla^2\vec{v}^{\,j}+\rho\vec{a}\left[1-\beta\left(T^j-T_{\text{ref}}\right)\right]\right\}\cdot\vec{n}_\Gamma \quad (32)$$

After calculating the pressure gradient field the velocity field at iteration level $j+1$ can be solved from the velocity Poisson equation

$$\nabla^2\vec{\hat{v}}^{j+1}=\frac{1}{\mu}\left\{\frac{1}{\Delta t}\left(\rho\vec{\hat{v}}^{j+1}-\rho\vec{v}_0\right)+\nabla\cdot\left(\rho\vec{v}^{\,j}\vec{v}^{\,j}\right) +\nabla P^{j+1}-\rho\vec{a}\left[1-\beta\left(T^j-T_{\text{ref}}\right)\right]\right\} \quad (33)$$

The "hat" on the velocity denotes that the velocity does not correspond to the mass conservation in general. The incompressibility is enforced through the pressure $\tilde{P}$ and velocity corrections $\vec{\tilde{v}}$, which ensure

$$\nabla\cdot\vec{v}^{\,j+1}=\nabla\cdot\left(\vec{\hat{v}}^{j+1}+\vec{\tilde{v}}^{j+1}\right) \quad (34)$$

Consider that the velocity corrections occur exclusively due to action of the pressure correction $\tilde{P}$

$$\varepsilon_{Pv}\frac{\rho}{\Delta t}\vec{\tilde{v}}^{j+1}=-\nabla\tilde{P}^{j+1} \quad (35)$$

where ε_{Pv} represents a heuristic velocity correction – pressure correction relaxation factor. The pressure correction can thus be calculated from the velocity field $\vec{\hat{v}}$ through the Pressure Correction Poisson Equation (PCPE)

$$\nabla^2\tilde{P}^{j+1}=\varepsilon_{Pv}\frac{\rho}{\Delta t}\vec{\hat{v}} \quad (36)$$

deduced from the equations (34) and (35). Since no correction is needed in the direction normal to the solid boundary, the following pressure correction boundary conditions are valid

$$\frac{\partial}{\partial n_\Gamma}\tilde{P}^{j+1}=0;\ \vec{p}\in\Gamma \quad (37)$$

Both PPE and PCPE are singular due to the presence of the Neumann boundary conditions over the whole boundary. One of the domain grid-points $\vec{p}_0$ is fixed to the reference pressure P_0 in case of the PPE, and to 0 in case of the PCPE, in order to avoid the singularity

$$P^{j+1}(\vec{p}_0)=P_0;\ \vec{p}_0\in\Omega \quad (38)$$

$$\tilde{P}^{j+1}(\vec{p}_0)=0;\ \vec{p}_0\in\Omega \quad (39)$$

The domain point which coincides with the point $\vec{p}_0$ is treated in a formally equivalent way as it would represent a boundary point with prescribed Dirichlet boundary conditions.

5.2 Solution of the energy equation

The energy equation is solved from

$$\nabla^2 T^{j+1} = \frac{1}{k}\left\{\frac{1}{\Delta t}\left(\rho c_P T^{j+1} - \rho c_P T_0\right) + \nabla\cdot\left(\rho\vec{v}^{j+1} c_P T^j\right)\right\} \tag{40}$$

At the end of the iteration, the body force is updated with the new value of the temperature and the solution is repeated until the iteration and steady state conditions are met for the pressure, velocity components, and temperature.

6 Numerical Results

6.1 Natural convection [De Vahl Davis (1983)] benchmark

Geometry is a square with dimension L. Two dimensional Cartesian coordinates p_x, p_y are used. The square is extending from $p_{x-} \le p_x \le p_{x+}$, $p_{y-} \le p_y \le p_{y+}$, $p_{xy\pm} = \pm L/2$. Upper and lower boundaries are insulated, the left boundary is subject to temperature T_+, the right boundary is subject to temperature T_-. The solution is performed for Prandtl number $\mathrm{Pr} = \mu c_P/k = 0.71$, and for Rayleigh numbers $\mathrm{Ra} = a\beta(T_+ - T_-)L^3\,\mathrm{Pr}\,\rho^2\mu^2 = 10^3$, 10^4, 10^5, 10^6. The square is discretized in an uniform grid 20×20 or 30×30. The first (second) grid involves 441 (961) points of which 80 (120) are on the boundary and 361 (841) in the domain. The dimensionless time-step used in the calculations has been set to $\Delta t = 10/\mathrm{Ra}$. The constant in multiquadrics has been set to $c = 1.75\Delta L$, where ΔL represents a typical grid distance. The global relaxation factor has been set to $\varepsilon_{rel} = 1$, the velocity correction – pressure correction relaxation factor has been set to $\varepsilon_{Pv} = 1$. The iteration criterion and the steady-state criterion have been set to $\varepsilon_{itr} = 10^{-5}$, $\varepsilon_{ste} = 10^{-4}$, respectively. The temperature is distributed linearly from the left to the right of the cavity and no flow is assumed at the initial time. The simulations have been performed with three different solution procedures: with the symmetric one (SYM) [Šarler, Perko Chen and Kuhn (2001)], with the a-symmetric one (ASY) and with the modified one (MOD). The dimensions of the Poisson equations are for the grids 20×20 equal to 441×441 in case of the ASY and SYM procedures, and equal to 521×521 in case of the MOD procedure. The dimensions of the Poisson equations are for the grids 30×30 equal to 961×961 in case of the ASY and SYM procedures, and equal to 1081×1081 in case of the MOD procedure.

6.2 Calculation of the streamfunction

The streamfunction Ψ is calculated as

$$\Psi = \int_{p_{y-}}^{p_y} v_x dp_y, \quad \Psi = -\int_{p_{x-}}^{p_x} v_y dp_x \tag{41}$$

A variation of the velocity components over the domain Ω and boundary Γ is based on the global approximation

$$v_x = \sum_{m=1}^{M} \varsigma_m^{v_x}\psi_m, \quad v_y = \sum_{m=1}^{M} \varsigma_m^{v_y}\psi_m \tag{42}$$

Respectively, Ψ can be calculated as

$$\Psi = \sum_{m=1}^{M} \varsigma_m^{v_x}\int_{p_{y-}}^{p_y}\psi_m dp_y, \quad \Psi = -\sum_{m=1}^{M} \varsigma_m^{v_y}\int_{p_{x-}}^{p_x}\psi_m dp_x \tag{43}$$

6.3 Calculation of the Nusselt number

The cavity Nusselt number Nu is calculated as

$$\mathrm{Nu} = \frac{1}{T_+ - T_-}\int_{p_{y-}}^{p_{y+}} \frac{\partial}{\partial p_x} T_m dp_y \tag{44}$$

A variation of the temperature over the domain Ω and boundary Γ is based on the global approximation

$$T = \sum_{m=1}^{M} \varsigma_m^{T}\psi_m \tag{45}$$

Respectively, Nu can be calculated as

$$\mathrm{Nu} = \frac{1}{T_+ - T_-}\sum_{m=1}^{M} \varsigma_m^{T}\int_{p_{y-}}^{p_{y+}} \frac{\partial}{\partial p_x}\psi_m dp_y \tag{46}$$

All involved integrals of RBFs in numerical evaluation of the Ψ and Nu are evaluated in an analytical way.

6.4 Numerical implementation

The numerical implementation is made double precision in Compaq Visual Fortran 90 with IMSL library. Test cases have been run on an HP Omnibook XE_3 laptop

with an Intel Pentium III 850MHz processor, 256MB ram and Windows 2000 Professional operating system. The system matrices have been first LU decomposed by using the routine DLFRGT and then solved by the routine DLFSRG at each iteration. The 30×30 MOD CPU time is 7.5 hours for $\mathrm{Ra} = 10^6$.

6.5 *Discussion of the results*

The number of required iterations remains almost the same in all three methods as seen in Table 1. Almost without exception, the SYM approach gives better results than the ASY one, and the MOD approach gives better results than the SYM one, as could be clearly concluded from the Tables 2-5. The exceptions might originate in the fact that the reference results are estimated to be accurate by less then 1% at $\mathrm{Ra} = 10^6$ and probably by less than a tenth of that at the lower Rayleigh numbers as stated in [De Vahl Davis (1983)].

Table 1 : The number of time-steps, and internal time-step iterations used as a function of the solution procedure and problem.

Ra	procedure	time-steps	Iterations
10^3	ASY	33	340
10^3	SYM	34	327
10^3	MOD	37	332
10^4	ASY	87	870
10^4	SYM	88	853
10^4	MOD	88	861
10^5	ASY	188	2062
10^5	SYM	191	1991
10^5	MOD	194	2007
10^6	ASY	295	3531
10^6	SYM	299	3501
10^6	MOD	304	3529

Table 2 : Natural convection in a square cavity. The maximum value of the vertical velocity $v_{y\,\max}$ on the horizontal midplane

Ra	Grid	ASY	SYM	MOD	DeVahl Davis 1983
10^3	20×20	3.551	3.562	3.668	3.697
10^3	30×30	3.566	3.583	3.680	3.697
10^4	20×20	18.97	19.13	19.54	19.62
10^4	30×30	19.04	19.20	19.59	19.62
10^5	20×20	66.60	66.90	67.03	68.59
10^5	30×30	67.59	68.01	68.27	68.59
10^6	20×20	205.11	207.67	209.31	219.36
10^6	30×30	211.67	213.81	215.56	219.36

Table 3 : Natural convection in a square cavity. The maximum value of the horizontal velocity $v_{x\,\max}$ on the vertical midplane.

Ra	grid	ASY	SYM	MOD	DeVahl Davis 1983
10^3	20×20	3.530	3.541	3.621	3.649
10^3	30×30	3.544	3.600	3.633	3.649
10^4	20×20	15.53	15.56	15.99	16.18
10^4	30×30	15.80	16.03	16.09	16.18
10^5	20×20	31.86	31.96	32.11	34.73
10^5	30×30	32.51	33.70	34.08	34.73
10^6	20×20	59.67	61.01	61.48	64.63
10^6	30×30	61.55	62.64	63.12	64.63

7 Conclusions

This paper explores three RBFCM strategies for dealing with transport phenomena. The asymmetric, symmetric and modified formulations are critically evaluated based on the classical De Vahl Davis benchmark test. Even the most simple of them, the a-symmetric formulation performs well for a broad spectra of CFD problems [Šarler, Perko, Chen, and Kuhn (2001), Kovačević, Poredoš and Šarler (2003), Šarler, Perko and Chen (2004)]. The other two formulations can be successfully applied to boost the performance of the first one on the expense of

Table 4 : Natural convection in a square cavity. Values of stream-function extreme Ψ at the center of the cavity as a function grid density and collocation scheme.

Ra	grid	ASY	SYM	MOD	DeVahl Davis 1983
10^3	20×20	1.153	1.161	1.165	1.174
10^3	30×30	1.165	1.167	1.170	1.174
10^4	20×20	4.833	4.841	4.850	5.071
10^4	30×30	4.971	4.980	4.995	5.071
10^5	20×20	8.861	8.880	8.901	9.111
10^5	30×30	8.907	8.950	9.073	9.111
10^6	20×20	15.03	15.41	15.88	16.32
10^6	30×30	15.91	16.07	16.22	16.32

Table 5 : Natural convection in a square cavity. Values of cavity Nusselt number Nu as a function of grid density and collocation scheme.

Ra	grid	ASY	SYM	MOD	DeVahl Davis 1983
10^3	20×20	1.111	1.119	1.125	1.118
10^3	30×30	1.100	1.108	1.114	1.118
10^4	20×20	2.236	2.247	2.249	2.243
10^4	30×30	2.241	2.244	2.246	2.243
10^5	20×20	4.647	4.640	4.573	4.519
10^5	30×30	4.530	4.528	4.523	4.519
10^6	20×20	8.921	8.909	8.876	8.800
10^6	30×30	8.869	8.845	8.834	8.800

slightly more complicated coding. These formulations include calculation of the higher order spatial derivatives which might impair the convergence rate and pose special smoothness requirements on the RBFs. Another possibility is approximation of the second derivatives by the expression (13) and employment of the first and second order integration to approximate first order derivative and function itself. This method is called the Indirect Radial Basis Function Collocation Method (IRBFCM) and has been explored for solution of the Navier-Stokes equations in [Mai-Dui and Tran-Cong (2001)]. The main disadvantage of the presented methods, including the IRBFCN are full matrices that are difficult and expensive to solve for large-scale problems. This issue might be in the future mitigated or overcome by the use of the compactly supported radial basis functions [Chen, Brebbia and Power (1999)], multilevel radial basis functions [Fasshauer (1998)], iterative solvers [Bulgakov, Šarler and Kuhn (1998)], adaptive grid or domain decomposition [Mai-Dui and Tran-Cong (2002)] or the local collocation [Lee, Liu and Fan (2003)]. It would be too ambitious to claim that the represented method can cope with a wide variety of large-scale computational science and engineering problems at this point. Additional research in this direction is definitely required.

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Appendix A

Transformations of the governing equations into Poisson form follow through the following definitions

Energy Equation

Variable Definition

Φ	T
$f(\Phi)$	$c_P T$
D	k
$D'_{\varsigma\xi}$	0
S	0
θ	$\rho\,[f(\Phi)-\rho f(\Phi_0)]/(\Delta t\,k)$
$\theta_{,\Phi}$	$\rho\,c_P f(\Phi)\,/(\Delta t\,k)$
Θ_x	$\rho\,v_x f(\Phi)$
Θ_y	$\rho\,v_y f(\Phi)$
$\Theta_{x,\Phi}$	$\rho\,v_x c_P$
$\Theta_{y,\Phi}$	$\rho\,v_y c_P$

Momentum Equation x-direction

Variable Definition

Φ	v_x
$f(\Phi)$	v_x
D	μ
$D'_{\varsigma\xi}$	0
S	$-P_{,x}+\rho a_x\left[1-\beta\left(T-T_{\text{ref}}\right)\right]$
θ	$\rho\left(\Phi-\Phi_0\right)/\left(\Delta t\,\mu\right)-S/\mu$
$\theta_{,\Phi}$	$\rho/\Delta t\,\mu$
Θ_x	$\rho\Phi^2/\mu$
Θ_y	$\rho v_y\Phi/\mu$
$\Theta_{x,\Phi}$	$2\rho\Phi/\mu$
$\Theta_{y,\Phi}$	$\rho v_y/\mu$

Momentum Equation y-direction

Variable Definition

Φ	v_y
$f(\Phi)$	v_y
D	μ
$D'_{\varsigma\xi}$	0
S	$-P_{,y}+\rho a_y\left[1-\beta\left(T-T_{\text{ref}}\right)\right]$
θ	$\rho\left(\Phi-\Phi_0\right)/\left(\Delta t\,\mu\right)-S/\mu$
$\theta_{,\Phi}$	$\rho/\Delta t\,\mu$
Θ_x	$\rho v_x\Phi/\mu$
Θ_y	$\rho\Phi^2/\mu$
$\Theta_{x,\Phi}$	$\rho v_x/\mu$
$\Theta_{y,\Phi}$	$2\rho\Phi/\mu$

Pressure Poisson Equation

Variable Definition

Φ	P
$f(\Phi)$	P
D	1
$D'_{\varsigma\xi}$	0
S	0
θ	0
$\theta_{,\Phi}$	0
Θ_x	$-\rho\left(v_x-v_{x0}\right)/\Delta t+\rho\left(v_xv_{x,x}-v_yv_{x,y}\right)$ $+\mu\left(v_{x,xx}+v_{x,yy}\right)+\rho a_x\left[1-\beta\left(T-T_{\text{ref}}\right)\right]$
Θ_y	$-\rho\left(v_y-v_{y0}\right)/\Delta t+\rho\left(v_xv_{y,x}-v_yv_{y,y}\right)$ $+\mu\left(v_{y,xx}+v_{y,yy}\right)+\rho a_y\left[1-\beta\left(T-T_{\text{ref}}\right)\right]$
$\Theta_{x,\Phi}$	0
$\Theta_{y,\Phi}$	0

Pressure Correction Poisson Equation

Variable Definition

Φ	$\tilde{P}$
$f(\Phi)$	$\tilde{P}$
D	1
$D'_{\varsigma\xi}$	0
S	0
θ	0
$\theta_{,\Phi}$	0
Θ_x	$-\rho v_x$
Θ_y	$-\rho v_y$
$\Theta_{x,\Phi}$	0
$\Theta_{y,\Phi}$	0