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On the use of a wave based prediction technique for steady-state structural-acoustic radiation analysis

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Abstract: Conventional element based methods for modelling structural-acoustic radiation problems are limited to low-frequency applications. Recently, a novel prediction technique has been developed based on the indirect Trefftz approach. This new wave based method is computationally more efficient than the element based methods and, as a consequence, can tackle problems also at higher frequencies. This paper discusses the basic principles of the new method and illustrates its performance for the two-dimensional radiation analysis of a bass-reflex loudspeaker.

keyword: structural-acoustics, radiation, Trefftz method, wave based method, mid-frequency analysis

1 Introduction

The use of element based prediction techniques such as the finite element (FE) method, the infinite element (IE) method and the boundary element (BE) method, is generally accepted for the steady-state dynamic analysis of coupled structural-acoustic radiation problems [Desmet, Pluymers, and Sas (2003)].

In the FE and the IE method, the problem domain is discretized into a finite number of small elements. *FE methods* [Zienkiewicz and Taylor (2000)] lead to system matrices which can be partly decomposed into symmetric, frequency independent, real, sparsely populated, banded submatrices. To be able to apply these FE methods for structural-acoustic radiation problems, an artificial truncation surface is introduced to truncate the unbounded acoustic domain (see figure 1) [Wolf and Song (1996)]. Some characteristic impedance model is applied at this truncation surface to prevent acoustic reflections [Keller and Givoli (1989); Coyette (1992); Nicholls and Nigam (2004)]. A distinction is made between local and global impedance models. Local impedance models apply characteristic impedance boundary conditions to the truncation surface. The impedance values are determined using locally defined properties. In this way, the sparsely populated, banded matrix structure is retained. Global impedance models, however, like for instance a Dirichlet to Neumann mapping [Harari, Patlashenko, and Givoli (1998)], relate all the degrees of freedom on the artificial truncation surface to each other such that the sparsity of the system matrices is significantly reduced.

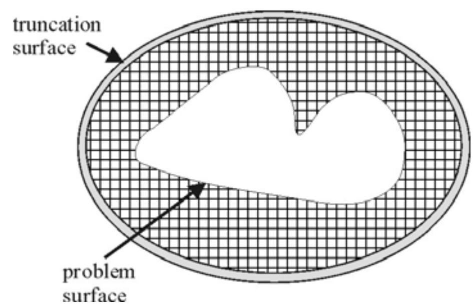

Figure 1 : An FE model with truncating boundary surface

The *IE method* [Bettess (1992); Astley, Macaulay, Coyette, and Cremers (1998); Gerdes (2000)] explicitly models the domain exterior to the artificial truncation surface. Infinite elements discretize the exterior domain and are coupled at the artificial truncation surface with the finite element discretization of the bounded interior domain (see figure 2). The shape functions used for modelling the acoustic field variables within the infinite elements combine a suitable amplitude decay and a wave-like variation for modelling outgoing travelling waves. By increasing the radial order of these shape functions, the accuracy increases, but so does the computational load. A distinction is made between *conjugated* infinite elements (or *wave envelope elements*) and *unconjugated* infinite elements, depending on whether the weighting functions in the integral formulation are, respectively, the complex conjugate of the shape functions or identical to

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the shape functions [Shirron and Babuska (1998); Ihlenburg (2000)]. With the conjugated formulation, the resulting system matrices are frequency independent and involve (a fairly simple) integration of polynomial functions but are not symmetric. The unconjugated formulation yields matrices that are symmetric but frequency dependent. Moreover, matrix coefficients result from more tedious numerical integrations [Astley (1999); Cremers, Fyfe, and Sas (2000); Harari, Barai, Barbone, and Slavutin (2001)].

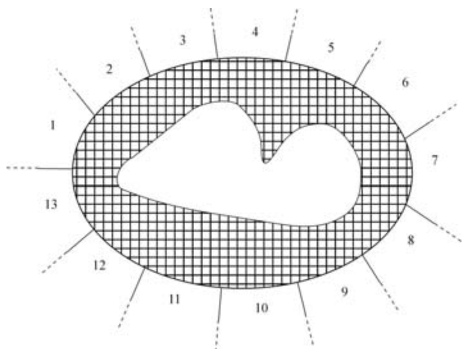

Figure 2 : An IE model with 13 infinite elements

BE methods [Kirkup (1998); Von Estorff (2000); Ochmann, Homm, Makarov, and Semenov (2003)] discretize only the problem boundary surface (see figure 3). Their solutions are based on a boundary integral formulation that relates the dynamic field variables in the (bounded or unbounded) domain to the distribution of some specific variables on the problem boundary surface. This integral formulation satisfies inherently the Sommerfeld radiation condition so that no special treatment is needed to account for the problem domain possibly being unbounded. Since only the boundary surface is discretized, BE models are small. However, drawbacks of these methods are the fully populated, frequency dependent, complex and not always symmetric system matrices which lead to computationally demanding calculations.

All element based methods express the dynamic response variables in terms of simple but approximating shape functions. Because of this approximation, a sufficient number of elements per wavelength is required to obtain reasonable accuracy. With increasing frequency, wavelengths shorten, so that the number of elements must increase accordingly to maintain the same level of accuracy. Therefore, computational resources restrict the

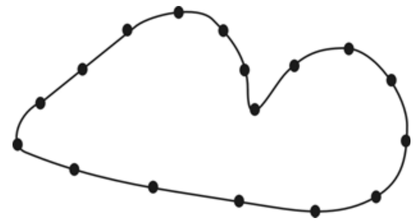

Figure 3 : A BE model

practical use of these element based methods to low-frequency applications [Ihlenburg and Babuska (1995)]. By applying certain approximations, like the Rayleigh and Kirchoff approximation, also high-frequency applications can be analyzed [Junger and Feit (1993); Herrin, Martinus, Wu, and Seybert (2003)]. These cost-efficient approximations yield accurate results when the radii of curvature of the problem boundary surface, measured in terms of acoustic wavelengths, becomes very large, which is the case for high-frequency applications.

In recent years, a vast amount of research has been done on alternative methods for the analysis of structural-acoustic radiation problems, in order to find an adequate method for the mid-frequency range; see for instance, ongoing research regarding boundary element formulations [Qian, Han, and Atluri (2004); Qian, Han, Ufimtsev, and Atluri (2004); Callsen, Von Estorff, and Zaleski (2004)], Galerkin least-squares methods [Thompson and Pinsky (1995); Babuska and Sauter (1997)], discontinuous Galerkin methods [Farhat, Wiedemann-Goiran, and Tezaur (2004)], element-free Galerkin methods [Babuska and Melenk (1997); Bouillard, Lacroix, and De Bel (2004)], meshless Petrov-Galerkin methods [Atluri (2004)] and Trefftz based methods [Fairweather, Karageorghis, and Martin (2003); Alves and Valtchev (2003)]. A recently developed wave based method (WBM) [Desmet (1998)] is part of the latter family of numerical methods. The WBM is based on the indirect Trefftz approach [Trefftz (1926)] and has proven to be successful for low- and mid-frequency applications [Desmet, Van Hal, Sas, and Vandepitte (2002); Pluymers, Desmet, Vandepitte, and Sas (2004)]. Instead of using simple, approximating shape functions, exact solutions of the governing differential equations are used to describe the dynamic variables. Fine discretization of the domains is not required for convex domains. Concave domains require a subdivision into convex subdomains, which is straightforward for domains of moderate geometrical complexity. Model size and subsequent computational efforts are much smaller than with element based methods. This allows to handle also mid-frequency applications. Recently, hybrid methods have been developed, coupling the WBM with conventional element based methods, in order to broaden the novel method applicability [Van Hal, Desmet, Vandepitte, and Sas (2003a); Van Hal, Desmet, Vandepitte, and Sas (2003b); Van Hal, Desmet, and Vandepitte (2005)].

This paper discusses how the WBM can be extended for radiation problems in unbounded domains. The technique is illustrated for the two-dimensional analysis of the sound radiation of a bass-reflex loudspeaker and its performance is compared with the conventional element based techniques.

2 Problem definition

Figure 4 shows a two-dimensional (2D) bass-reflex loudspeaker. The thin flexible membrane Γ_s is excited by a normal line force F at circular frequency ω and radiates sound into the loudspeaker and its unbounded surroundings. The loudspeaker back-cavity panels Γ_0 are rigid. The time-harmonic normal displacement of the thin membrane is given by $w(\underline{r}_s,t) = w(\underline{r}_s,\omega)e^{j\omega t}$ with j the imaginary unit $\sqrt{-1}$ and with t denoting the time. The time-harmonic acoustic pressure response is given by $p(\underline{r},t) = p(\underline{r},\omega)e^{j\omega t}$. From here onwards, the steady-state solutions $w(\underline{r}_s,\omega)$ and $p(\underline{r},\omega)$ are abbreviated as $w(\underline{r}_s)$ and $p(\underline{r})$, respectively.

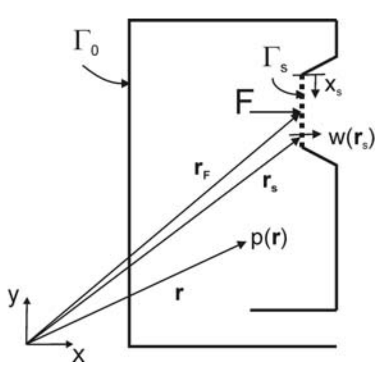

Figure 4 : A 2D bass-reflex loudspeaker

The steady-state normal displacement $w(\underline{r}_s)$ of the loudspeaker membrane is governed by the dynamic plate equation

$$(\nabla^4 - k_b^4)w(\underline{r}_s) = \frac{F}{D}\delta(\underline{r}_s,\underline{r}_F) + \frac{p_{diff}(\underline{r}_s)}{D} \tag{1}$$

with the bilaplacian operator $\nabla^4 = \frac{\partial^4}{\partial x_s^4} + 2\frac{\partial^4}{\partial x_s^2 \partial y_s^2} + \frac{\partial^4}{\partial y_s^4}$, $(\underline{r}_s = (x_s, y_s))$. $k_b = \sqrt[4]{\frac{\rho_s h \omega^2}{D}}$ is the bending wavenumber and $D = \frac{Eh^3(1+j\eta)}{12(1-\nu^2)}$ is the bending stiffness, with h the plate thickness, ρ_s the density, E the elasticity modulus, ν the Poisson coefficient and η the loss factor. $p_{diff}(\underline{r}_s)$ is the pressure difference over the membrane, acting as an external load.

To uniquely define the normal displacement, four structural boundary conditions must be specified, i.e. two conditions at both plate edges. For clamped plates, for instance, the boundary conditions are $(\underline{r}_s \rightarrow x_s \in [0,L])$

$$\begin{aligned} w(0) = w(L) = 0 \\ \frac{dw(0)}{dx_s} = \frac{dw(L)}{dx_s} = 0 \end{aligned} \tag{2}$$

Assuming that the system is linear, non-viscous, and adiabatic, the steady-state acoustic pressure $p(\underline{r})$ is governed by the homogeneous Helmholtz equation

$$\nabla^2 p(\underline{r}) + k^2 p(\underline{r}) = 0 \tag{3}$$

with $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ the laplacian operator, $(\underline{r} = (x,y))$, $k = \omega/c$ the acoustic wavenumber and c the speed of sound.

The acoustic boundary conditions for the rigid back-cavity panels are:

$$\underline{r} \in \Gamma_0 : \frac{j}{\rho_0\omega}\frac{\partial p(\underline{r})}{\partial n} = 0 \tag{4}$$

with ρ_0 the ambient fluid density and $\partial/\partial n$ the derivative in the normal direction.

To ensure the normal velocity continuity along the fluid-plate coupling interface Γ_s, the following relationship must apply at the interface:

$$\underline{r}_s \in \Gamma_s : \frac{j}{\rho_0\omega}\frac{\partial p(\underline{r}_s)}{\partial n} = j\omega w(\underline{r}_s) \tag{5}$$

The Sommerfeld radiation condition for outgoing waves ensures that no reflections occur at infinity [Ihlenburg (1998); Colton and Kress (1998)],

$$\lim_{|\underline{r}| \rightarrow \infty} \left(|\underline{r}| \frac{\partial p(\underline{r})}{\partial |\underline{r}|} + jkp(\underline{r}) \right) = 0 \tag{6}$$

3 The Wave Based Method

The Wave Based Method (WBM) adopts an indirect Trefftz approach [Trefftz (1926)] in that the approximations of the dynamic response variables w and p exactly satisfy the governing differential equations (1) and (3) and, where appropriate, the Sommerfeld radiation condition (6).

3.1 Partitioning into subdomains

A sufficient condition for the WBM approximations to converge towards the exact solution is convexity of the considered problem domains [Desmet (1998)]. In the 2D bass-reflex loudspeaker problem, see figure 4, the structural problem domain is convex. However, the acoustic problem domain Ω is a non-convex domain and has to be partitioned into a number N_Ω of non-overlapping, convex subdomains Ω_i, $\left(\Omega = \bigcup_{i=1}^{N_\Omega} \Omega_i\right)$. The boundary of a convex subdomain Ω_i is denoted as $\delta\Omega_i$. Furthermore, the following notations are introduced

- $\Gamma_{0,i} = \Gamma_0 \bigcap \delta\Omega_i$ indicates the part of the boundary of subdomain Ω_i on which rigid boundary conditions are applied.
- $\Gamma_{s,i} = \Gamma_s \bigcap \delta\Omega_i$ indicates the part of the boundary of subdomain Ω_i which belongs to the fluid-plate coupling interface.
- $\Gamma_{ij} = \delta\Omega_i \bigcap \delta\Omega_j$ indicates the part of the boundary of subdomain Ω_i which is in contact with part of the boundary of subdomain Ω_j.
- $\Gamma_{I,i} = \bigcup_{j=1, i\neq j}^{N_\Omega} \Gamma_{ij}$ indicates the total part of the boundary of subdomain Ω_i which is in contact with any another subdomain.

Each subdomain boundary can thus be decomposed into three parts

$$\delta\Omega_i = \Gamma_{0,i} \bigcup \Gamma_{s,i} \bigcup \Gamma_{I,i} \tag{7}$$

At the subdomain boundaries $\Gamma_{0,i}$ and $\Gamma_{s,i}$ acoustic boundary conditions (4) and (5), respectively, are imposed. At each of the induced subdomain interfaces Γ_{ij} continuity conditions have to be applied [Pluymers, Desmet, Vandepitte, and Sas (2003)]. In order to describe the continuity condition, the following 'equivalent velocity' operator is introduced

$$\mathbf{R}_{int} = \left(\frac{j}{\rho_0\omega}\frac{\partial}{\partial n} - \frac{1}{\overline{Z}_{int}}\right) \tag{8}$$

with n the boundary normal direction, with positive orientation away from the subdomain.

The continuity condition for subdomain Ω_i applied on Γ_{ij} to ensure continuity with subdomain Ω_j is

$$\mathbf{R}_{int}\left[p_i(\underline{r}_i)\right] = \mathbf{R}_{int}\left[p_j(\underline{r}_i)\right], \quad (\underline{r}_i \in \Gamma_{ij}) \tag{9}$$

with p_i and p_j the acoustic pressure in subdomain Ω_i and Ω_j, respectively. $\overline{Z}_{int}$ is a weighting factor which determines the relative importance of the velocity term compared to the pressure term in the 'equivalent velocity' and is chosen to be the characteristic acoustic impedance value $\rho_0 c$. It is shown in [Pluymers, Desmet, Vandepitte, and Sas (2003)] that choosing $\overline{Z}_{int} = \rho_0 c$ is beneficial for the convergence ratio.

The following sections describe both the approximations of the acoustic pressure response and the structural displacement response, resulting in a coupled WBM model.

3.2 Acoustic pressure approximation

The acoustic problem domain is divided into two regions similar to the IE method (see figure 5): an unbounded region exterior to a circular truncation boundary surface (subdomain 12) and a bounded region inside the truncation surface (subdomains 1 - 11). The non-convex region inside the circular truncation surface of the considered loudspeaker problem is partitioned into 11 convex subdomains to ensure convergence of the WBM approximations (see figure 5). In total, the acoustic problem domain is subdivided into $N_\Omega = 12$ subdomains.

The steady-state pressure fields $p_i(\underline{r})$ $(i = 1...11)$ in the 11 bounded acoustic subdomains are approximated as solution expansions $\widehat{p}_i(\underline{r})$,

$$p_i(\underline{r}) \simeq \widehat{p}_i(\underline{r}) = \sum_{a=1}^{m_i} p_{i,a}\Phi_{i,a}(\underline{r}) \tag{10}$$

Each function $\Phi_{i,a}(\underline{r})$ is an acoustic wave function, which satisfies the homogeneous Helmholtz equation:

$$\Phi_{i,a}(\underline{r}(x,y)) = \begin{cases} \Phi_{i,a_r}(x,y) = cos(k_{xi,a_r}x)e^{-jk_{yi,a_r}y} \\ \Phi_{i,a_s}(x,y) = e^{-jk_{xi,a_s}x}cos(k_{yi,a_s}y) \end{cases} \tag{11}$$

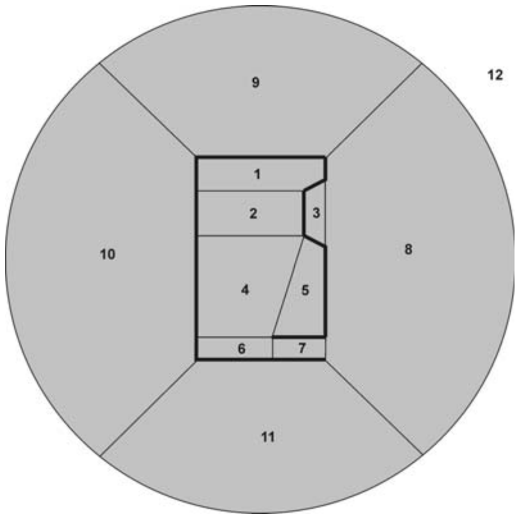

Figure 5 : Domain decomposition

Since the only requirement for the wave number components (k_{xi,a_r}, k_{yi,a_r}) and (k_{xi,a_s}, k_{yi,a_s}) is that $k^2_{xi,a_r} + k^2_{yi,a_r} = k^2_{xi,a_s} + k^2_{yi,a_s} = k_i^2$ with $k_i = \omega / c_i$, an infinite number of wave functions (11) can be defined for the expansion (10). It is proposed to select the following wavenumber components,

$$(k_{xi,a_r}, k_{yi,a_r}) = \left(\frac{e_{i,a_r}\pi}{L_{x_i}}, \pm\sqrt{k_i^2 - \left(\frac{e_{i,a_r}\pi}{L_{x_i}}\right)^2} \right) \tag{12}$$

$$(k_{xi,a_s}, k_{yi,a_s}) = \left(\pm\sqrt{k_i^2 - \left(\frac{f_{i,a_s}\pi}{L_{y_i}}\right)^2}, \frac{f_{i,a_s}\pi}{L_{y_i}} \right)$$

with $e_{i,a_r}, f_{i,a_s} = 0, 1, 2, \ldots$ The dimensions L_{x_i} and L_{y_i} in (12) represent the dimensions of the (smallest) rectangular domain, enclosing the considered subdomain. From here onwards, both k_{xi,a_r} and k_{xi,a_s} are denoted as $k_{xi,a}$ and both k_{yi,a_r} and k_{yi,a_s} are denoted as $k_{yi,a}$.

The wave function contributions $p_{i,a}$ in (10) are the unknowns.

In the unbounded subdomain outside the circular boundary surface, i.e. subdomain 12, the steady-state pressure $p_{12}(\underline{r})$ is approximated as a solution expansion of wave functions that are solutions of the Helmholtz equation (3) and that satisfy the Sommerfeld radiation condition for outgoing waves (6). It has been proven in [Herrera (1984)] that, for a 2D acoustic domain, exterior to a circular boundary surface, the expansion

$$p_{12}(r,\theta) \simeq \widehat{p}_{12}(r,\theta) = p_{c,0}H_0^{(2)}(kr) + \sum_{a=1}^{N} p_{c,a}H_a^{(2)}(kr)cos(a\theta) + p_{s,a}H_a^{(2)}(kr)sin(a\theta) \tag{13}$$

converges for $N \to \infty$. $H_a^{(2)}(*)$ is the a-th order Hankel function of the second kind. The contributions $p_{c,0}$, $p_{c,a}$ and $p_{s,a}$ are the $m_{12} = 2N+1$ unknowns.

The set of all $n_{Adof} = \sum_{i=1}^{12} m_i$ acoustic wave function contributions $p_{i,a}$ $(i = 1 \ldots 11)$, $p_{c,0}$, $p_{c,a}$ and $p_{s,a}$ is denoted as p_{set}.

3.3 Structural displacement approximation

Based on the above pressure approximations, the steady-state normal displacement w is approximated as a solution expansion $\widehat{w}$

$$w(x_s) \simeq \widehat{w}(x_s) = \sum_{s=1}^{4} w_s \Psi_s(x_s) + \widehat{w}_F(x_s) + \sum_{i=1}^{N_\Omega} \zeta_i \widehat{w}_{A_i}(x_s)$$
$$\text{with} \quad \begin{array}{ll} \zeta_i = 0 & \text{if} \quad \Gamma_{s,i} = \emptyset \\ \zeta_i = \underline{n}_i^T . \underline{n}_s & \text{if} \quad \Gamma_{s,i} \neq \emptyset \end{array} \tag{14}$$

where $\underline{n}_s$ and $\underline{n}_i$ represent the unit normal vectors of the plate and of subdomain Ω_i, respectively. $.^T$ is the transpose operator. $\widehat{w}_{A_i}$ represents a particular solution of the dynamic plate equation (1) due to the pressure loading from the acoustic field along the cavity-plate interface $\Gamma_{s,i}$. For the considered bass-reflex example, see figure 4, the following expression can be used

$$\widehat{w}_{A_i}(x_s) = \sum_{a=1}^{m_i} w_{A_{i,a}}(x_s) \tag{15}$$

with

$$w_{A_{i,a}}(x_s) = p_{i,a} \frac{\Phi_{i,a}(x_s)}{D(k^4_{xi,a} - k_b^4)} \tag{16}$$

The four structural wave functions Ψ_s are four linearly independent solutions of the homogeneous part of the fourth-order dynamic plate equation (1),

$$\Psi_s(x_s) = e^{-j^s k_b x_s}, \ (s = 1 \ldots 4) \tag{17}$$

and $\widehat{w}_F$ is the normal displacement of an infinite plate, excited by a normal line force F,

$$\widehat{w}_F(x_s) = \frac{-jF}{4Dk_b^3}(e^{-jk_b|x_s - x_F|} - je^{-k_b|x_s - x_F|}) \tag{18}$$

The 4 structural wave function contributions w_s ($s = 1...4$) are the structural unknowns and are denoted as w_{set}.

3.4 Coupled structural-acoustic wave model

With the use of the proposed pressure and displacement expansions (10), (13) and (14), the dynamic plate equation (1), the Helmholtz equation (3) and the Sommerfeld radiation condition (6) are always exactly satisfied, irrespective of the values of the $n_{DOF} = (4 + n_{Adof})$ unknown wave function contributions w_{set} and p_{set}. These contributions are merely determined by the structural and acoustic boundary conditions. There are two types of boundary conditions

- For a structural domain in a 2D problem, structural boundary conditions are specified at discrete edge locations. The boundary conditions (2) can be imposed exactly using expansion (14). This leads to 4 algebraic equations in the n_{DOF} unknown wave function contributions

$$\left[\begin{array}{cc} A_{SS} & C_{SA} \end{array}\right] \left\{\begin{array}{c} w_{set} \\ p_{set} \end{array}\right\} = \left\{\begin{array}{c} f_S \end{array}\right\} \tag{19}$$

The (4×4) matrix $[A_{SS}]$ results from the application of the structural boundary conditions (2) on the 4 structural wave functions (17)

$$[A_{SS}] = \left[\begin{array}{cccc} \Psi_1(0) & \Psi_2(0) & \Psi_3(0) & \Psi_4(0) \\ \Psi_1(L) & \Psi_2(L) & \Psi_3(L) & \Psi_4(L) \\ \frac{d\Psi_1(0)}{dx_s} & \frac{d\Psi_2(0)}{dx_s} & \frac{d\Psi_3(0)}{dx_s} & \frac{d\Psi_4(0)}{dx_s} \\ \frac{d\Psi_1(L)}{dx_s} & \frac{d\Psi_2(L)}{dx_s} & \frac{d\Psi_3(L)}{dx_s} & \frac{d\Psi_4(L)}{dx_s} \end{array}\right] \tag{20}$$

The $\left(4 \times n_{Adof}\right)$ matrix $[C_{SA}]$ results from the application of the structural boundary conditions (2) on the particular solution terms $w_{A_{i,a}}$ (16) ($a = 1...m_i, i = 1...N_\Omega$) which are due to the acoustic loading on the structural domain. Each column corresponds with an acoustic wave function and is calculated as

$$\left\{\begin{array}{c} \zeta_i\, w_{A_{i,a}}(0) \\ \zeta_i\, w_{A_{i,a}}(L) \\ \zeta_i \frac{dw_{A_{i,a}}(0)}{dx_s} \\ \zeta_i \frac{dw_{A_{i,a}}(L)}{dx_s} \end{array}\right\} \tag{21}$$

Note that, for the considered loudspeaker case, only the columns associated with wave functions of subdomains 2 and 3 will be non-zero.

The (4×1) vector $\{f_S\}$ results from the application of the structural boundary conditions (2) on the particular solution term (18)

$$\left\{\begin{array}{c} f_S \end{array}\right\} = \left\{\begin{array}{c} -\widehat{w}_F(0) \\ -\widehat{w}_F(L) \\ -\frac{d\widehat{w}_F(0)}{dx_s} \\ -\frac{d\widehat{w}_F(L)}{dx_s} \end{array}\right\} \tag{22}$$

- Due to the introduction of $N_\Omega = 12$ acoustic subdomains, continuity conditions (9) along the subdomain interfaces Γ_{ij} must be taken into account, in addition to the problem boundary conditions (4) and (5). Since both the acoustic boundary conditions and the continuity conditions are defined at an infinite number of boundary positions, while only finite sized prediction models are amenable to numerical implementation, the acoustic boundary and the continuity conditions are transformed into a weighted residual formulation.

Applying a weighted residual formulation on each subdomain Ω_i separately, yields, for every subdomain Ω_i, a set of m_i algebraic equations in the n_{DOF} unknown wave function contributions. In the proposed weighted residual formulation three residual error functions occur:

$$R_{0_i}(\underline{r}_0) = \frac{j}{\rho_0 \omega} \frac{\partial \widehat{p}_i(\underline{r}_0)}{\partial n} - 0 \quad , \quad (\underline{r}_0 \in \Gamma_{0,i}) \tag{23}$$

$$R_{s_i}(\underline{r}_s) = \frac{j}{\rho_0 \omega} \frac{\partial \widehat{p}_i(\underline{r}_s)}{\partial n} - j\omega \widehat{w}(\underline{r}_s), \quad (\underline{r}_s \in \Gamma_{s,i}) \tag{24}$$

$$R_{ij}(\underline{r}_i) = \mathbf{R}_{int}\left[\widehat{p}_i(\underline{r}_i)\right] - \mathbf{R}_{int}\left[\widehat{p}_j(\underline{r}_i)\right], \quad (\underline{r}_i \in \Gamma_{ij}) \tag{25}$$

These errors are orthogonalised with respect to a weighting function $\tilde{p}_i$. The integral formulation is expressed as

$$\int_{\Gamma_{0,i}} \tilde{p}_i R_{0_i} d\Gamma + \int_{\Gamma_{s,i}} \tilde{p}_i R_{s,i} d\Gamma + \sum_{j=1, i \neq j}^{N_\Omega} \int_{\Gamma_{ij}} \tilde{p}_i R_{ij} d\Gamma = 0 \tag{26}$$

Like in the Galerkin weighting procedure, used in the FEM, the weighting function $\tilde{p}_i$ is expanded in terms of the same set of acoustic wave functions used in the field variable expansions (10) and (13) for the considered subdomain Ω_i.

Substituting the field variable expansions (10), (13) and (14) and the weighting function expansion into the weighted residual formulation (26) leads to m_i equations in n_{DOF} unknowns

$$\begin{bmatrix} C_{AS_i} & A_{AA_i} \end{bmatrix} \begin{Bmatrix} w_{set} \\ p_{set} \end{Bmatrix} = \begin{Bmatrix} f_{A_i} \end{Bmatrix} \tag{27}$$

The element on row a and column s of the $(m_i \times 4)$ matrix $[C_{AS_i}]$ is

$$[C_{AS_i}]_{as} = -\zeta_i \int_{\Gamma_{s,i}} j\omega \Phi_{i,a} \Psi_s \, d\Gamma \tag{28}$$

The element on row a_u and column a_v of the $(m_i \times m_i)$ matrix $[A_{AA_i}]$ is

$$\begin{aligned} [A_{AA_i}]_{a_u a_v} = & \int_{\Gamma_{0,i} \bigcup \Gamma_{s,i}} \frac{j}{\rho_0 \omega} \Phi_{i,a_u} \frac{\partial \Phi_{i,a_v}}{\partial n} d\Gamma \\ & -\zeta_i \int_{\Gamma_{s,i}} j\omega \Phi_{i,a_u} w_{A_{i,a_v}} \, d\Gamma \\ & + \sum_{j=1, i\neq j}^{N_\Omega} \int_{\Gamma_{ij}} \frac{j}{\rho_0 \omega} \Phi_{i,a_u} \frac{\partial \Phi_{i,a_v}}{\partial n} - \frac{1}{\overline{Z}_{int}} \Phi_{i,a_u} \Phi_{i,a_v} d\Gamma \\ & + \sum_{j=1, i\neq j}^{N_\Omega} \int_{\Gamma_{ij}} \frac{j}{\rho_0 \omega} \Phi_{i,a_u} \frac{\partial \Phi_{j,a_v}}{\partial n} + \frac{1}{\overline{Z}_{int}} \Phi_{i,a_u} \Phi_{j,a_v} d\Gamma \end{aligned} \tag{29}$$

The element on row a of the $(m_i \times 1)$ vector $\{f_{A_i}\}$ is

$$\{f_{A_i}\}_a = \int_{\Gamma_{s,i}} \Phi_{i,a} j\omega \widehat{w}_F \, d\Gamma \tag{30}$$

Combining N_Ω matrix equations of type (27) associated with each of the N_Ω subdomains Ω_i results in a system of n_{Adof} algebraic equations in n_{DOF} unknowns.

$$\begin{bmatrix} C_{AS} & A_{AA} \end{bmatrix} \begin{Bmatrix} w_{set} \\ p_{set} \end{Bmatrix} = \begin{Bmatrix} f_A \end{Bmatrix} \tag{31}$$

The combination of the equations resulting from the structural boundary conditions (19) and those resulting from the acoustic boundary and continuity conditions (31) yields a square matrix equation in the n_{DOF} unknown wave function contributions w_{set} and p_{set}

$$\begin{bmatrix} A_{SS} & C_{SA} \\ C_{AS} & A_{AA} \end{bmatrix} \begin{Bmatrix} w_{set} \\ p_{set} \end{Bmatrix} = \begin{Bmatrix} f_S \\ f_A \end{Bmatrix} \tag{32}$$

As for the BE method and in contrast with the FE method, the proposed technique yields a fully populated matrix, whose elements are complex and which cannot be decomposed into frequency independent submatrices. The big advantage of the WBM is, however, that the system matrices are substantially smaller in comparison with the element based techniques. This property, combined with the fast convergence of the WBM, make it a less computationally demanding method for dynamic response calculations, which creates opportunities to tackle problems also in the mid-frequency range. The beneficial convergence characteristics of the WBM for radiation problems, in comparison with the IE and BE methods, are illustrated in the next section.

4 Numerical results

4.1 Validation example

In order to illustrate the high accuracy that can be obtained with the WBM, a 2D bass-reflex, as shown in figure 6, is considered. A unit normal line force F is applied at the center of the loudspeaker membrane ($E = 70.10^9 \frac{N}{m^2}$, $\rho_s = 700 \frac{kg}{m^3}$, $\nu = 0.3$, $t = 3mm$). The membrane edges are clamped. The loudspeaker is surrounded with air ($c = 340 \frac{m}{s}$, $\rho_0 = 1.225 \frac{kg}{m^3}$).

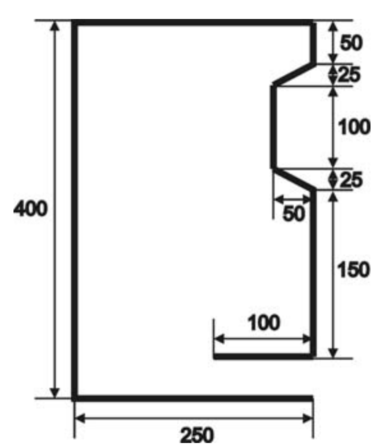

Figure 6 : Dimensions (in *mm*) of the bass-reflex loudspeaker

Figures 7 and 8 show the calculated pressure field and the calculated active intensity field at $120Hz$. These results are obtained with a wave model which consists of 495 wave functions with a truncation boundary surface having a radius of $0.5m$. The pressure contour plots show that the rigid boundary conditions are correctly taken into account by the WBM since the pressure contour lines are perpendicular to the rigid walls. Also, no pressure field discontinuities are observed, which indicates that the continuity conditions at the subdomain interfaces and the infinite domain interfaces are correctly taken into account. Figure 8 shows that active intensity flows from both the membrane and the reflex-channel towards infinity which clearly illustrates the working principle of a bass-reflex channel. For the given loudspeaker dimensions, the considered frequency of $120Hz$ indeed corresponds to the reflex-frequency of the loudspeaker.

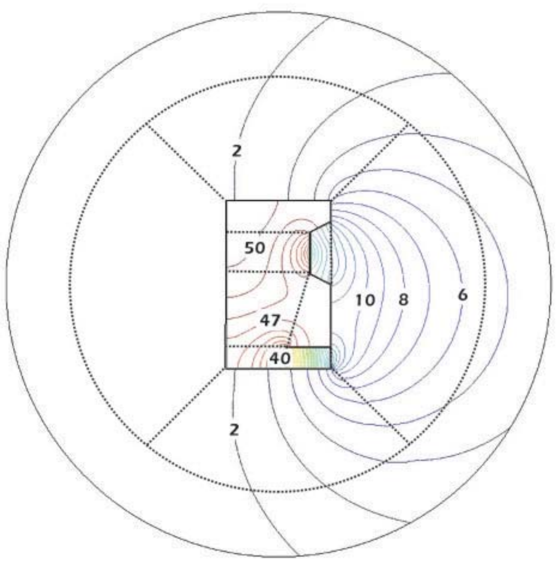

Figure 7 : Contour plot of the pressure amplitudes at $120Hz$ ($10^{-4}Pa$)

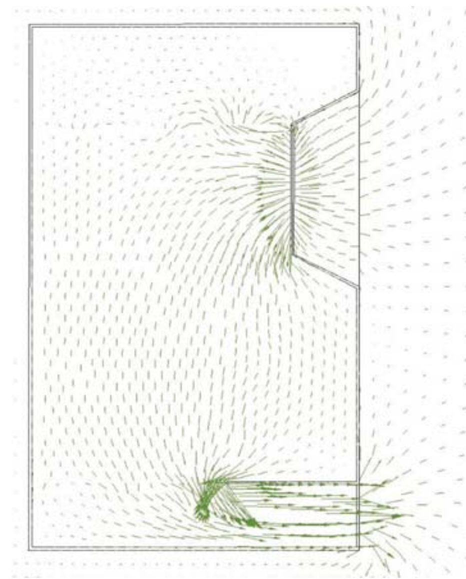

Figure 8 : Vector plot of the active intensity at $120Hz$

4.2 Comparison with element based techniques

To compare the performances of the WBM and the existing element based techniques, several coupled FE/indirect BE and FE/(8^{th} order conjugated) IE models of the considered problem have been solved using LMS/SYSNOISE Rev.5.5. The structural FE meshes consist of 2-noded plate elements, the acoustic BE meshes of 2-noded linear fluid elements and the acoustic FE meshes of 3- and 4-noded linear fluid elements. Tables 1 and 2 show the number of elements used to model the considered loudspeaker problem (see figure 6).

Table 1 : FE/IE model sizes

♯ acoustic FE inside cavity	♯ acoustic FE outside cavity	♯ structural FE	♯ IE	total ♯ dofs
2835	7786	34	312	14567
4832	15973	68	394	25031
9492	26525	136	524	42552

Table 2 : BE and WBM model sizes

♯ ac. BE	♯ str. BE	tot. ♯ dofs
472	34	593
944	68	1167
1888	136	2315
3776	272	4611

♯ ac. WBM dofs	♯ str. WBM dofs	tot. ♯ WBM dofs
131	4	135
195	4	199
327	4	331
587	4	591
853	4	857

Figure 9 plots the relative prediction errors for the radiated sound power W at $5000Hz$ against the CPU time needed for a direct response calculation at one frequency on a Windows XP system (INTEL $1.8GHz, 1Gb$ RAM) and against the number of degrees of freedom to model the problem with the different methods. The indicated CPU times for the WBM and the FE/BE models include both the times for construction of the model and for solution of the resulting matrix equation since the matrices are frequency dependent. This is in contrast with the frequency independent FE/conjugated IE models where only the solution time is taken into account. Both in terms of the CPU time and in terms of the number of degrees of freedom, the WBM has a beneficial convergence rate, compared with the element based techniques, which confirms the findings of [Desmet (1998); Desmet, Van Hal, Sas, and Vandepitte (2002); Pluymers, Desmet, Vandepitte, and Sas (2004)].

To increase the accuracy of the FE/IE models, a finer element discretization is required. However, since the artificial boundary, which truncates the unbounded domain, was chosen to be a circle, the greater part of the acoustic finite elements are needed to model the enclosed area between the artificial truncation boundary surface and the cavity, which is not very efficient and which leads to prohibitively large calculations. A more efficient way of modelling the problem would be to use a more close-fitting truncation boundary, such as for instance an ellipsoidal boundary, so that the area between the truncation boundary and the problem boundary becomes much smaller.

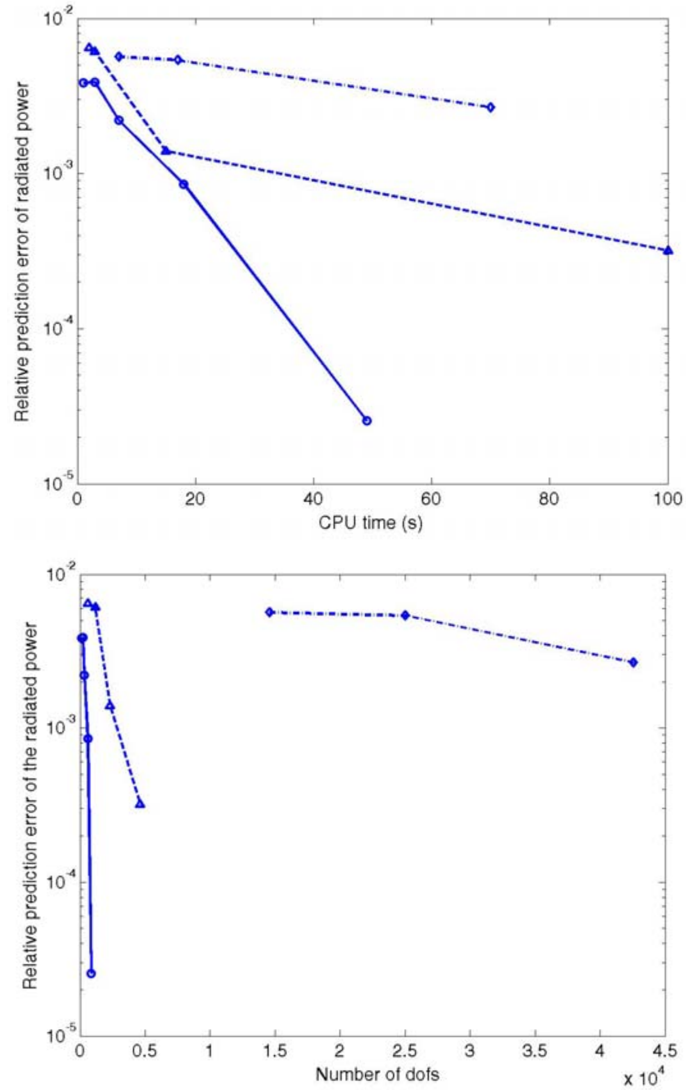

Figure 9 : Convergence curves at $5000Hz$ in terms of CPU time and in terms of number of degrees of freedom,(solid (o): WBM, dashed ($\triangle$): FE/BE, dot-dashed ($\diamondsuit$): FE/IE)

5 Conclusions

This paper applies a novel wave based method for the steady-state dynamic analysis of structural-acoustic problems with unbounded fluid domains. It is illustrated through a bass-reflex loudspeaker example that the method converges towards the exact solution. A comparison with corresponding FE/indirect BE and FE/conjugated IE models indicates the enhanced convergence rate of the new method. In this way, the proposed technique offers an adequate way to comply with the current challenge in structural-acoustic modelling. Due to its beneficial convergence rate, the practical frequency threshold may become substantially higher than for the element based methods, resulting in a significant narrowing of the currently existing mid-frequency twilight zone.

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