

# Advances in Computational Methods for Multibody System Dynamics

R.L. Huston<sup>1</sup> and C.-Q. Liu<sup>2</sup>

**Abstract:** This paper presents a summary of recent developments in computational methods for multibody dynamics analyses. The developments are presented within the context of an automated numerical analysis. The intent of the paper is to provide a basis for the easy development of computational algorithms. The principal concepts discussed are: differentiation algorithms, partial velocities and partial angular velocities, generalized speeds, Euler parameters, Kane's equations, orthogonal complement arrays, lower body arrays and accuracy testing functions.

**keyword:** Multibody dynamics, numerical methods, algorithmic development.

## 1 Introduction

In the early 1960's, as engineers were developing methods for matrix structural analysis, few would have imagined that within a few years, advances in vastly different fields would come together to enable today's design technologies. Complex designs may currently be developed collaboratively by engineers in various countries around the world. At the core of these designs is the finite element method (FEM) stemming directly from the 1960's matrix structural analyses. The associated enabling technologies include: increasingly capable computer hardware, advances in matrix inversion algorithms, advances in computer graphics, and high-speed internet connections.

From a mechanical design perspective, these combined technologies are now applicable with problems in solid mechanics, fluid mechanics, heat transfer, vibration, and control with applications to optimal design and failure/life analyses. Paradoxically a long envisioned application in studying the dynamics of large multibody systems has not yet been fully realized.

In the 1970s and 1980s Dr. Clifford Astill, through the National Science Foundation, encouraged and supported research efforts aimed at modeling the dynamics of large mechanical systems via a lumped-mass, or "finite-segment," systems approach, just as structures were modeled by finite elements. But whereas finite element modeling was a natural extension of matrix structural analysis, finite-segment modeling of dynamical systems was not a natural extension of Newton-Euler formulations nor of Lagrange/Hamiltonian formulations. Indeed, with large systems (later to be known as "multibody" systems), both the Newton-Euler methods and the Lagrange/Hamilton (energy) methods become increasingly tedious and detailed as the system size increases. They are thus not easily converted into computational procedures.

For multibody dynamics analyses to enjoy comparable technological advances and applications to that of FEM there needs to be: 1) automated modeling of systems; 2) efficient dynamical analysis; and 3) computer graphics representation of both the modeling and the subsequent analysis results. As with FEM the ensuing applications will be widespread providing representation for any entity having motion.

In this paper we present concepts and procedures which have advanced multibody dynamics analyses to the point where the long-envisioned applications can occur.

The paper is divided into ten parts, each documenting an important advance and methodology for computational analysis. The final part also presents a discussion and concluding remarks.

## 2 Lower body arrays

A principal difficulty in large system multibody dynamics analysis is being able to organize the complex geometry. The difficulty increases dramatically as the number of bodies increases.

A useful method for overcoming this difficulty, no matter how large the system becomes, is to use lower body

---

<sup>1</sup>Mechanical, Industrial and Nuclear Engineering, University of Cincinnati, Cincinnati, OH 45221-0072, U.S.A.

<sup>2</sup>NVH Development and Engineering, CIMS 481-47-10 Daimler-Chrysler Corp., Auburn Hills, MI 48326-2757, U.S.A

arrays to describe the connection configuration. These arrays, originally introduced by Huston, Passerello, and Harlow (1978), provide a means for automatically (numerically) computing the governing differential equations of motion. The concept and associated procedures are remarkably simple: To illustrate the ideas, consider a multibody system such as that in Fig. 1. The system has 10 bodies which may be assumed to be

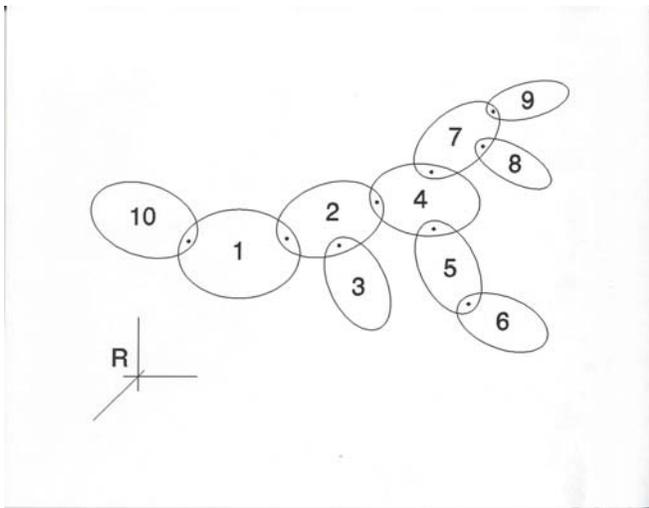


Figure 1 : An illustrative multibody system

connected by spherical joints and without closed loops – a so-called “open-chain” or “open-tree” system. Creating a lower body array begins with the labeling or numbering of the bodies of the system as follows: Select a body, any body, but preferably a large or major body, and label or number it as body 1, or simply  $B_1$ . Next, number the other bodies in ascending progression away from body 1 through the branches of the system, in a clockwise or counterclockwise procedure, as the view of the system is projected onto a plane. Fig. 1 illustrates such a labeling procedure. With this numbering procedure each body (except body 1) is connected to one, and only one, adjacent lower numbered body. The array listing these lower number labels in the lower body array:  $L(K)$ . By inspection of Fig. 1, we immediately see that  $L(K)$  for that system is:

$$\begin{matrix} K & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ L(K) & 0 & 1 & 2 & 2 & 4 & 5 & 4 & 7 & 7 & 1 \end{matrix} \quad (1)$$

where 0 is assigned as the lower numbered body number of body 1. The inertia frame R is also associated with 0.

It happens that  $L(K)$  contains complete information about the connection configuration of the system. To see this observe that in  $L(K)$  some of the body numbers do not occur, some are repeated, and some occur only once. Those body numbers which do not occur are associated with extremity bodies. Those numbers occurring more than once are numbers of branching bodies. Those numbers appearing once and only once represent intermediate bodies in a branch of the system.

Next  $L(K)$  can be used to automatically develop kinematic descriptions of the system. This is accomplished by viewing  $L(K)$  as an operator on the integers  $K$  which assigns a lower body number for each body integer. By viewing  $L(\bullet)$  as an operator we can obtain a lower body number for each body integer, with  $L(0)$  defined as 0. Thus we can obtain a lower body array of  $L(K)$ , that is,  $L(L(K))$  [or  $L^2(K)$ ] as:

$$\begin{matrix} L^0(K) & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ L^1(K) & 0 & 1 & 2 & 2 & 4 & 5 & 4 & 7 & 7 & 1 \\ L^2(K) & 0 & 0 & 1 & 1 & 2 & 4 & 2 & 4 & 4 & 0 \end{matrix} \quad (2)$$

where  $L^0(K)$  is assigned the array  $K$ .

In this way we can construct a series of higher order lower body arrays until all the entries are 0. Table 1 provides such a listing.

Table 1 : Lower body arrays for the multibody system of Fig. 1

$L^0(K)$	1	2	3	4	5	6	7	8	9	10
$L^1(K)$	0	1	2	2	4	5	4	7	7	1
$L^2(K)$	0	0	1	1	2	4	2	4	4	0
$L^3(K)$	0	0	0	0	1	2	1	2	2	0
$L^4(K)$	0	0	0	0	0	1	0	1	1	0
$L^5(K)$	0	0	0	0	0	0	0	0	0	0

As an illustration of one of the algorithmic uses of the lower body arrays, observe that the columns of Tab. 1 define the branching of the system. For example, in column 9 we have the numbers: 9, 7, 4, 2, 1, 0. These are the body numbers of the bodies in the path through the system from body 9 to the inertia frame R (See Fig. 1).

To further illustrate the utility of the arrays, let the angular velocity of body  $B_k$  relative to its adjoining lower numbered body  $B_j$  ( ${}^{B_j}\omega^{B_k}$ ) be written simply as  $\hat{\omega}_k$ . Similarly, let the angular velocity of  $B_k$  in R (“absolute”

angular velocity:  ${}^R\omega^{\mathbf{B}_k}$ ) be written simply as  $\omega_k$  (no overhat). Then  $\omega_9$  may be written as:

$$\omega_9 = \hat{\omega}_9 + \hat{\omega}_7 + \hat{\omega}_4 + \hat{\omega}_2 + \hat{\omega}_1 \quad (3)$$

(Observe that the indices in this expression are identical to the entries in column 9 of Tab. 1.)

Equation 3 may be written in a more general and more compact form as:

$$\omega_k = \sum_{p=0}^r \hat{\omega}_q, \quad q = L^p(k) \quad (4)$$

and where  $r$  is the integer such that  $L^r(k)$  is 1.

Observe that by knowing  $L(K)$ , Eq. 4 provides an algorithm for finding the angular velocities of all the bodies of a multibody system. Moreover, the angular velocities in turn may be used to compute mass center velocities and accelerations.

### 3 Differentiation algorithm.

Perhaps the greatest advance in computational methods for multibody dynamics is the development and implementation of differentiation algorithms which provide for an efficient automated development of the governing differential equations. The algorithms are remarkably simple: They are based upon a fundamental formula of elementary mechanics: Specifically, if a vector  $\mathbf{c}$  is fixed in a body  $B$  which in turn is moving in a reference frame  $R$ , the derivative of  $\mathbf{c}$  in  $R$  is simply

$${}^R d\mathbf{c}/dt = \omega \times \mathbf{c} \quad (5)$$

where  $\omega$  is the angular velocity of  $B$  in  $R$ .

The utility of Eq. 5 is that a derivative is calculated by a multiplication – a routine computation procedure as opposed to, say, finite differences.

By developing the applications of Eq. 5, we can calculate all necessary kinematic quantities as well as matrix derivatives. To illustrate this, consider the differentiation of an orthogonal transformation matrix: Recall that an orthogonal transformation matrix enables the expression of vectors relative to different unit vector sets. That is, if  $B$  is a body moving in a fixed frame  $R$  and if  $\mathbf{n}_i$  and  $N_i$  ( $i=1,2,3$ ) are mutually perpendicular unit vectors fixed in  $B$  and  $R$ , then the  $N_i$  and the  $\mathbf{n}_i$  may be related by an orthogonal transformation matrix  $S$  with elements  $S_{ij}$  as:

$$N_i = S_{ij}n_j \quad \text{and} \quad n_j = S_{ij}N_i \quad (6)$$

(with summation over repeated indices) where the  $S_{ij}$  are defined as

$$S_{ij} = N_i \bullet n_j \quad (7)$$

Consequently if  $V$  is any vector expressed in terms of the  $N_i$  and the  $n_i$  as

$$V = V_i N_i = v_j n_j \quad (8)$$

then the components  $V_i$  and  $v_i$  are related by the expressions:

$$V_i = S_{ij}v_j \quad \text{and} \quad v_j = S_{ij}V_i \quad (9)$$

(Note the similarity of Eqs. 6 and 9.)

Consider now the derivatives of  $S$ : Since the  $N_i$  are fixed in  $R$ , we have

$$dS_{ij}/dt = d(N_i \cdot n_j)/dt = N_i \cdot dn_j/dt = N_i \cdot \omega \times n_j \quad (10)$$

where the last equality follows from Eq. 5 since the  $\mathbf{n}_j$  are fixed in  $B$ .

Next, let  $T$  and the  $\mathbf{n}_j$  be expressed in terms of the  $N_i$  as:

$$\omega = \Omega_k N_k \quad \text{and} \quad n_j = S_{\ell j} N_\ell \quad (11)$$

Then by substituting into Eq. 10 we obtain

$$dS_{ij}/dt = N_i \cdot \Omega_k N_k \times S_{\ell j} N_\ell = e_{ik\ell} \Omega_k S_{\ell j} \quad (12)$$

where  $e_{ik\ell}$  is the usual permutation symbol [see, for example, Josephs and Huston (2002)].

Equation 12 may be written in the forms:

$$dS_{ij}/dt = W_{i\ell} S_{\ell j} \quad \text{and} \quad dS/dt = WS \quad (13)$$

where the  $W_{i\ell}$  are the elements of the angular velocity matrix defined by inspection of Eq. 12 as:

$$W_{i\ell} = e_{ik\ell} \Omega_k = \begin{bmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{bmatrix} \quad (14)$$

Again, the derivative is computed by multiplication – a convenient numerical procedure.

#### 4 Partial velocities and partial angular velocities

Consider a generic multibody system such as in Fig. 1. If the bodies are connected by spherical joints, then in the absence of other constraints, the system will have  $3N+3$  degrees of freedom, where  $N$  is the number of bodies. Each body has three rotation degrees of freedom and the system as a whole has three translation degrees of freedom in the inertia frame  $R$ . The rotations are conveniently described in terms of orientation changes of the bodies relative to their adjacent lower numbered bodies. The translations are conveniently defined in terms of the movement of a reference point of body 1 (perhaps its mass center) relative to the origin of  $R$ .

If the system has constraints, such as closed loops, specified motion, or partially restricted joint motion, the convenient numerical approach is to represent these constraints by constraint equations which are later appended to the dynamical equations. (An even more general approach is to initially allow each body to have six degrees of freedom and then to configure the system via the constraint equations. This procedure, however, is not usually practical or convenient for most physical systems.)

The degrees of freedom are usually represented by variables  $q_r$  ( $r=1, \dots, 3N+3$ ) called “generalized coordinates” which then become the dependent variables in the governing differential equations. (An alternative, more efficient procedure numerically is to use generalized speeds as discussed in the next section.)

For a typical body  $B_k$  of the system, the principal kinematic quantities of interest are together with the angular velocity and the angular acceleration. Then for an  $N$ -body system, there are thus  $4N$  kinematic quantities of interest. If the system has  $n$  degrees of freedom ( $n=3N+3$ ) the mass center velocity and the angular velocity may be expressed as [see for example Huston (1990), Huston and Liu (2001), and Josephs and Huston (2002)]:

$$v_k = v_{k\ell m} \dot{q}_\ell n_{om} \quad \text{and} \quad \omega_k = \omega_{k\ell m} \dot{q}_\ell n_{om} \quad k = 1, \dots, N \quad (15)$$

where the  $n_{om}$  ( $m=1, 2, 3$ ) are mutually perpendicular unit vectors fixed in  $R$ .

The derivatives of  $v_k$  and  $\omega_k$  with respect to the  $\dot{q}_\ell$  are called “partial velocity” and “partial angular velocity” vectors and in view of the linear dependence of  $v_k$  and  $\omega_k$  on the  $\dot{q}_\ell$ , the partial velocity and partial angular ve-

locity vectors are simply the coefficients of the  $\dot{q}_\ell$  in Eq. 15. That is,

$$\partial v_k / \partial \dot{q}_\ell \stackrel{D}{=} v_{k,\ell} = v_{k\ell m} n_{om} \quad \text{and}$$

$$\partial \omega_k / \partial \dot{q}_\ell = \omega_{k,\ell} = \omega_{k\ell m} n_{om} \quad k = 1, \dots, N; \ell = 1, \dots, n \quad (16)$$

The coefficients  $v_{k\ell m}$  and  $\omega_{k\ell m}$  form  $N \times n \times 3$  block arrays which form the “building blocks” of the kinematic and dynamic analysis. Interestingly these arrays may be automatically generated numerically once the connection configuration (via the lower body array) is known and the geometric and inertial properties of the bodies are known. The partial velocity and partial angular velocity arrays are analogous to base vectors used in classical continuum mechanics analyses.

Finally, using differentiation algorithms, the mass center accelerations and the body angular accelerations are readily obtained as:

$$a_k = (v_{k\ell m} \ddot{q}_\ell + \dot{v}_{k\ell m} \dot{q}_\ell) n_{om} \quad \text{and}$$

$$\alpha_k = (\omega_{k\ell m} \ddot{q}_\ell + \dot{\omega}_{k\ell m} \dot{q}_\ell) n_{om} \quad (17)$$

#### 5 Generalized speeds

The usual selection of general coordinates for rotational degrees of freedom is orientation angles. But when orientation angles are used in this role there occur nonlinearities in the form of trigonometric functions which in turn can allow singularities to occur in the numerical solution of the governing equations.

To see this, consider again a body  $B$  moving in a reference frame  $R$ . If dextral (Bryan) orientation angles, say  $\forall, \exists, \zeta$ , are used to define the orientation of  $B$  in  $R$ , then the angular velocity  $T$  of  $B$  in  $R$  may be expressed as [see for example Josephs and Huston (2002)]:

$$\omega = (\dot{\alpha} + \dot{\gamma} s_\beta) n_1 + (\dot{\beta} c_\alpha - \dot{\gamma} c_\beta s_\alpha) n_2 + (\dot{\beta} s_\alpha + \dot{\gamma} c_\beta c_\alpha) n_3 = \omega_i n_i \quad (18)$$

where  $n_1, n_2$ , and  $n_3$  are mutually perpendicular unit vectors, fixed in  $R$  and where  $s$  and  $c$  represent sine and cosine. Then by solving for  $\dot{\alpha}$ ,  $\dot{\beta}$ , and  $\dot{\gamma}$  in terms of  $\omega_1, \omega_2$ , and  $\omega_3$  we obtain

$$\dot{\alpha} = \omega_1 + s_\beta (\omega_2 s_\alpha - \omega_3 c_\alpha) / c_\beta,$$

$$\dot{\beta} = \omega_2 c_\alpha + \omega_3 s_\alpha,$$



### 6 Euler parameters

Euler parameters play two principal roles in multibody dynamics analyses: First they are a substitute for orientation angles in defining the orientations of the bodies of the system. Unlike orientation angles, Euler parameters do not give rise to singularities as in Eqs. 19. The singularities are avoided by using four variables, instead of three, to define the orientation of a body. This in turn leads to linear relations and quadratic forms, and these singularities do not occur. Also, as a result of the simplified kinematic expressions the additional variable does not create a computational burden.

Second, Euler parameters are an aid in formulating systems of first order equations for numerical integration.

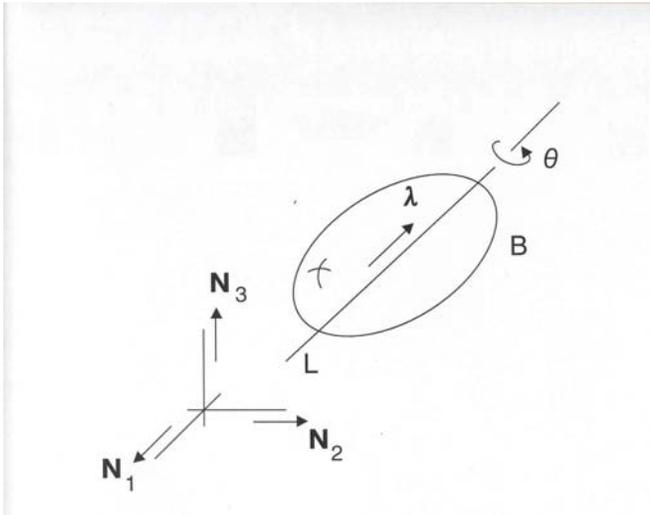


Figure 3 : A body B moving in a fixed frame R

To briefly explore and document these concepts, consider again the movement of a body B in a fixed frame R as in Fig. 3 where  $N_1, N_2$  and  $N_3$  are mutually perpendicular unit vectors fixed in R. There is a long established theorem in kinematics which states that if B has any given orientation in R it can be brought into any other given orientation by a single rotation about a line L fixed in both B and R [see, for example, Whittaker (1937) and Huston (1990)]. Then if  $\lambda$  is a unit vector parallel to L, and if  $\theta$  is the rotation angle, four Euler parameters  $\epsilon_i (i=1, \dots, 4)$  may be defined as

$$\begin{aligned} \epsilon_1 &= \lambda_1 \sin(\theta/2), \quad \epsilon_2 = \lambda_2 \sin(\theta/2), \\ \epsilon_3 &= \lambda_3 \sin(\theta/2), \quad \epsilon_4 = \cos(\theta/2) \end{aligned} \tag{26}$$

where the  $\lambda_i (i=1,2,3)$  are the  $N_i$  components of  $\lambda$ .

Observe that the four parameters of Eq. 26 are not independent but instead they are related by the expression

$$\epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2 + \epsilon_4^2 = 1 \tag{27}$$

With the definitions of Eq. 26 we can readily express the rotational kinematics in terms of the Euler parameters: For example, let  $n_i (i=1,2,3)$  be unit vectors fixed in B, which at the beginning of an orientation change are mutually aligned with the unit vectors  $N_i$  of R. Then as B rotates to some general orientation, the  $n_i$  will have a general orientation relative to the  $N_i$ . With the  $n_i$  fixed in B, the orientation of B, in R, is then defined by the orientations of the  $n_i$  in R, and specifically relative to the  $N_i$ . The transformation matrix S relating the  $n_i$  and the  $N_i$ , as in Eqs. 6 and 7, is then [see Huston (1990)]:

$$S = \begin{bmatrix} (\epsilon_1^2 - \epsilon_2^2 - \epsilon_3^2 + \epsilon_4^2) & 2(\epsilon_1\epsilon_2 + \epsilon_3\epsilon_4) & & \\ 2(\epsilon_1\epsilon_2 + \epsilon_3\epsilon_4) & (-\epsilon_1^2 + \epsilon_2^2 - \epsilon_3^2 + \epsilon_4^2) & & \\ 2(\epsilon_1\epsilon_3 - \epsilon_2\epsilon_4) & & 2(\epsilon_2\epsilon_3 + \epsilon_1\epsilon_4) & \\ & & 2(\epsilon_1\epsilon_3 + \epsilon_2\epsilon_4) & \\ & & 2(\epsilon_2\epsilon_3 - \epsilon_1\epsilon_4) & \\ & & (-\epsilon_1^2 - \epsilon_2^2 + \epsilon_3^2 + \epsilon_4^2) & \end{bmatrix} \tag{28}$$

Alternatively, if dextral orientation angles  $\forall, \exists, (\$  are used to define the orientation of B in R, the transformation matrix S is

$$S = \begin{bmatrix} c_\beta c_\gamma & -c_\beta s_\gamma & s_\beta & \\ (c_\alpha s_\gamma + s_\alpha s_\beta c_\gamma) & (c_\alpha c_\gamma - s_\alpha s_\beta s_\gamma) & -s_\alpha c_\beta & \\ (s_\alpha s_\gamma - c_\alpha s_\beta c_\gamma) & (s_\alpha c_\gamma + c_\alpha s_\beta s_\gamma) & c_\alpha c_\beta & \end{bmatrix} \tag{29}$$

Observe the simplicity and uniformity of the quadratic forms in the elements of S in Eq. 28 as compared with nonuniform and nonlinear trigonometric products in the elements in Eq. 29.

Next, let  $\epsilon$  be a column array of Euler parameters whose transpose  $\epsilon^T$  is:  $[\epsilon_1 \epsilon_2 \epsilon_3 \epsilon_4]$ , and let T be an array of angular velocity components (of B in R relative to the  $N_i$ ) with transpose  $\omega^T$ :  $[\omega_1 \omega_2 \omega_3 0]$ . Then  $\epsilon$  and  $\omega$  are related by the simple expressions:

$$\dot{\epsilon} = 2E\omega \quad \text{and} \quad \omega = (1/2)E^T\dot{\epsilon} \tag{30}$$

where E is the orthogonal matrix

$$E = \begin{bmatrix} \epsilon_4 & -\epsilon_3 & \epsilon_2 & -\epsilon_1 \\ \epsilon_3 & \epsilon_4 & -\epsilon_1 & -\epsilon_2 \\ -\epsilon_2 & \epsilon_1 & \epsilon_4 & -\epsilon_3 \\ \epsilon_1 & \epsilon_2 & \epsilon_3 & -\epsilon_4 \end{bmatrix} \tag{31}$$

[See also Josephs and Huston (2002).]

Observe the absence of singularities in Eqs. 30 as compared with those of Eq. 19.

Finally, recall that most numerical integration routines (or “solvers”) of systems of ordinary differential equations are written for systems of the form:  $dy/dx=f(x,y)$  where  $y$  and  $f$  are column arrays and  $x$  is the independent variable. Thus if we wish to numerically solve a system of second order equations, as normally occur in dynamic analyses, we need to convert the second order equation to the first order form. This is usually done by introducing an additional set of dependent variables representing the first derivatives of the original dependent variables. But when generalized speeds are used as the dependent variables, the dynamical equations are already first order equations and the auxiliary equations are then provided through the Euler parameters as in the first expression in Eq. 30, thus avoiding the need to introduce a new set of dependent variables.

## 7 Kane’s equations

Kane’s equations, originally called “Lagrange’s Form of d’Alembert’s Principle,” were introduced by T. R. Kane in 1961 as an aid in studying nonholonomic systems. They received relatively little attention for nearly 30 years until they became a principle of choice for studying large multibody systems [see Huston (1991)].

With Kane’s equations the analysis advantages of Lagrange’s equations and Newton’s laws are simultaneously incorporated but without incorporating their corresponding disadvantages. Specifically “non-working” internal constraint forces are automatically eliminated without the need to conduct tedious differentiation of energy functions. Indeed by using Kane’s equations, the governing dynamical equations can be written in closed form by employing the concepts of generalized applied (“active”) and inertia (“passive”) forces. Kane and Levinson (1980) provide a comprehensive comparison of Kane’s equations and other commonly used dynamics principles.

To briefly illustrate the use of Kane’s equations with multibody systems, consider a typical body  $B_k$  of a system  $S$  as in Fig. 4 where  $G_k$  is the mass center of  $B_k$  and  $R$  is the inertia frame. Let  $B_k$  be subjected to an applied force field (gravity and contact forces) and let this field be

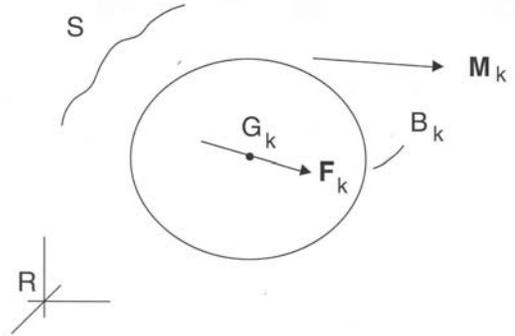


Figure 4 : A typical body of a multibody system

replaced by an equivalent force field consisting of a single force  $F_k$  passing through  $G_k$  together with a couple with torque  $M_k$ . Then the generalized active (applied) force  $F_\lambda$  on  $B_k$  for a generalized speed  $y_\lambda$  is:

$$F_\ell = v_{k\ell m} F_{km} + \omega_{k\ell m} M_{km} \quad (32)$$

where as before  $v_{k\ell m}$  and  $\omega_{k\ell m}$  are partial velocity and partial angular velocity components for  $G_k$  and  $B_k$  referred to the unit vectors  $n_{om}$  fixed in  $R$  for the generalized speed  $y_\ell$ , and where  $F_{km}$  and  $M_{km}$  are the  $n_{om}$  components of  $F_k$  and  $M_k$ .

Similarly, the generalized inertia forces  $F_\ell^*$  on  $B_k$  for  $y_\ell$  are

$$F_\ell^* = v_{k\ell m} F_{km}^* + \omega_{k\ell m} M_{km}^* \quad (33)$$

From d’Alembert’s principle the inertia force  $F_k^*$  and couple torque  $M_k^*$  on  $B_k$  may be expressed as [see Kane and Levinson (1985)]:

$$F_k^* = -m_k a_k, \quad M_k^* = -I_k \cdot \alpha_k - \omega_k \times (I_k \cdot \omega_k) \quad (34)$$

where  $m_k$  is the mass of  $B_k$ ,  $I_k$  is the central inertia dyadic of  $B_k$ , and  $\alpha_k$ ,  $\omega_k$ , and  $\dot{\omega}_k$  are the acceleration of  $G_k$ , the angular velocity of  $B_k$  and the angular acceleration of  $B_k$  respectively as given by Eqs. 15 and 17.

Kane’s equations simply state that the sum of the generalized forces is zero for each generalized speed  $y_\ell$ . That is

$$F_\ell + F_\ell^* = 0 \quad \ell = 1, \dots, n \quad (35)$$

where  $n$  is the number of degrees of freedom of the system.

By substituting from Eqs. 32, 33, and 34 into 35 we see that Kane's equations for multibody systems can be written in the compact form:

$$a_{\ell p} \dot{y}_p = f_\ell \quad (36)$$

where  $a_{\ell p}$  and  $f_\ell$  are

$$a_{\ell p} = m_k v_{k\ell m} v_{kpm} + I_{kmn} \omega_{k\ell m} \omega_{kpn} \quad (37)$$

and

$$\begin{aligned} f_\ell = & v_{k\ell m} F_{km} + \omega_{k\ell m} M_{km} - (m_k v_{k\ell m} \dot{v}_{kpm} y_p \\ & + I_{kmn} \omega_{k\ell m} \omega_{kpn} \dot{y}_p + I_{kmn} \omega_{k\ell m} \dot{\omega}_{kpn} y_p \\ & + e_{rsm} I_{ksn} \omega_{k\ell m} \omega_{kqr} \omega_{kpn} y_p y_q) \end{aligned} \quad (38)$$

where the  $I_{kmn}$  are  $n_{om}$ ,  $n_{on}$  components of  $I_k$ .

## 8 Constraints

When a multibody system is used to model a physical system, there are usually constraints on the physical system. Typically these constraints are due to closed loops (geometric) or due to some specified motion (kinematic). If the constraints are geometric (holonomic) they may be written in the form:

$$\phi_i(q_j) = 0 \quad i = 1, \dots, m; \quad j = 1, \dots, n \quad (m < n) \quad (39)$$

where  $m$  is the number of constraints. By differentiating Eq. 39 may be written in the form:

$$(\partial \phi_i / \partial q_j) \dot{q}_j = 0 \quad \text{or} \quad c_{ij} \dot{q}_j = 0 \quad (40)$$

where the  $c_{ij}$  are defined by inspection. By using Eqs. 20, Eq. 40 may be written in terms of the generalized speeds as:

$$b_{p\ell} y_\ell = g_\ell \quad (41)$$

If the constraints are kinematic (non-holonomic) that is, motion or velocity constraints, they may have the form

$$\hat{c}_{ij} \dot{q}_j = \hat{d}_i \quad (42)$$

or in terms of generalized speeds as

$$\hat{b}_{p\ell} y_\ell = \hat{g}_\ell \quad (43)$$

Eqs. 41 and 43 may readily be combined into a single matrix equation as:

$$By = g \quad (44)$$

When constraints are exerted on a multibody system, those constraints are established and maintained through constraining forces and moments. An analyst may or may not be interested in knowing the values of these constraining force and moment components. If they are not of interest it is convenient to eliminate them from the analysis, thus reducing the amount of numerical computation. Alternatively if they are of interest, they need to be efficiently exhibited.

To this end, it is usually most convenient analytically, and also computationally, to formulate the governing dynamical equations as though the system were unconstrained (as in Eq. 36) and to represent the constraints as forces exerted on the system. This procedure then requires the inclusion of these constraining forces (and moments) in the dynamics equations. This inclusion is readily attained by the use of generalized constraint forces, computed in the same manner as the generalized applied and inertia forces.

Specifically, if the generalized applied, inertia, and constraint forces are assembled into column arrays as  $F$ ,  $F^*$ , and  $F'$  respectively, then Kane's equations have the matrix form:

$$F + F^* + F' = 0 \quad (45)$$

Interestingly, it has been shown [see Huston (1999)] that the generalized constraint force array  $F'$  is directly proportional to the matrix of coefficients  $B$  of the constraint equations in Eq. 44. That is,

$$F' = B^T \lambda \quad (46)$$

where  $\lambda$  is an array of constraining force and moment components. Thus the dynamics equations become:

$$F + F^* + B^T \lambda = 0 \quad (47)$$

Taken together, Eqs. 44 and 47 become the governing equations for the multibody systems. They form a set of  $m+n$  equation for the  $y_\ell$  ( $\ell = 1, \dots, n$ ) and the  $m$  constraint force and moment components in the  $\lambda$  array.

## 9 Numerical procedures

If an unconstrained multibody system has  $n$  degrees of freedom represented by  $n$  generalized speeds  $y_\ell$  ( $\ell = 1, \dots, n$ ) and if then  $m$  constraints are imposed on the system via  $m$  constraining force and moment components, these then will be  $n + m$  unknowns consisting of the  $n$ ys and the  $m$  force and moment components. In this case Eqs. 47 and 44 then represent  $n$  dynamics equations and a constraint equation for the  $n + m$  unknowns. These equations, however, are nonlinear differential algebraic equations where closed form solutions in terms of elementary functions generally will not exist. Therefore numerical solutions are mandated.

There are two commonly used methods for obtaining these numerical solutions. In the first of these, the constraint force and moment array 8 is eliminated between the two sets of equations leading to a solution for the ys in terms of a pseudo inverse function. In the second method 8 is eliminated by orthogonal complement matrices, thus reducing the number of equations to be solved. In both procedures the force and moment components, if desired, are obtained by back substitution.

In the first method, Eqs. 35, 37, and 38 are used to rewrite Eq. 47 in the form:

$$A\dot{y} = \phi + F + B^T\lambda \quad (48)$$

where  $y$  is the column array of generalized speeds,  $A$  is the array whose elements are  $a_{\ell p}$  of Eq. 37 and where  $N$  is defined by inspection of Eq. 38. Then  $\dot{y}$  is:

$$\dot{y} = A^{-1}(\phi + F + B^T\lambda) \quad (49)$$

By substituting into the differentiated form of Eq. 44 we then obtain

$$B\dot{y} = BA^{-1}(\phi + F + B^T\lambda) = \dot{g} - \dot{B}y \quad (50)$$

Then by solving for  $(BA^{-1}B^T)\lambda$  we have

$$(BA^{-1}B^T)\lambda = \dot{g} - \dot{B}y - BA^{-1}\phi - BA^{-1}F \quad (51)$$

But  $BA^{-1}B^T$  is non-singular. Thus we can solve Eq. 51 for  $\lambda$  and substitute into Eq. 49 and obtain

$$\dot{y} = A^{-1}(\phi + F) + A^{-1}B^T(BA^{-1}B^T)^{-1} \times (\dot{g} - \dot{B}y - BA^{-1}\phi - BA^{-1}F) \quad (52)$$

[The term:  $A^{-1}B^T(BA^{-1}B^T)^{-1}$  is sometimes called the “weighted pseudo inverse” of  $B$ .] See, for example, Lawson and Hanson (1974).

Equation 52 represents a set of  $n$  ordinary differential equations for the  $n$  generalized speeds. Upon solution, Eq. 51 then provides the constraint force and moment components through the array 8.

The second method using orthogonal complement arrays is less cumbersome and consequently more efficient: In this method an  $n \times (n - m)$  array  $C$  (an “orthogonal complement” to  $B$ ) is sought such that

$$BC = 0 \quad \text{or} \quad C^TB^T = 0 \quad (53)$$

Then by premultiplying in Eq. 48 by  $C^T$  we have

$$C^TA\dot{y} = C^T(\phi + F) \quad (54)$$

Then by combining these equations with the differentiated form of Eq. 44 we have again a set of  $n$  ordinary differential equations for the  $n$  generalized speeds. Upon solution 8 can be obtained by back substitution, as before.

A principal advantage of this second method is that efficient algorithms have been written to obtain orthogonal complement arrays. One such algorithm proposed by Walton and Steeves (1969) uses a zero eigenvalues theorem to generate the orthogonal complement array.

## 10 Discussion

Our objective in this paper is two-fold: 1) to provide a listing of developments leading to the development of computational multibody dynamics; and 2) to provide a concise but yet comprehensive outline of procedures for the computer modeling of large multibody systems. The outlined modeling readily leads to numerical algorithms for developing and solving the governing dynamical equations.

The computational efficiency stems from the use of: 1) lower body arrays for organizing the geometry; 2) differentiation algorithms where derivatives are computed by multiplication; 3) partial velocities and partial angular velocities which eliminate non-working internal constraint forces; 4) generalized speeds and Euler parameters which eliminate singularities arising in the numerical solutions; 5) Kane’s equations for providing the governing dynamical equations; and 6) orthogonal complement

arrays for accommodating constraints on the system. Once the connection configuration (via the lower body array) is known; and the geometric and inertial properties of the bodies are known; and the applied forces and constraints are given; and finally, the initial conditions are given; the complete system of governing differential equations can be developed and solved numerically.

A question arising in this procedure is: How accurate are the results? To answer this question Liu and Huston (1995) have developed a testing function for evaluating the solution accuracy. Specifically, they established that the kinetic energy  $K$  of the system is related to the generalized speeds and generalized applied forces by the simple relation:

$$dK/dt = \sum_{\ell=1}^n F_{\ell} y_{\ell} \quad (55)$$

At each integration time step the kinetic energy can readily be calculated, and then since the  $y_{\ell}$  and  $F_{\ell}$  are also known, Eq. (55) can be integrated to obtain a comparison value of the kinetic energy.

Finally, as these concepts are further developed and incorporated into computer software, real time simulations of mechanical systems is feasible. Immediate application will occur with human body modeling, robotic analyses, vehicle dynamics, and naval systems.

**Acknowledgement:** The senior author is particularly grateful for the vision, encouragement, and support provided by Dr. Astill in developing virtually all of the concepts outlined in this paper. The support was provided through a series of grants from the National Science Foundation from 1974 through 1986.

## References

- Huston, R. L.; Passerello, C.E.; Harlow, M.W.** (1978): Dynamics of multi-rigid-body systems. *J Appl Mech*, vol. 45, pp. 889-894.
- Josephs, H.; Huston, R.L.** (2002): *Dynamics of mechanical systems*, CRC Press, Boca Raton, pp. 353-438, 605-660.
- Huston, R. L.; Liu, C. Q.** (2001): *Formulas for dynamic analysis*, Marcel Dekker, New York, pp. 543-613.
- Kane, T. R.; Wang, C. F.** (1965): On the derivation of equations of motion. *J Soc Ind Appl Math*, vol. 14, pp. 487-492.
- Kane, T. R.** (1961): Dynamics of nonholonomic systems, *J Appl Mech*, vol. 28, pp. 574.
- Huston, R. L.** (1991): Multibody dynamics – modeling and analysis methods, Feature Article, *Appl Mech Rev*, vol. 44, pp. 109-117.
- Kane, T. R.; Levinson, D. A.** (1980): Formulation of equations of motion for complex spacecraft, *J Guid Cont*, vol. 3, pp. 99-112.
- Huston, R. L.** (1999): Constraint forces and undetermined multipliers in constrained multibody systems, *Multibody Sys Dyn*, vol. 3, pp. 381-389.
- Lawson, C. L.; Hanson, R. J.** (1974): *Solving least squares problems*, Prentice Hall, pp. 36-40.
- Walton, W. C., Jr.; Steeves, E. C.** (1969): A new matrix theorem and its application for establishing independent coordinates for complex dynamical systems with constraints, NASA Technical Report TR-326.
- Liu, C. Q.; Huston, R. L.** (1995): An energy theorem for developing testing functions for the numerical simulation of dynamical systems, *J Dyn Sys Meas and Contrl*, vol. 117, pp. 193-198.
- Huston, R. L.** (1990): *Multibody dynamics*, Butterworth-Heinemann.
- Whittaker, E. T.** (1937): *Analytical dynamics*, Cambridge, pp. 8, 15.
- Kane, T. R.; Likins, P. W.; Levinson, D. A.** (1983): *Spacecraft dynamics*, McGraw Hill, pp. 422-431.
- Kane, T. R.; Levinson, D. A.** (1985): *Dynamics: theory and applications*, McGraw Hill, pp. 124-127.