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BIE Method for 3D Problems of Rigid Disk-Inclusion and Crack Interaction in Elastic Matrix

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Abstract: The 3D elastostatic problem for an infinite remotely loaded matrix containing a finite number of arbitrarily located rigid disk-inclusions and plane cracks is solved by the boundary integral equation (BIE) method. Its boundary integral formulation is achieved by the superposition principle with the subsequent integral representations of superposition terms through surface integrals, which should satisfy the displacement linearity conditions in the inclusion domains and load-free conditions in the crack domains. The subtraction technique in the conjunction with mapping technique under taking into account the structure of the solution at the edges of inhomogeneities is applied for the regularization of BIE obtained. The discrete analogue of equations is constructed by using the collocation scheme. The mixed mode stress intensity factors as functions of angular coordinates of front points for the interacting pairs of circular inclusions and cracks, which have different mutual orientations, distances and sizes, are calculated. The reinforcing properties of dispersed phase are estimated for involved models.

keyword: Boundary integral equation method, Three-dimensional elastic matrix, Interacting disk-inclusions and cracks, Mixed mode stress intensity factors.

1 Introduction

The use in practice of composite materials with wide spectrum of stiff characteristics of the constituents and complex inclusion forms and architecture stimulates the development of methods based on numerical simulation of mechanical response of such composites including fracture parameters. The BIE and related boundary element method has the remarkable advantages in this respect due to the reduction of problem dimension, the accurate evaluation of stress concentration near the interfaces and the possibility of involving in the analysis of multiple inclusions and damages of different kinds. Earlier this technique was successfully implemented under micro- and macrodescription (by the introduction of representative elements) of elastic matrix containing interacting elastic inclusions and voids by Noda and Matsuo, 1998; Fu, Klimkowski, Rodin, Berger, Browne, Singer, Geijn and Vemaganti, 1998; Liu, Xu nd Luo, 2000; Buryachenko and Pagano, 2000; Liu and Chen, 2003; Okada, Fukui and Kumazava, 2004; Wang and Yao, 2005. Improved weakly-singular boundary integral formulations, which are suitable for the analysis of such objects, are proposed by Han and Atluri, 2003; Atluri, Han and Shen, 2003. Considerably less articles deal with the BIE method for problems on the neighborhood of inclusions and crack-like defects, which concern 2D statements (see Wen and Lam, 1994; Lee and Mal, 1997; Noda and Matsuo, 1997; Petrova, Tamuzs and Romalis, 2000; Selvadurai, 2002). It was shown, that the behavior of crack in a composite and homogeneous material is essentially different, especially when its dimension is proportional to the dimension of inhomogeneity. This is due to local stress redistribution in the system of interacting cracks and particles of a reinforcing phase. Introduction in the matrix material of stiff inclusions and fibers blocks the crack propagation, and, hence increases its resistance to fracture. Analogous reinforcing effect of volumetric dispersed phase on the mode I crack in 3D case was fixed by the multiple expansion method by Kushch, 1998. However, the general numerical modeling and estimation of 3D particle composites with many mixed mode cracks was not carried out.

In this paper, a BIE method is presented for microanalysis of the cracked composite solid, in which the rigid disk-inclusions are used as fillers (it is known, that armouring the matrix material by inclusions of such type is the most efficient way to increase its stiffness, see Kanaun and Levin, 1993; Altenbach, Altenbach and

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Kissing, 2004). No restrictions on the number, shapes and mutual localization of inhomogeneities in an infinite remotely loaded matrix are foreseen. Perfect bonding is assumed between the matrix and inclusions, which allow translations and rotations. The displacement and stress components in such a solid are given in terms of surface integrals, where the jumps of interfacial stresses across the inclusions and displacement jumps across the crack faces are the unknown densities. Then the BIE for these functions are deduced by satisfying the effective boundary conditions of displacement linearity in the inclusion domains and load-free boundary conditions in the crack domains. They are accompanied by the equilibrium equations for each inclusion as a rigid unit. Accounting the stress concentration in the vicinities of inhomogeneities is provided by square-root extraction from the functions, which are determined. The regularization procedures include both analytical evaluation of integrals with singularities of different orders in the source points and mapping of the circular domains into rectangular domains to avoid the contour singularities. As to interaction between inclusions and cracks, it is described in the BIE by the regular kernels, which are written in the explicit form. Finally, the discrete analogue of BIE as a system of linear algebraic equations is obtained by means of collocation scheme. As numerical example, the interaction of inclined circular inclusion and crack due to the uniaxial tension of solid is studied. The crack-inclusion configurations with the maximal reinforcing effects concerning fracture parameters are revealed.

2 Statement of problem by the integral presentations and equations

Consider an infinite isotropic solid (matrix) with a system of arbitrarily located N^I thin absolutely rigid inclusions and N^C cracks, occupying the non-intersecting plane regions $S^{(k)I}\left(k=1,2,\ldots,N^I\right)$ and $S^{(n)C}\left(n=1,2,\ldots,N^C\right)$, respectively. The conditions of ideal contact between a matrix and inclusions are realized. Let the solid be under outer mechanical disturbances, which is described by the displacement field $\mathbf{u}^{(0)}$ in a homogeneous solid (without the stress concentrators). This main field, and connected with it by the Hooke's law the stress components $\sigma_{ij}^{(0)}$, is known.

The governing equation for the elastic matrix with inclusions and cracks is the equilibrium equation in the vector form

$$\nabla^2\mathbf{u}+\frac{1}{1-2\nu}\nabla\left(\nabla\cdot\mathbf{u}\right)=0\,, \tag{1}$$

where ∇ – is a three-dimensional nabla-vector, ν – is Poisson's ratio of matrix material.

For convenience of writing the boundary conditions and subsequent representations of unknown functions N^I+N^C Cartesian coordinate systems are introduced, so that the coordinate system $O^{(k)I}x_1^{(k)I}x_2^{(k)I}x_3^{(k)I}$ $\left(k=1,2,\ldots,N^I\right)$ is connected with the kth inclusion, and the coordinate system $O^{(n)C}x_1^{(n)C}x_2^{(n)C}x_3^{(n)C}$ $\left(n=1,2,\ldots,N^C\right)$ with the nth crack by the location of inhomogeneities in the planes $x_3^{(k)I}=0$ and $x_3^{(n)C}=0$, respectively. Besides, the coincidence of the center of coordinate system with the geometric center of the region, occupied by the corresponding inclusions or crack, is stipulated (Fig. 1). Then mutual location of the kth and nth inhomogeneities is described by the vectors $\mathbf{O}^{(k)I}\mathbf{O}^{(n)I}$, $\mathbf{O}^{(k)I}\mathbf{O}^{(n)C}$, $\mathbf{O}^{(k)C}\mathbf{O}^{(n)I}$ or $\mathbf{O}^{(k)C}\mathbf{O}^{(n)C}$ and direction cosines $l_j^{(kn)}$, $m_j^{(kn)}$ and $p_j^{(kn)}$ of the axes of the kth coordinate system with respect to the nth system (Appendix A). In addition, the point of solid with position vector $\mathbf{x}^{(n)I}\left(x_1^{(n)I},\,x_2^{(n)I},\,x_3^{(n)I}\right)$ or $\mathbf{x}^{(n)C}\left(x_1^{(n)C},\,x_2^{(n)C},\,x_3^{(n)C}\right)$ in the nth coordinate system, in the kth system is defined as

$$\begin{aligned}
&\mathbf{x}^{(kn)I}=\mathbf{O}^{(k)I}\mathbf{O}^{(n)I}+\mathbf{x}^{(n)I},\\
&\mathbf{x}^{(kn)I}=\mathbf{x}^{(kn)I}\left(x_1^{(kn)I},x_2^{(kn)I},x_3^{(kn)I}\right),\\
&\mathbf{x}^{(kn)IC}=\mathbf{O}^{(k)I}\mathbf{O}^{(n)C}+\mathbf{x}^{(n)C},\\
&\mathbf{x}^{(kn)IC}=\mathbf{x}^{(kn)IC}\left(x_1^{(kn)IC},x_2^{(kn)IC},x_3^{(kn)IC}\right),\\
&\mathbf{x}^{(kn)CI}=\mathbf{O}^{(k)C}\mathbf{O}^{(n)I}+\mathbf{x}^{(n)I},\\
&\mathbf{x}^{(kn)CI}=\mathbf{x}^{(kn)CI}\left(x_1^{(kn)CI},x_2^{(kn)CI},x_3^{(kn)CI}\right),\\
&\mathbf{x}^{(kn)C}=\mathbf{O}^{(k)C}\mathbf{O}^{(n)C}+\mathbf{x}^{(n)C},\\
&\mathbf{x}^{(kn)C}=\mathbf{x}^{(kn)C}\left(x_1^{(kn)C},x_2^{(kn)C},x_3^{(kn)C}\right).
\end{aligned} \tag{2}$$

Accounting the inclusion properties as rigid objects leads to the boundary conditions on the displacement components of the form

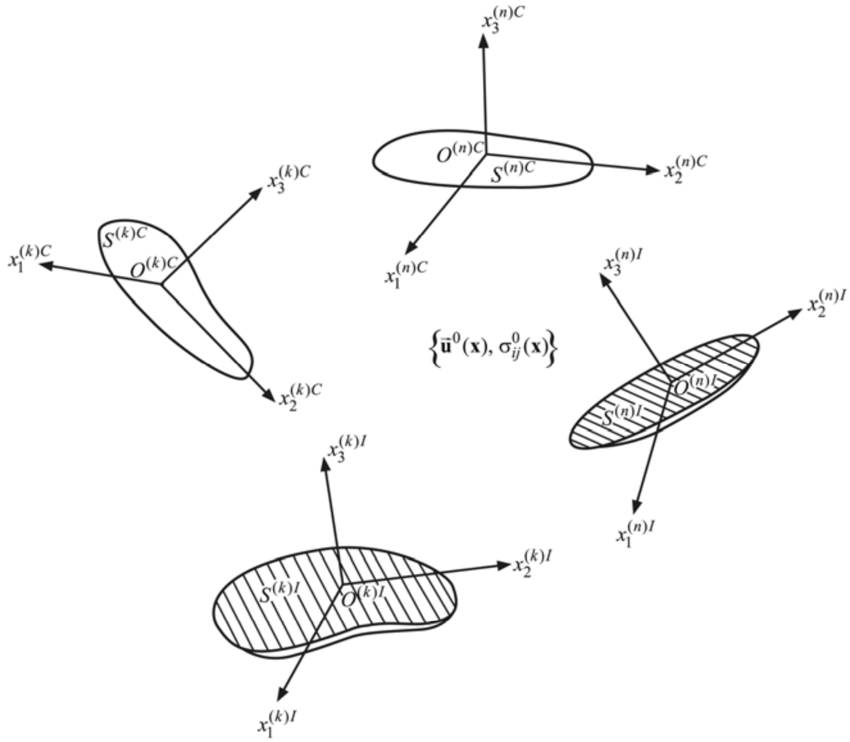

Figure 1 : Geometry of problem

$$
\begin{aligned}
u_j\left(\mathbf{x}^{(n)I}\right) = {} & U_j^{(n)} + (\delta_{1j}+\delta_{2j})(-1)^j \Omega_3^{(n)} x_{3-j}^{(n)I} \\
& +\delta_{3j}\left(\Omega_1^{(n)} x_2^{(n)I} - \Omega_2^{(n)} x_1^{(n)I}\right), \\
& \mathbf{x}^{(n)I} \in S^{(n)I}, j=1,2,3, \quad n=1,2,...,N^I,
\end{aligned} \tag{3}
$$

where δ_{ij} is the Kronecker symbol, the constants $U_j^{(n)}$ and $\Omega_j^{(n)}$ characterize the translation and rotation of the nth inclusion relative to the axis $O^{(n)}x_j^{(n)I}$, respectively. Also for each inclusion should be satisfied the conditions of its equilibrium, which consist of equality to zero of the principal vector and moment of tractions, acting on the inclusion from the side of matrix.

Because the crack faces are load-free, the following boundary conditions on the stress components take place

$$
\begin{aligned}
\sigma_{j3}\left(\mathbf{x}^{(n)C}\right) = 0, \qquad & \mathbf{x}^{(n)C} \in S^{(n)C}, \\
j=1,2,3, \quad & n=1,2,\ldots,N^C.
\end{aligned} \tag{4}
$$

In accordance to the superposition principle, the total field in the solid can be formed from the main (primary) field and disturbance fields due to the presence of the system of inhomogeneities, i.e.

$$
\mathbf{u} = \mathbf{u}^{(0)} + \sum_{k=1}^{N^I} \mathbf{u}^{(k)I} + \sum_{k=1}^{N^C} \mathbf{u}^{(k)C}. \tag{5}
$$

All terms in relation (5) must satisfy Eq. (1), the terms under summation signs must vanish at infinity. Besides, the component $\mathbf{u}^{(k)I}$ must provide the stress jumps in the regions of the kth inclusion, and the component $\mathbf{u}^{(k)C}$ – the displacement jumps in the region of the kth crack.

The integral representations for the displacements $\mathbf{u}^{(k)I}$ and $\mathbf{u}^{(k)C}$, considering the above peculiarities in the region of the kth inclusion or crack, can be obtained either using the Newtonian potentials as solutions of the corresponding mixed problem for a half-space (see Kit and Khaj, 1989; Alexandrov, Smetanin and Sobol, 1993) or applying the Somigliano formula (see Balas, Sladek and Sladek, 1989). As a result of both approaches, the following integral presentations of displacement components $\mathbf{u}^{(k)I}\left(u_1^{(k)I}, u_2^{(k)I}, u_3^{(k)I}\right)$ and $\mathbf{u}^{(k)C}\left(u_1^{(k)C}, u_2^{(k)C}, u_3^{(k)C}\right)$ are obtained in the kth coordinate system

$$u_j^{(k)I}\left(\mathbf{x}^{(k)I}\right)=\frac{1}{G}\int\int_{S^{(k)I}}\frac{\Delta\sigma_j^{(k)}\left(\mathbf{y}^{(k)I}\right)}{\left|\mathbf{x}^{(k)I}-\mathbf{y}^{(k)I}\right|}dS_{\mathbf{y}}$$
$$-\frac{1}{4(1-\nu)G}\frac{\partial}{\partial x_j^{(k)I}}\int\int_{S^{(k)I}}\left[\Delta\sigma_1^{(k)}\left(\mathbf{y}^{(k)I}\right)\frac{\partial}{\partial x_1^{(k)I}}\right.$$
$$\left.+\Delta\sigma_2^{(k)}\left(\mathbf{y}^{(k)I}\right)\frac{\partial}{\partial x_2^{(k)I}}+\Delta\sigma_3^{(k)}\left(\mathbf{y}^{(k)I}\right)\frac{\partial}{\partial x_3^{(k)I}}\right]$$
$$\left|\mathbf{x}^{(k)I}-\mathbf{y}^{(k)I}\right|dS_{\mathbf{y}},\qquad j=1,2,3,$$

$$u_j^{(k)C}\left(\mathbf{x}^{(k)C}\right)=\frac{\partial}{x_3^{(k)C}}\int\int_{S^{(k)C}}\frac{\Delta u_j^{(k)}\left(\mathbf{y}^{(k)C}\right)}{\left|\mathbf{x}^{(k)C}-\mathbf{y}^{(k)C}\right|}dS_{\mathbf{y}}$$
$$-\frac{1-2\nu}{2(1-\nu)}\frac{\partial}{\partial x_j^{(k)C}}\int\int_{S^{(k)C}}\frac{\Delta u_3^{(k)C}\left(\mathbf{y}^{(k)C}\right)}{\left|\mathbf{x}^{(k)C}-\mathbf{y}^{(k)C}\right|}dS_{\mathbf{y}}$$
$$+\frac{1-2\nu}{2(1-\nu)}\delta_{j3}\int\int_{S^{(k)C}}\left[\Delta u_1^{(k)}\left(\mathbf{y}^{(k)C}\right)\frac{\partial}{\partial x_1^{(k)C}}\right.$$
$$\left.+\Delta u_2^{(k)}\left(\mathbf{y}^{(k)C}\right)\frac{\partial}{\partial x_2^{(k)C}}+\Delta u_3^{(k)}\left(\mathbf{y}^{(k)C}\right)\frac{\partial}{\partial x_3^{(k)C}}\right]$$
$$\frac{dS_{\mathbf{y}}}{\left|\mathbf{x}^{(k)C}-\mathbf{y}^{(k)C}\right|}$$
$$-\frac{x_3^{(k)C}}{2(1-\nu)}\frac{\partial}{\partial x_j^{(k)C}}\int\int_{S^{(k)C}}\left[\Delta u_1^{(k)}\left(\mathbf{y}^{(k)C}\right)\frac{\partial}{\partial x_1^{(k)C}}\right.$$
$$\left.+\Delta u_2^{(k)}\left(\mathbf{y}^{(k)C}\right)\frac{\partial}{\partial x_2^{(k)C}}++\Delta u_3^{(k)}\left(\mathbf{y}^{(k)C}\right)\frac{\partial}{\partial x_3^{(k)C}}\right]\tag{6}$$
$$\frac{dS_{\mathbf{y}}}{\left|\mathbf{x}^{(k)C}-\mathbf{y}^{(k)C}\right|},\qquad j=1,2,3,$$

where G is the shear modulus, $|\mathbf{x}-\mathbf{y}|$ is the distance between a field point $\mathbf{x}(x_1,x_2,x_3)$ and the integration point $\mathbf{y}(y_1,y_2)$ in the corresponding coordinate system, $\Delta\sigma_j^{(k)},\Delta u_j^{(k)}$ $(j=1,2,3)$ are the unknown densities of potentials, having the physical meaning. Thus, the analysis of limit properties of displacements in the region $S^{(k)C}$ shows, that the density $\Delta u_j^{(k)}$ characterizes the kth crack opening in the direction of the axe $O^{(k)C}x_j^{(k)C}$, namely

$$\Delta u_j^{(k)}\left(\mathbf{y}^{(k)C}\right)=\frac{1}{4\pi}\left[u_j^-\left(y_1^{(k)C},y_2^{(k)C}\right)\right.$$
$$\left.-u_j^+\left(y_1^{(k)C},y_2^{(k)C}\right)\right],\qquad j=1,2,3,$$
$$\mathbf{y}^{(k)C}\left(y_1^{(k)C},y_2^{(k)C}\right)\in S^{(k)C},$$
$$u_j^{\pm}\left(x_1^{(k)C},x_2^{(k)C}\right)=\lim_{x_3^{(k)C}\to\pm0}u_j\left(\mathbf{x}^{(k)C}\right).\tag{7}$$

Having defined by the Hooke's law the stresses $\sigma_{ij}^{(k)I}$ and $\sigma_{ij}^{(k)C}$, which correspond to the displacements of the form (6), we come to understanding the densities $\Delta\sigma_j^{(k)}$ as contact traction between the matrix and kth inclusion in the direction of axe $O^{(k)I}x_j^{(k)I}$, namely

$$\Delta\sigma_j^{(k)}\left(\mathbf{y}^{(k)I}\right)=\frac{1}{4\pi}\left[\sigma_{j3}^-\left(y_1^{(k)I},y_2^{(k)I}\right)\right.$$
$$\left.-\sigma_{j3}^+\left(y_1^{(k)I},y_2^{(k)I}\right)\right],\qquad j=1,2,3,$$
$$\mathbf{y}^{(k)I}\left(y_1^{(k)I},y_2^{(k)I}\right)\in S^{(k)I},$$
$$\sigma_{j3}^{\pm}\left(x_1^{(k)I},x_2^{(k)I}\right)=\lim_{x_3^{(k)I}\to\pm0}\sigma_{j3}\left(\mathbf{x}^{(k)I}\right).\tag{8}$$

Thus, combination of relations (5), (6) makes it possible to describe the stress-strain state of the solid with inclusions and cracks in integral form in terms of functions of contact tractions between matrix and inclusions and functions of the crack opening. To define these functions the boundary conditions (3), (4) can be used. With this purpose the total displacements and stresses by components should be written. For displacement components in the coordinate systems, connected with inclusions, we have

$$u_j\left(\mathbf{x}^{(n)I}\right)=u_j^{(0)}\left(\mathbf{x}^{(n)I}\right)$$
$$+\sum_{k=1}^{N^I}\left[u_1^{(k)I}\left(\mathbf{x}^{(kn)I}\right)l_j^{(kn)I}+u_2^{(k)I}\left(\mathbf{x}^{(kn)I}\right)m_j^{(kn)I}\right.$$
$$\left.+u_3^{(k)I}\left(\mathbf{x}^{(kn)I}\right)p_j^{(kn)I}\right]$$
$$+\sum_{k=1}^{N^C}\left[u_1^{(k)C}\left(\mathbf{x}^{(kn)CI}\right)l_j^{(kn)CI}\right.$$
$$\left.+u_2^{(k)C}\left(\mathbf{x}^{(kn)CI}\right)m_j^{(kn)CI}+u_3^{(k)C}\left(\mathbf{x}^{(kn)CI}\right)p_j^{(kn)CI}\right]$$
$$j=1,2,3,\qquad n=1,2,\ldots,N^I.\tag{9}$$

For the stress components in the coordinate systems, connected with cracks, we have

$$\sigma_{j3}^{(n)}\left(\mathbf{x}^{(n)C}\right)=\sigma_{j3}^{(0)}\left(\mathbf{x}^{(n)C}\right)$$
$$+\sum_{k=1}^{N^I}\left[\sigma_{11}^{(k)I}\left(\mathbf{x}^{(kn)IC}\right)l_{j3}^{(kn)IC}+\sigma_{22}^{(k)I}\left(\mathbf{x}^{(kn)IC}\right)m_{j3}^{(kn)IC}\right.$$
$$+\sigma_{33}^{(k)I}\left(\mathbf{x}^{(kn)IC}\right)p_{j3}^{(kn)IC}+\sigma_{12}^{(k)I}\left(\mathbf{x}^{(kn)IC}\right)l_{j3*}^{(kn)IC}$$
$$\left.+\sigma_{23}^{(k)I}\left(\mathbf{x}^{(kn)IC}\right)m_{j3*}^{(kn)IC}++\sigma_{13}^{(k)I}\left(\mathbf{x}^{(kn)IC}\right)p_{j3*}^{(kn)IC}\right]$$

$$+\sum_{k=1}^{N^C}\left[\sigma_{11}^{(k)C}\left(\mathbf{x}^{(kn)C}\right)l_{j3}^{(kn)C}+\sigma_{22}^{(k)C}\left(\mathbf{x}^{(kn)C}\right)m_{j3}^{(kn)C}\right.$$
$$+\sigma_{33}^{(k)C}\left(\mathbf{x}^{(kn)C}\right)p_{j3}^{(kn)C}+\sigma_{12}^{(k)C}\left(\mathbf{x}^{(kn)C}\right)l_{j3*}^{(kn)C}$$
$$\left.+\sigma_{23}^{(k)C}\left(\mathbf{x}^{(kn)C}\right)m_{j3*}^{(kn)C}+\sigma_{13}^{(k)C}\left(\mathbf{x}^{(kn)C}\right)p_{j3*}^{(kn)C}\right],$$
$$j=1,2,3,\qquad n=1,2,...,N^C, \tag{10}$$

where the coefficients are expressed in terms of geometrical parameters of inhomogeneity location as

$$l_{j3}^{(kn)IC}=l_j^{(kn)IC}l_3^{(kn)IC},$$
$$m_{j3}^{(kn)IC}=m_j^{(kn)IC}m_3^{(kn)IC},$$
$$p_{j3}^{(kn)IC}=p_j^{(kn)IC}p_3^{(kn)IC},$$
$$l_{j3}^{(kn)C}=l_j^{(kn)C}l_3^{(kn)C},$$
$$m_{j3}^{(kn)C}=m_j^{(kn)C}m_3^{(kn)C},$$
$$p_{j3}^{(kn)C}=p_j^{(kn)C}p_3^{(kn)C},$$
$$l_{j3*}^{(kn)IC}=l_j^{(kn)IC}m_3^{(kn)IC}+m_j^{(kn)IC}l_3^{(kn)IC},$$
$$m_{j3*}^{(kn)IC}=m_j^{(kn)IC}p_3^{(kn)IC}+p_j^{(kn)IC}m_3^{(kn)IC},$$
$$p_{j3*}^{(kn)IC}=p_j^{(kn)IC}l_3^{(kn)IC}+l_j^{(kn)IC}p_3^{(kn)IC},$$
$$l_{j3*}^{(kn)C}=l_j^{(kn)C}m_3^{(kn)C}+m_j^{(kn)C}l_3^{(kn)C},$$
$$m_{j3*}^{(kn)C}=m_j^{(kn)C}p_3^{(kn)C}+p_j^{(kn)C}m_3^{(kn)C},$$
$$p_{j3*}^{(kn)C}=p_j^{(kn)C}l_3^{(kn)C}+l_j^{(kn)C}p_3^{(kn)C}. \tag{11}$$

On having satisfied by the relations (9), (10) of the boundary conditions (3), (4) and considering the boundary properties of the potentials in the representations of displacements (6) and corresponding stresses, we arrive to the system of $3\left(N^I+N^C\right)$ BIE with respect to function

$$\Delta\sigma_j^{(n)}\left(j=1,2,3,\ n=1,2,\ldots,N^I\right),$$
$$\Delta u_j^{(n)}\left(j=1,2,3,\ n=1,2,\ldots,N^C\right)$$

and constants

$$U_j^{(n)},\ \Omega_j^{(n)}\ \left(j=1,2,3,\ n=1,2,\ldots,N^I\right)$$

in the form

$$(3-4\nu)\iint\limits_{S^{(n)I}}\frac{\Delta\sigma_j^{(n)}\left(\mathbf{y}^{(n)I}\right)}{\left|\mathbf{x}^{(n)I}-\mathbf{y}^{(n)I}\right|}dS_{\mathbf{y}}$$
$$+(\delta_{1j}+\delta_{2j})\iint\limits_{S^{(n)I}}\left[\frac{\left(x_j^{(n)I}-y_j^{(n)I}\right)^2}{\left|\mathbf{x}^{(n)I}-\mathbf{y}^{(n)I}\right|^2}\Delta\sigma_j^{(n)}\left(\mathbf{y}^{(n)I}\right)\right.$$
$$\left.+\frac{\left(x_1^{(n)I}-y_1^{(n)I}\right)\left(x_2^{(n)I}-y_2^{(n)I}\right)}{\left|\mathbf{x}^{(n)I}-\mathbf{y}^{(n)I}\right|^2}\Delta\sigma_{3-j}^{(n)}\left(\mathbf{y}^{(n)I}\right)\right]$$
$$\frac{dS_{\mathbf{y}}}{\left|\mathbf{x}^{(n)I}-\mathbf{y}^{(n)I}\right|}+\sum_{k=1}^{N^I}(1-\delta_{kn})\sum_{r=1}^{3}\iint\limits_{S^{(k)I}}\Delta\sigma_r^{(k)}\left(\mathbf{y}^{(k)I}\right)$$
$$\times R_{jr}^{(kn)I}\left(\mathbf{x}^{(kn)I},\mathbf{y}^{(k)I}\right)dS_{\mathbf{y}}$$
$$+\sum_{k=1}^{N^C}\sum_{r=1}^{3}\iint\limits_{S^{(k)C}}\Delta u_r^{(k)}\left(\mathbf{y}^{(k)C}\right)L_{jr}^{(kn)CI}\left(\mathbf{x}^{(kn)CI},\mathbf{y}^{(k)C}\right)dS_{\mathbf{y}}$$
$$-4(1-\nu)G\left[U_j^{(n)}+\delta_{3j}\left(\Omega_1^{(n)}x_2^{(n)I}-\Omega_2^{(n)}x_1^{(n)I}\right)\right.$$
$$\left.+(1-\delta_{3j})(-1)^j\Omega_3^{(n)}x_{3-j}^{(n)I}\right]$$
$$=-4(1-\nu)Gu_j^{(0)}\left(\mathbf{x}^{(n)I}\right),$$
$$\mathbf{x}^{(n)I}\left(x_1^{(n)I},x_2^{(n)I}\right)\in S^{(n)I},$$
$$j=1,2,3,\ n=1,2,\ldots,N^I,$$

$$\left[\delta_{3j}+(1-2\nu)(\delta_{1j}+\delta_{2j})\right]\iint\limits_{S^{(n)C}}\frac{\Delta u_j^{(n)}\left(\mathbf{y}^{(n)C}\right)}{\left|\mathbf{x}^{(n)C}-\mathbf{y}^{(n)C}\right|^3}dS_{\mathbf{y}}$$
$$+3\nu(\delta_{1j}+\delta_{2j})\iint\limits_{S^{(n)C}}\left[\frac{\left(x_j^{(n)C}-y_j^{(n)C}\right)^2}{\left|\mathbf{x}^{(n)C}-\mathbf{y}^{(n)C}\right|^2}\Delta u_j^{(n)}\left(\mathbf{y}^{(n)C}\right)\right.$$
$$\left.+\frac{\left(x_1^{(n)C}-y_1^{(n)C}\right)\left(x_2^{(n)C}-y_2^{(n)C}\right)}{\left|\mathbf{x}^{(n)C}-\mathbf{y}^{(n)C}\right|^2}\Delta u_{3-j}^{(n)}\left(\mathbf{y}^{(n)C}\right)\right]$$
$$\frac{dS_{\mathbf{y}}}{\left|\mathbf{x}^{(n)C}-\mathbf{y}^{(n)C}\right|^3}+\sum_{k=1}^{N^C}(1-\delta_{kn})\sum_{r=1}^{3}\iint\limits_{S^{(k)C}}\Delta u_r^{(k)}\left(\mathbf{y}^{(k)C}\right)$$

$$\times R_{jr}^{(kn)C}\left(\mathbf{x}^{(kn)C},\mathbf{y}^{(k)C}\right)dS_{\mathbf{y}}$$

$$+\sum_{k=1}^{N^I}\sum_{r=1}^{3}\iint_{S^{(k)I}}\Delta\sigma_r^{(k)}\left(\mathbf{y}^{(k)I}\right)L_{jr}^{(kn)IC}\left(\mathbf{x}^{(kn)IC},\mathbf{y}^{(k)I}\right)dS_{\mathbf{y}}$$

$$=-\tfrac{1-\nu}{G}\sigma_{j3}^{(0)}\left(\mathbf{x}^{(n)C}\right),$$

$$\mathbf{x}^{(n)C}\left(x_1^{(n)C},x_2^{(n)C}\right)\in S^{(n)C},$$

$$j=1,2,3,\qquad n=1,2,\ldots,N^C, \tag{12}$$

which, for the completeness are accompanied by the $6N^I$ equilibrium conditions for each inclusion as a rigid unit

$$\iint_{S^{(n)I}}\Delta\sigma_j^{(n)}\left(\mathbf{y}^{(n)I}\right)dS_{\mathbf{y}}=0,$$
$$j=1,2,3,\qquad n=1,2,\ldots,N^I,$$

$$\iint_{S^{(n)I}}y_{3-j}^{(n)I}\Delta\sigma_3^{(n)}\left(\mathbf{y}^{(n)I}\right)dS_{\mathbf{y}}=0,$$
$$j=1,2,\qquad n=1,2,\ldots,N^I,$$

$$\iint_{S^{(n)I}}\left[y_2^{(n)I}\Delta\sigma_1^{(n)}\left(\mathbf{y}^{(n)I}\right)-y_1^{(n)I}\Delta\sigma_2^{(n)}\left(\mathbf{y}^{(n)I}\right)\right]dS_{\mathbf{y}}=0,$$
$$n=1,2,\ldots,N^I. \tag{13}$$

In the first $3N^I$ equations of the system (12), defined on the inclusion domains, the weakly singular integrals in the source point $\mathbf{y}^{(n)I}=\mathbf{x}^{(n)I}$ form the operator of BIE of 3D problem on a single inclusion in an infinite body given by Rahman, 1999. In the next $3N^C$ equations the hypersingular integrals in the source point $\mathbf{y}^{(n)C}=\mathbf{x}^{(n)C}$ coincide with the operator of BIE of 3D problems on single crack in an infinite solid given by Kit and Khaj, 1989. The regular kernels $R_{jr}^{(kn)I}$ and $L_{jr}^{(kn)CI}$ describe the influence on the nth inclusion of kth inclusion and kth crack, while the regular kernels $R_{jr}^{(kn)C}$ and $L_{jr}^{(kn)IC}$ describe the influence on the nth crack of kth crack and kth inclusion, respectively. The explicit expressions for these kernels are written in Appendix B.

Because the Eqs. (12) contain the integrals with peculiarities, their adaptation to the numerical solution is needed with considering the solution behavior in the vicinity of the front of each inhomogeneity.

3 Regularization and discretization procedures

For the sake of brevity let's illustrate the general scheme for the case of presence in a composite solid of circular disk-shaped inclusions of radii $a^{(n)}$ $\left(n=1,2,\ldots,N^I\right)$ and circular cracks of radii $b^{(n)}$ $\left(n=1,2,\ldots,N^C\right)$. It should be mentioned, that the cases of inhomogeneities of more complicated configurations can be easily taken into account by methodology of Sladek, Sladek, Mykhas'kiv, Stankevych, 2003. To satisfy analitycally the conditions of displacement continuity near the inclusion and crack fronts, the following ansatzs are used in the corresponding coordinate system

$$\Delta\sigma_j^{(n)}\left(\mathbf{x}^{(n)I}\right)=$$
$$\alpha_j^{(n)}\left(\mathbf{x}^{(n)I}\right)\Big/\sqrt{\left(a^{(n)}\right)^2-\left(x_1^{(n)I}\right)^2-\left(x_2^{(n)I}\right)^2},$$
$$\mathbf{x}^{(n)I}\in S^{(n)I},\qquad j=1,2,3,\qquad n=1,2,\ldots,N^I,$$

$$\Delta u_j^{(n)}\left(\mathbf{x}^{(n)C}\right)=$$
$$\sqrt{\left(b^{(n)}\right)^2-\left(x_1^{(n)C}\right)^2-\left(x_2^{(n)C}\right)^2}\,\beta_j^{(n)}\left(\mathbf{x}^{(n)C}\right),$$
$$\mathbf{x}^{(n)C}\in S^{(n)C},\quad j=1,2,3,\quad n=1,2,\ldots,N^C, \tag{14}$$

where $\alpha_j^{(n)}$ and $\beta_j^{(n)}$ are the new unknown functions of sufficient smoothness.

Substituting Eqs. (14) into Eqs. (12) leads to the integral equations which, in addition to the different order singularities in the source points, have the root singularities on the contours of domains $S^{(n)I}$. These circumstances define the way of subsequent regularization.

The initial step foresees the interpretation of the particular (at source points) integrals of Eqs. (12) in the domains $S^{(n)I}$ and $S^{(n)C}$ in the sense:

$$\iint_{S^{(n)I}}\frac{\alpha_j^{(n)}\left(\mathbf{y}^{(n)I}\right)}{\sqrt{\left(a^{(n)}\right)^2-\left(y_1^{(n)I}\right)^2-\left(y_2^{(n)I}\right)^2}\left|\mathbf{x}^{(n)I}-\mathbf{y}^{(n)I}\right|}dS_{\mathbf{y}}=$$

$$M^{(n)}\left(\mathbf{x}^{(n)I}\right)\alpha_j^{(n)}\left(\mathbf{x}^{(n)I}\right)$$

$$+\iint_{S^{(n)I}}\frac{\alpha_j^{(n)}\left(\mathbf{y}^{(n)I}\right)-\alpha_j^{(n)}\left(\mathbf{x}^{(n)I}\right)}{\sqrt{\left(a^{(n)}\right)^2-\left(y_1^{(n)I}\right)^2-\left(y_2^{(n)I}\right)^2}\left|\mathbf{x}^{(n)I}-\mathbf{y}^{(n)I}\right|}dS_{\mathbf{y}},$$

$$j=1,2,3,$$

$$\int\int_{S^{(n)I}} \frac{\alpha_j^{(n)}\left(\mathbf{y}^{(n)I}\right)}{\sqrt{\left(a^{(n)}\right)^2-\left(y_1^{(n)I}\right)^2-\left(y_2^{(n)I}\right)^2}} \frac{\left(x_1^{(n)I}-y_1^{(n)I}\right)^i\left(x_2^{(n)I}-y_2^{(n)I}\right)^{2-i}}{\left|\mathbf{x}^{(n)I}-\mathbf{y}^{(n)I}\right|^2} dS_{\mathbf{y}}$$

$$= I_i\left(\mathbf{x}^{(n)I}\right)\alpha_j^{(n)}\left(\mathbf{x}^{(n)I}\right)$$

$$+\int\int_{S^{(n)I}} \frac{\alpha_j^{(n)}\left(\mathbf{y}^{(n)I}\right)-\alpha_j^{(n)}\left(\mathbf{x}^{(n)I}\right)}{\sqrt{\left(a^{(n)}\right)^2-\left(y_1^{(n)I}\right)^2-\left(y_2^{(n)I}\right)^2}}$$

$$\frac{\left(x_1^{(n)I}-y_1^{(n)I}\right)^i\left(x_2^{(n)I}-y_2^{(n)I}\right)^{2-i}}{\left|\mathbf{x}^{(n)I}-\mathbf{y}^{(n)I}\right|^2} dS_{\mathbf{y}},$$

$$j=1,2, \quad i=0,1,2,$$

$$\int\int_{S^{(n)C}} \frac{\sqrt{\left(b^{(n)}\right)^2-\left(y_1^{(n)C}\right)^2-\left(y_2^{(n)C}\right)^2}}{\left|\mathbf{x}^{(n)C}-\mathbf{y}^{(n)C}\right|^3}\beta_j^{(n)}\left(\mathbf{y}^{(n)C}\right)dS_{\mathbf{y}}$$

$$= P_{00}^{(n)}\left(\mathbf{x}^{(n)C}\right)\beta_j^{(n)}\left(\mathbf{x}^{(n)C}\right)$$

$$+P_{10}^{(n)}\left(\mathbf{x}^{(n)C}\right)\frac{\partial\beta_j^{(n)}\left(\mathbf{x}^{(n)C}\right)}{\partial x_1^{(n)C}} + P_{01}^{(n)}\left(\mathbf{x}^{(n)C}\right)\frac{\partial\beta_j^{(n)}\left(\mathbf{x}^{(n)C}\right)}{\partial x_2^{(n)C}}$$

$$+\frac{1}{2}P_{20}^{(n)}\left(\mathbf{x}^{(n)C}\right)\frac{\partial^2\beta_j^{(n)}\left(\mathbf{x}^{(n)C}\right)}{\partial x_1^{(n)C2}}$$

$$+\frac{1}{2}P_{02}^{(n)}\left(\mathbf{x}^{(n)C}\right)\frac{\partial^2\beta_j^{(n)}\left(\mathbf{x}^{(n)C}\right)}{\partial x_2^{(n)C2}}$$

$$+P_{11}^{(n)}\left(\mathbf{x}^{(n)C}\right)\frac{\partial^2\beta_j^{(n)}\left(\mathbf{x}^{(n)C}\right)}{\partial x_1^{(n)C}\partial x_2^{(n)C}}$$

$$+\int\int_{S^{(n)C}} \frac{\sqrt{\left(b^{(n)}\right)^2-\left(y_1^{(n)C}\right)^2-\left(y_2^{(n)C}\right)^2}}{\left|\mathbf{x}^{(n)C}-\mathbf{y}^{(n)C}\right|^3}\left[\beta_j^{(n)}\left(\mathbf{y}^{(n)C}\right)\right.$$

$$-\beta_j^{(n)}\left(\mathbf{x}^{(n)C}\right) - \left(y_1^{(n)C}-x_1^{(n)C}\right)\frac{\partial\beta_j^{(n)}\left(\mathbf{x}^{(n)C}\right)}{\partial x_1^{(n)C}}$$

$$-\left(y_2^{(n)C}-x_2^{(n)C}\right)\frac{\partial\beta_j^{(n)}\left(\mathbf{x}^{(n)C}\right)}{\partial x_2^{(n)C}}$$

$$-\frac{1}{2}\left(y_1^{(n)C}-x_1^{(n)C}\right)^2\frac{\partial^2\beta_j^{(n)}\left(\mathbf{x}^{(n)C}\right)}{\partial x_1^{(n)C2}}$$

$$-\frac{1}{2}\left(y_2^{(n)C}-x_2^{(n)C}\right)^2\frac{\partial^2\beta_j^{(n)}\left(\mathbf{x}^{(n)C}\right)}{\partial x_2^{(n)C2}}$$

$$\left.-\left(y_1^{(n)C}-x_1^{(n)C}\right)\left(y_2^{(n)C}-x_2^{(n)C}\right)\frac{\partial^2\beta_j^{(n)}\left(\mathbf{x}^{(n)C}\right)}{\partial x_1^{(n)C}\partial x_2^{(n)C}}\right] dS_{\mathbf{y}},$$

$$j=1,2,3,$$

$$\int\int_{S^{(n)C}} \sqrt{\left(b^{(n)}\right)^2-\left(y_1^{(n)C}\right)^2-\left(y_2^{(n)C}\right)^2}$$

$$\frac{\left(x_1^{(n)C}-y_1^{(n)C}\right)^i\left(x_2^{(n)C}-y_2^{(n)C}\right)^{2-i}}{\left|\mathbf{x}^{(n)C}-\mathbf{y}^{(n)C}\right|^5}\beta_j^{(n)}\left(\mathbf{y}^{(n)C}\right)dS_{\mathbf{y}}$$

$$= J_{00i}^{(n)}\left(\mathbf{x}^{(n)C}\right)\beta_j^{(n)}\left(\mathbf{x}^{(n)C}\right) + J_{10i}^{(n)}\left(\mathbf{x}^{(n)C}\right)\frac{\partial\beta_j^{(n)}\left(\mathbf{x}^{(n)C}\right)}{\partial x_1^{(n)C}}$$

$$+J_{01i}^{(n)}\left(\mathbf{x}^{(n)C}\right)\frac{\partial\beta_j^{(n)}\left(\mathbf{x}^{(n)C}\right)}{\partial x_2^{(n)C}}$$

$$+\frac{1}{2}J_{20i}^{(n)}\left(\mathbf{x}^{(n)C}\right)\frac{\partial^2\beta_j^{(n)}\left(\mathbf{x}^{(n)C}\right)}{\partial x_1^{(n)C2}}$$

$$+\frac{1}{2}J_{02i}^{(n)}\left(\mathbf{x}^{(n)C}\right)\frac{\partial^2\beta_j^{(n)}\left(\mathbf{x}^{(n)C}\right)}{\partial x_2^{(n)C2}}$$

$$+J_{11i}^{(n)}\left(\mathbf{x}^{(n)C}\right)\frac{\partial^2\beta_j^{(n)}\left(\mathbf{x}^{(n)C}\right)}{\partial x_1^{(n)C}\partial x_2^{(n)C}}$$

$$+\int\int_{S^{(n)C}} \sqrt{\left(b^{(n)}\right)^2-\left(y_1^{(n)C}\right)^2-\left(y_2^{(n)C}\right)^2}$$

$$\frac{\left(x_1^{(n)C}-y_1^{(n)C}\right)^i\left(x_2^{(n)C}-y_2^{(n)C}\right)^{2-i}}{\left|\mathbf{x}^{(n)C}-\mathbf{y}^{(n)C}\right|^5}$$

$$\left[\beta_j^{(n)}\left(\mathbf{y}^{(n)C}\right) - \beta_j^{(n)}\left(\mathbf{x}^{(n)C}\right)\right.$$

$$-\left(y_1^{(n)C}-x_1^{(n)C}\right)\frac{\partial\beta_j^{(n)}\left(\mathbf{x}^{(n)C}\right)}{\partial x_1^{(n)C}}$$

$$-\left(y_2^{(n)C}-x_2^{(n)C}\right)\frac{\partial\beta_j^{(n)}\left(\mathbf{x}^{(n)C}\right)}{\partial x_2^{(n)C}}$$

$$-\frac{1}{2}\left(y_1^{(n)C}-x_1^{(n)C}\right)^2\frac{\partial^2\beta_j^{(n)}\left(\mathbf{x}^{(n)C}\right)}{\partial x_1^{(n)C2}}$$

$$-\frac{1}{2}\left(y_2^{(n)C}-x_2^{(n)C}\right)^2\frac{\partial^2\beta_j^{(n)}\left(\mathbf{x}^{(n)C}\right)}{\partial x_2^{(n)C2}}$$

$$\left.-\left(y_1^{(n)C}-x_1^{(n)C}\right)\left(y_2^{(n)C}-x_2^{(n)C}\right)\frac{\partial^2\beta_j^{(n)}\left(\mathbf{x}^{(n)C}\right)}{\partial x_1^{(n)C}\partial x_2^{(n)C}}\right] dS_{\mathbf{y}},$$

$$j=1,2,\quad i=0,1,2. \tag{15}$$

Here the integrals $M^{(n)}$, $I_i^{(n)}$, $P_{jk}^{(n)}$ and $J_{jki}^{(n)}$ are determined by the formulae

$$M^{(n)}\left(\mathbf{x}^{(n)I}\right)=\int\!\!\int_{S^{(n)I}}\frac{dS_{\mathbf{y}}}{\sqrt{\left(a^{(n)}\right)^2-\left(y_1^{(n)I}\right)^2-\left(y_2^{(n)I}\right)^2}\left|\mathbf{x}^{(n)I}-\mathbf{y}^{(n)I}\right|},$$

$$\mathbf{x}^{(n)I}\in S^{(n)I},$$

$$I_i^{(n)}\left(\mathbf{x}^{(n)I}\right)=\int\!\!\int_{S^{(n)I}}\frac{\left(x_1^{(n)I}-y_1^{(n)I}\right)^i\left(x_2^{(n)I}-y_2^{(n)I}\right)^{2-i}}{\sqrt{\left(a^{(n)}\right)^2-\left(y_1^{(n)I}\right)^2-\left(y_2^{(n)I}\right)^2}\left|\mathbf{x}^{(n)I}-\mathbf{y}^{(n)I}\right|^3}dS_{\mathbf{y}},$$

$$\mathbf{x}^{(n)I}\in S^{(n)I},\quad i=0,1,2,$$

$$P_{jk}^{(n)}\left(\mathbf{x}^{(n)C}\right)=\int\!\!\int_{S^{(n)C}}\sqrt{\left(b^{(n)}\right)^2-\left(y_1^{(n)C}\right)^2-\left(y_2^{(n)C}\right)^2}$$

$$\frac{\left(y_1^{(n)C}-x_1^{(n)C}\right)^j\left(y_2^{(n)C}-x_2^{(n)C}\right)^k}{\left|\mathbf{x}^{(n)C}-\mathbf{y}^{(n)C}\right|^3}dS_{\mathbf{y}},$$

$$\mathbf{x}^{(n)C}\in S^{(n)C},\qquad k+j=0,1,2,$$

$$J_{jki}^{(n)}\left(\mathbf{x}^{(n)C}\right)=\int\!\!\int_{S^{(n)C}}\sqrt{\left(b^{(n)}\right)^2-\left(y_1^{(n)C}\right)^2-\left(y_2^{(n)C}\right)^2}$$

$$\frac{\left(y_1^{(n)C}-x_1^{(n)C}\right)^{j+i}\left(y_2^{(n)C}-x_2^{(n)C}\right)^{2+k-i}}{\left|\mathbf{x}^{(n)C}-\mathbf{y}^{(n)C}\right|^5}dS_{\mathbf{y}},$$

$$\mathbf{x}^{(n)C}\in S^{(n)C},\qquad k+j=0,1,2,\qquad i=0,1,2. \tag{16}$$

These integrals can be evaluated in close form by referring to the local polar coordinate systems with centers at the points $\mathbf{x}^{(n)I}$ and $\mathbf{x}^{(n)C}$, respectivelly, and subsequent integration by parts in the cases of $P_{jk}^{(n)}$ and $J_{jki}^{(n)}$. Their values are cited in Appendix C.

The last integrals in the right parts of Eqs. (15) are regular, what follows from the analysis of integrands in the limit $\mathbf{y}^{(n)I}\to\mathbf{x}^{(n)I}$ and $\mathbf{y}^{(n)C}\to\mathbf{x}^{(n)C}$. Therefore, their numerical integration can be carried out in ordinary sense along the domains $S_0^{(n)I}$ and $S_0^{(n)C}$, which are the domains $S^{(n)I}$ and $S^{(n)C}$ without the small domains around the points $\mathbf{x}^{(n)I}$ and $\mathbf{x}^{(n)C}$. In the domains $S_0^{(n)I}$ and $S_0^{(n)C}$ the source points and the integration points do not coincide.

The next step of regularization bases on the mapping of the domains $S^{(n)I}$ $\left(n=1,2,\ldots,N^I\right)$ into rectangular domains to avoid the contour root singularities. To this end the change of variables is applied in the form

$$x_1^{(n)I}=a^{(n)}\sin\tilde{x}_1^{(n)I}\cos\tilde{x}_2^{(n)I},$$

$$x_2^{(n)I}=a^{(n)}\sin\tilde{x}_1^{(n)I}\sin\tilde{x}_2^{(n)I},$$

$$y_1^{(n)I}=a^{(n)}\sin\tilde{y}_1^{(n)I}\cos\tilde{y}_2^{(n)I},$$

$$y_2^{(n)I}=a^{(n)}\sin\tilde{y}_1^{(n)I}\sin\tilde{y}_2^{(n)I},$$

$$n=1,2,\ldots n^I, \tag{17}$$

where $\tilde{\mathbf{x}}^{(n)I}\left(\tilde{x}_1^{(n)I},\tilde{x}_2^{(n)I}\right)$ and $\tilde{\mathbf{y}}^{(n)I}\left(\tilde{y}_1^{(n)I},\tilde{y}_2^{(n)I}\right)$ are new variables, which vary in the rectangular domain $\tilde{S}^{(n)I}\left\{0\le\tilde{x}_1^{(n)I},\tilde{y}_1^{(n)I}\le\pi/2;\quad 0\le\tilde{x}_2^{(n)I},\tilde{y}_2^{(n)I}\le 2\pi\right\}$. The Jacobian of the transformation (17) eliminates the singularities on the contours of the domains $S^{(n)I}$ in the integrands of Eqs. (12).

By combining relations (13)-(17) the following regular analogue of Eqs. (12) is obtained:

i) on the mapped inclusion domains as

$$\Big[(3-4\nu)M^{(n)*}\left(\tilde{\mathbf{x}}^{(n)I}\right)+\delta_{1j}I_2^{(n)*}\left(\tilde{\mathbf{x}}^{(n)I}\right)$$

$$+\delta_{2j}I_0^{(n)*}\left(\tilde{\mathbf{x}}^{(n)I}\right)\Big]\tilde{\alpha}_j^{(n)}\left(\tilde{\mathbf{x}}^{(n)I}\right)$$

$$+\left(\delta_{1j}+\delta_{2j}\right)I_1^{(n)*}\left(\tilde{\mathbf{x}}^{(n)I}\right)\tilde{\alpha}_{3-j}^{(n)}\left(\tilde{\mathbf{x}}^{(n)I}\right)$$

$$+(3-4\nu)\int\!\!\int_{\tilde{S}_0^{(n)I}}\frac{\sin\tilde{y}_1^{(n)I}}{d\left(\tilde{\mathbf{x}}^{(n)I},\tilde{\mathbf{y}}^{(n)I}\right)}\tilde{\alpha}_j^{(n)}\left(\tilde{\mathbf{y}}^{(n)I}\right)dS_{\tilde{\mathbf{y}}}$$

$$+\left(\delta_{1j}+\delta_{2j}\right)\int\!\!\int_{\tilde{S}_0^{(n)I}}\Big\{\left[f_j\left(\tilde{\mathbf{x}}^{(n)I},\tilde{\mathbf{y}}^{(n)I}\right)\right]^2\tilde{\alpha}_j^{(n)}\left(\tilde{\mathbf{y}}^{(n)I}\right)$$

$$+f_1\left(\tilde{\mathbf{x}}^{(n)I},\tilde{\mathbf{y}}^{(n)I}\right)f_2\left(\tilde{\mathbf{x}}^{(n)I},\tilde{\mathbf{y}}^{(n)I}\right)\tilde{\alpha}_{3-j}^{(n)}\left(\tilde{\mathbf{y}}^{(n)I}\right)\Big\}$$

$$\times \frac{\sin\tilde{y}_1^{(n)I}}{\left[d\left(\tilde{\mathbf{x}}^{(n)I},\tilde{\mathbf{y}}^{(n)I}\right)\right]^3} dS_{\tilde{\mathbf{y}}} + \sum_{k=1}^{N^I}(1-\delta_{kn})$$

$$\sum_{r=1}^{3}\int\int_{\tilde{S}^{(k)I}} \tilde{\alpha}_r^{(k)}\left(\tilde{\mathbf{y}}^{(k)I}\right)\tilde{R}_{jr}^{(kn)I}\left(\tilde{\mathbf{x}}^{(kn)I},\tilde{\mathbf{y}}^{(k)I}\right) dS_{\tilde{\mathbf{y}}}$$

$$+\sum_{k=1}^{N^C}\sum_{r=1}^{3}\int\int_{S^{(k)C}} \sqrt{\left(b^{(k)}\right)^2-\left(y_1^{(k)C}\right)^2-\left(y_2^{(k)C}\right)^2}$$

$$\beta_r^{(k)}\left(\mathbf{y}^{(k)C}\right)\tilde{L}_{jr}^{(kn)CI}\left(\tilde{\mathbf{x}}^{(kn)I},\tilde{\mathbf{y}}^{(k)I}\right) dS_{\mathbf{y}}$$

$$-4(1-\nu)G\left[U_j^{(n)}+\delta_{3j}a^{(n)}\sin\tilde{x}_1^{(n)I}(\Omega_1^{(n)}\sin\tilde{x}_2^{(n)I}\right.$$

$$-\Omega_2^{(n)}\cos\tilde{x}_2^{(n)I})+a^{(n)}\sin\tilde{x}_1^{(n)I}\Omega_3^{(n)}$$

$$\left.\times\left(\delta_{2j}\cos\tilde{x}_2^{(n)I}-\delta_{1j}\sin\tilde{x}_2^{(n)I}\right)\right]$$

$$=-4(1-\nu)G\tilde{u}_j^{(0)}\left(\tilde{\mathbf{x}}^{(n)I}\right),$$

$$\tilde{\mathbf{x}}^{(n)I}\in\tilde{S}^{(n)I},\quad j=1,2,3,\quad n=1,2,\ldots,N^I; \tag{18}$$

ii) on the crack domains as

$$[\delta_{3j}+(1-2\nu)(\delta_{1j}+\delta_{2j})]$$

$$\left[P_{00}^{(n)*}\left(\mathbf{x}^{(n)C}\right)\beta_j^{(n)}\left(\mathbf{x}^{(n)C}\right)+P_{10}^{(n)*}\left(\mathbf{x}^{(n)C}\right)\frac{\partial\beta_j^{(n)}\left(\mathbf{x}^{(n)C}\right)}{\partial x_1^{(n)C}}+\right.$$

$$+P_{01}^{(n)*}\left(x^{(n)C}\right)\frac{\partial\beta_j^{(n)}\left(\mathbf{x}^{(n)C}\right)}{\partial x_2^{(n)C}}$$

$$+\frac{1}{2}P_{20}^{(n)*}\left(\mathbf{x}^{(n)C}\right)\frac{\partial^2\beta_j^{(n)}\left(\mathbf{x}^{(n)C}\right)}{\partial x_1^{(n)C^2}}$$

$$+\frac{1}{2}P_{02}^{(n)*}\left(\mathbf{x}^{(n)C}\right)\frac{\partial^2\beta_j^{(n)}\left(\mathbf{x}^{(n)C}\right)}{\partial x_2^{(n)C^2}}+$$

$$\left.+P_{11}^{(n)*}\left(\mathbf{x}^{(n)C}\right)\frac{\partial^2\beta_j^{(n)}\left(\mathbf{x}^{(n)C}\right)}{\partial x_1^{(n)C}\partial x_2^{(n)C}}\right]$$

$$+3\nu\left\{\left[\delta_{1j}J_{002}^{(n)*}\left(\mathbf{x}^{(n)C}\right)\right.\right.$$

$$\left.+\delta_{2j}J_{000}^{(n)*}\left(\mathbf{x}^{(n)C}\right)\right]\beta_j^{(n)}\left(\mathbf{x}^{(n)C}\right)$$

$$+\left[\delta_{1j}J_{102}^{(n)*}\left(x^{(n)C}\right)+\delta_{2j}J_{100}^{(n)*}\left(\mathbf{x}^{(n)C}\right)\right]\frac{\partial\beta_j^{(n)}\left(\mathbf{x}^{(n)C}\right)}{\partial x_1^{(n)C}}$$

$$+\left[\delta_{1j}J_{012}^{(n)*}\left(\mathbf{x}^{(n)C}\right)+\delta_{2j}J_{010}^{(n)*}\left(\mathbf{x}^{(n)C}\right)\right]\frac{\partial\beta_j^{(n)}\left(\mathbf{x}^{(n)C}\right)}{\partial x_2^{(n)C}}$$

$$+\frac{1}{2}\left[\delta_{1j}J_{202}^{(n)*}\left(\mathbf{x}^{(n)C}\right)+\delta_{2j}J_{200}^{(n)*}\left(\mathbf{x}^{(n)C}\right)\right]\frac{\partial^2\beta_j^{(n)}\left(\mathbf{x}^{(n)C}\right)}{\partial x_1^{(n)C^2}}$$

$$+\frac{1}{2}\left[\delta_{1j}J_{022}^{(n)*}\left(\mathbf{x}^{(n)C}\right)+\delta_{2j}J_{020}^{(n)*}\left(\mathbf{x}^{(n)C}\right)\right]\frac{\partial^2\beta_j^{(n)}\left(\mathbf{x}^{(n)C}\right)}{\partial x_2^{(n)C^2}}+$$

$$\left.+\left[\delta_{1j}J_{112}^{(n)*}\left(x^{(n)C}\right)+\delta_{2j}J_{110}^{(n)*}\left(\mathbf{x}^{(n)C}\right)\right]\frac{\partial^2\beta_j^{(n)}\left(\mathbf{x}^{(n)C}\right)}{\partial x_1^{(n)C}\partial x_2^{(n)C}}\right\}$$

$$+3\nu(\delta_{1j}+\delta_{2j})\left[J_{001}^{(n)*}\left(\mathbf{x}^{(n)C}\right)\beta_{3-j}^{(n)}\left(\mathbf{x}^{(n)C}\right)+\right.$$

$$+J_{101}^{(n)*}\left(\mathbf{x}^{(n)C}\right)\frac{\partial\beta_{3-j}^{(n)}\left(\mathbf{x}^{(n)C}\right)}{\partial x_1^{(n)C}}+J_{011}^{(n)*}\left(\mathbf{x}^{(n)C}\right)\frac{\partial\beta_{3-j}^{(n)}\left(\mathbf{x}^{(n)C}\right)}{\partial x_2^{(n)C}}$$

$$+\frac{1}{2}J_{201}^{(n)*}\left(\mathbf{x}^{(n)C}\right)\frac{\partial^2\beta_{3-j}^{(n)}\left(\mathbf{x}^{(n)C}\right)}{\partial x_1^{(n)C^2}}+$$

$$+\frac{1}{2}J_{021}^{(n)*}\left(\mathbf{x}^{(n)C}\right)\frac{\partial^2\beta_{3-j}^{(n)}\left(\mathbf{x}^{(n)C}\right)}{\partial x_2^{(n)C^2}}$$

$$\left.+J_{111}^{(n)*}\left(\mathbf{x}^{(n)C}\right)\frac{\partial^2\beta_{3-j}^{(n)}\left(\mathbf{x}^{(n)C}\right)}{\partial x_1^{(n)C}\partial x_2^{(n)C}}\right]$$

$$+[\delta_{3j}+(1-2\nu)(\delta_{1j}+\delta_{2j})]$$

$$\int\int_{S_0^{(n)C}}\frac{\sqrt{\left(b^{(n)}\right)^2-\left(y_1^{(n)C}\right)^2-\left(y_2^{(n)C}\right)^2}}{\left|\mathbf{x}^{(n)C}-\mathbf{y}^{(n)C}\right|^3}\beta_j^{(n)}\left(\mathbf{y}^{(n)C}\right)dS_{\mathbf{y}}$$

$$+3\nu(\delta_{1j}+\delta_{2j})$$

$$\int\int_{S_0^{(n)C}}\sqrt{\left(b^{(n)}\right)^2-\left(y_1^{(n)C}\right)^2-\left(y_2^{(n)C}\right)^2}$$

$$\left[\frac{\left(x_j^{(n)C}-y_j^{(n)C}\right)^2}{\left|\mathbf{x}^{(n)C}-\mathbf{y}^{(n)C}\right|^2}\beta_j^{(n)}\left(\mathbf{y}^{(n)C}\right)+\right.$$

$$\left.+\frac{\left(x_1^{(n)C}-y_1^{(n)C}\right)\left(x_2^{(n)C}-y_2^{(n)C}\right)}{\left|\mathbf{x}^{(n)C}-\mathbf{y}^{(n)C}\right|^2}\beta_{3-j}^{(n)}\left(\mathbf{y}^{(n)C}\right)\right]$$

$$\frac{dS_y}{\left|\mathbf{x}^{(n)C}-\mathbf{y}^{(n)C}\right|^3}+\sum_{k=1}^{N^C}(1-\delta_{kn})$$

$$\times\sum_{r=1}^{3}\int\int_{S^{(k)C}}\sqrt{\left(b^{(k)}\right)^2-\left(y_1^{(k)C}\right)^2-\left(y_2^{(k)C}\right)^2}$$

$$\beta_r^{(k)}\left(\mathbf{y}^{(k)C}\right)R_{jr}^{(kn)C}\left(\mathbf{x}^{(kn)C},\mathbf{y}^{(k)C}\right)dS_{\mathbf{y}}$$

$$+\sum_{k=1}^{N^I}\sum_{r=1}^{3}\int\int_{\tilde{S}^{(k)I}}\tilde{\alpha}_r^{(k)}\left(\tilde{\mathbf{y}}^{(k)I}\right)\tilde{L}_{jr}^{(kn)IC}\left(\mathbf{x}^{(kn)IC},\tilde{\mathbf{y}}^{(k)I}\right)dS_{\tilde{\mathbf{y}}}$$

$$=-\tfrac{1-\nu}{G}\sigma_{j3}^{0}\left(\mathbf{x}^{(n)C}\right),$$

$$x^{(n)C}\in S^{(n)C},\quad j=1,2,3,\quad n=1,2,\ldots,N^C. \tag{19}$$

In Eqs. (18), (19) $\tilde{S}_0^{(n)I}$ is the mapping of domain $S_0^{(n)I}$ by the relations (17), where $\tilde{\mathbf{x}}^{(n)I}\neq\tilde{\mathbf{y}}^{(n)I}$, the composite functions of new arguments are introduced by

$$\tilde{\alpha}_j^{(n)I}\left(\tilde{\mathbf{x}}^{(n)I}\right)=\alpha_j^{(n)}\left(\mathbf{x}^{(n)I}\right)\Big|\begin{array}{l}x_1^{(n)I}=a^{(n)}\sin\tilde{x}_1^{(n)I}\cos\tilde{x}_2^{(n)I};\\ x_2^{(n)I}=a^{(n)}\sin\tilde{x}_1^{(n)I}\sin\tilde{x}_2^{(n)I},\end{array}$$

$$\tilde{u}_j^{(0)}\left(\tilde{\mathbf{x}}^{(n)I}\right)=u_j^{(0)}\left(\mathbf{x}^{(n)I}\right)\Big|\begin{array}{l}x_1^{(n)I}=a^{(n)}\sin\tilde{x}_1^{(n)I}\cos\tilde{x}_2^{(n)I};\\ x_2^{(n)I}=a^{(n)}\sin\tilde{x}_1^{(n)I}\sin\tilde{x}_2^{(n)I},\end{array}$$

$$\tilde{R}_{jr}^{(kn)I}\left(\tilde{\mathbf{x}}^{(kn)I},\tilde{\mathbf{y}}^{(k)I}\right)=a^{(k)}\sin\tilde{y}_1^{(k)I}\;R_{jr}^{(kn)I}\left(\mathbf{x}^{(kn)I},\mathbf{y}^{(k)I}\right)\Big|\begin{array}{l}x_1^{(n)I}=a^{(n)}\sin\tilde{x}_1^{(n)I}\cos\tilde{x}_2^{(n)I},\\ y_1^{(k)I}=a^{(k)}\sin\tilde{y}_1^{(k)I}\cos\tilde{y}_2^{(k)I};\\ x_2^{(n)I}=a^{(n)}\sin\tilde{x}_1^{(n)I}\sin\tilde{x}_2^{(n)I},\\ y_2^{(k)I}=a^{(k)}\sin\tilde{y}_1^{(k)I}\sin\tilde{y}_2^{(k)I},\end{array}$$

$$\tilde{L}_{jr}^{(kn)CI}\left(\tilde{\mathbf{x}}^{(kn)CI},\mathbf{y}^{(k)C}\right)=L_{jr}^{(kn)CI}\left(\mathbf{x}^{(kn)CI},\mathbf{y}^{(k)C}\right)\Big|\begin{array}{l}x_1^{(n)I}=a^{(n)}\sin\tilde{x}_1^{(n)I}\cos\tilde{x}_2^{(n)I};\\ x_2^{(n)I}=a^{(n)}\sin\tilde{x}_1^{(n)I}\sin\tilde{x}_2^{(n)I},\end{array}$$

$$\tilde{L}_{jr}^{(kn)IC}\left(\mathbf{x}^{(kn)IC},\tilde{\mathbf{y}}^{(k)I}\right)=a^{(k)}\sin\tilde{y}_1^{(k)I}\;L_{jr}^{(kn)IC}\left(\mathbf{x}^{(kn)IC},\mathbf{y}^{(k)I}\right)\Big|\begin{array}{l}y_1^{(k)I}=a^{(k)}\sin\tilde{y}_1^{(k)I}\cos\tilde{y}_2^{(k)I};\\ y_2^{(k)I}=a^{(k)}\sin\tilde{y}_1^{(k)I}\sin\tilde{y}_2^{(k)I}.\end{array} \tag{20}$$

Also the following denotations are used

$$d\left(\tilde{\mathbf{x}}^{(n)I},\tilde{\mathbf{y}}^{(n)I}\right)=\Big[\sin^2\tilde{x}_1^{(n)I}+\sin^2\tilde{y}_1^{(n)I}-2\sin\tilde{x}_1^{(n)I}\sin\tilde{y}_1^{(n)I}\cos\left(\tilde{x}_2^{(n)I}-\tilde{y}_2^{(n)I}\right)\Big]^{\frac{1}{2}},$$

$$f_1\left(\tilde{\mathbf{x}}^{(n)I},\tilde{\mathbf{y}}^{(n)I}\right)=\sin\tilde{x}_1^{(n)I}\cos\tilde{x}_2^{(n)I}-\sin\tilde{y}_1^{(n)I}\cos\tilde{y}_2^{(n)I},$$

$$f_2\left(\tilde{\mathbf{x}}^{(n)I},\tilde{\mathbf{y}}^{(n)I}\right)=\sin\tilde{x}_1^{(n)I}\sin\tilde{x}_2^{(n)I}-\sin\tilde{y}_1^{(n)I}\sin\tilde{y}_2^{(n)I},$$

$$M^{(n)*}\left(\tilde{\mathbf{x}}^{(n)I}\right)=M^{(n)}\left(\mathbf{x}^{(n)I}\right)\Big|\begin{array}{l}x_1^{(n)I}=a^{(n)}\sin\tilde{x}_1^{(n)I}\cos\tilde{x}_2^{(n)I};\\ x_2^{(n)I}=a^{(n)}\sin\tilde{x}_1^{(n)I}\sin\tilde{x}_2^{(n)I},\end{array}-\int\int_{\tilde{S}_0^{(n)I}}\frac{\sin\tilde{y}_1^{(n)I}}{d\left(\tilde{\mathbf{x}}^{(n)I},\tilde{\mathbf{y}}^{(n)I}\right)}dS_{\tilde{\mathbf{y}}},$$

$$I_i^{(n)*}\left(\tilde{\mathbf{x}}^{(n)I}\right)=I_i^{(n)}\left(\mathbf{x}^{(n)I}\right)\Big|\begin{array}{l}x_1^{(n)I}=a^{(n)}\sin\tilde{x}_1^{(n)I}\cos\tilde{x}_2^{(n)I};\\ x_2^{(n)I}=a^{(n)}\sin\tilde{x}_1^{(n)I}\sin\tilde{x}_2^{(n)I},\end{array}-\int\int_{\tilde{S}_0^{(n)I}}\frac{\sin\tilde{\mathbf{y}}_1^{(n)I}\left[f_1\left(\tilde{\mathbf{x}}^{(n)I},\tilde{\mathbf{y}}^{(n)I}\right)\right]^i\left[f_2\left(\tilde{\mathbf{x}}^{(n)I},\tilde{\mathbf{y}}^{(n)I}\right)\right]^{2-i}}{\left[d\left(\tilde{\mathbf{x}}^{(n)I},\tilde{\mathbf{y}}^{(n)I}\right)\right]^3}dS_{\tilde{\mathbf{y}}},$$

$$i=0,1,2,$$

$$P_{jk}^{(n)*}\left(\mathbf{x}^{(n)C}\right)=P_{jk}^{(n)}\left(\mathbf{x}^{(n)C}\right)-\int\int_{S_0^{(n)C}}\sqrt{\left(b^{(n)}\right)^2-\left(y_1^{(n)C}\right)^2-\left(y_2^{(n)C}\right)^2}\;\frac{\left(y_1^{(n)C}-x_1^{(n)C}\right)^j\left(y_2^{(n)C}-x_2^{(n)C}\right)^k}{\left|\mathbf{x}^{(n)C}-\mathbf{y}^{(n)C}\right|^3}dS_{\mathbf{y}},$$

$$k+j=0,1,2,$$

$$J_{jki}^{(n)*}\left(\mathbf{x}^{(n)C}\right)=J_{jki}^{(n)}\left(\mathbf{x}^{(n)C}\right)-\int\int_{S_0^{(n)C}}\sqrt{\left(b^{(n)}\right)^2-\left(y_1^{(n)C}\right)^2-\left(y_2^{(n)C}\right)^2}\;\frac{\left(y_1^{(n)C}-x_1^{(n)C}\right)^{j+i}\left(y_2^{(n)C}-x_2^{(n)C}\right)^{2+k-i}}{\left|\mathbf{x}^{(n)C}-\mathbf{y}^{(n)C}\right|^5}dS_{\mathbf{y}},$$

$$k+j=0,1,2,\quad i=0,1,2. \tag{21}$$

In accordance to the transformations (14) and (17), the Eqs. (13) of inclusions equilibrium are written in the form

$$\iint\limits_{\tilde{S}^{(n)I}} \sin\tilde{y}_1^{(n)I}\tilde{\alpha}_j^{(n)}\left(\tilde{\mathbf{y}}^{(n)I}\right)dS_{\tilde{\mathbf{y}}}=0,$$
$$j=1,2,3,\quad n=1,2,\ldots,N^I,$$

$$\iint\limits_{\tilde{S}^{(n)I}} \sin^2\tilde{y}_1^{(n)I}\left(\delta_{1j}\sin\tilde{y}_2^{(n)I}+\delta_{2j}\cos\tilde{y}_2^{(n)I}\right)\tilde{\alpha}_3^{(n)}\left(\tilde{\mathbf{y}}^{(n)I}\right)dS_{\tilde{\mathbf{y}}}=0,$$
$$j=1,2,\quad n=1,2,\ldots,N^I,$$

$$\iint\limits_{\tilde{S}^{(n)I}} \sin^2\tilde{y}_1^{(n)I}\left[\sin\tilde{y}_2^{(n)I}\tilde{\alpha}_1^{(n)}\left(\tilde{\mathbf{y}}^{(n)I}\right)-\cos\tilde{y}_2^{(n)I}\tilde{\alpha}_2^{(n)}\left(\tilde{\mathbf{y}}^{(n)I}\right)\right]dS_{\tilde{\mathbf{y}}}=0, \tag{22}$$
$$n=1,2,\ldots,N^I.$$

The regularity of the kernels in the completed system of Eqs. (18), (19), (22) provide the boundary element parametrization of problem by the following approach. Each rectangular domain $\tilde{S}^{(n)I}$ $\left(n=1,2,\ldots,N^I\right)$ is uniformly divided into $Q^{(n)I}$ rectangular elements $\tilde{S}_q^{(n)I}$ $\left(\tilde{S}^{(n)I}=\bigcup\limits_{q=1}^{Q^{(n)I}}\tilde{S}_q^{(n)I}\right)$ and each circular domain $S^{(n)C}$ $\left(n=1,2,\ldots,N^C\right)$ is divided into $Q^{(n)C}$ quadrilateral elements $S_q^{(n)C}$ $\left(S^{(n)C}=\bigcup\limits_{q=1}^{Q^{(n)C}}S_q^{(n)C}\right)$ of equal length in the polar coordinate directions. Then the unknown functions $\tilde{\alpha}_j^{(n)}$ and $\beta_j^{(n)}$ are approximated by the interpolation formulas

$$\tilde{\alpha}_j^{(n)}\left(\tilde{\mathbf{x}}^{(n)I}\right)=\sum_{q=1}^{Q^{(n)I}}\tilde{\alpha}_{jq}^{(n)}\theta_q^{(n)I}\left(\tilde{\mathbf{x}}^{(n)I}\right),$$
$$\tilde{\mathbf{x}}^{(n)I}\in S^{(n)I},\quad j=1,2,3,\quad n=1,2,\ldots,N^I,$$

$$\beta^{(n)}\left(\mathbf{x}^{(n)C}\right)=\sum_{q=1}^{Q^{(n)C}}\beta_{jq}^{(n)}\theta_q^{(n)C}\left(\mathbf{x}^{(n)C}\right),$$
$$\mathbf{x}^{(n)C}\in S^{(n)C},\quad j=1,2,3,\quad n=1,2,\ldots,N^C, \tag{23}$$

in which $\tilde{\alpha}_{jq}^{(n)}=\tilde{\alpha}_j^{(n)}\left(\tilde{\mathbf{x}}_q^{(n)I}\right)$ and $\beta_{jq}^{(n)}=\beta_j^{(n)}\left(\mathbf{x}_q^{(n)C}\right)$ represent the unknown functions at the centroids $\tilde{\mathbf{x}}_q^{(n)I}\left(\tilde{x}_{1q}^{(n)I},\tilde{x}_{2q}^{(n)I}\right)$ of the qth element $\tilde{S}_q^{(n)I}$ on the nth inclusion image domain and $\mathbf{x}_q^{(n)C}\left(x_{1q}^{(n)C},x_{2q}^{(n)C}\right)$ of the qth element $S_q^{(n)C}$ on the nth crack domain, respectively. The shape functions $\theta_q^{(n)I}$ and $\theta_q^{(n)C}$ have the properties $\theta_q^{(n)I}\left(\tilde{\mathbf{x}}_i^{(n)I}\right)=\theta_q^{(n)C}\left(\mathbf{x}_i^{(n)C}\right)=\delta_{qi}$.

Substituting Eqs. (23) into Eqs. (18) and (19) considered at collocation points $\tilde{\mathbf{x}}_i^{(n)I}\left(i=1,2,\ldots,Q^{(n)I}\right)$ and $\mathbf{x}_i^{(n)C}\left(i=1,2,\ldots,Q^{(n)C}\right)$, respectively, and into Eqs. (22) we arrive at a system of $3\left(\sum\limits_{k=1}^{N^I}Q^{(k)I}+\sum\limits_{k=1}^{N^C}Q^{(k)C}+2N^I\right)$ linear algebraic equations relative to the values of functions $\tilde{\alpha}_j^{(n)}$ and $\beta_j^{(n)}$ in nodal points

$$\sum_{r=1}^{3}\sum_{q=1}^{Q^{(n)I}}g_{jirq}^{(n)I}\tilde{\alpha}_{rq}^{(n)}+\sum_{k=1}^{N^I}\left(1-\delta_{kn}\right)\sum_{r=1}^{3}\sum_{q=1}^{Q^{(k)I}}e_{jirq}^{(kn)I}\tilde{\alpha}_{rq}^{(k)}+\sum_{k=1}^{N^C}\sum_{r=1}^{3}\sum_{q=1}^{Q^{(k)C}}e_{jirq}^{(kn)CI}\beta_{rq}^{(k)}-4\left(1-\nu\right)G\left[U_j^{(n)}+\sum_{r=1}^{3}c_{jir}^{(n)}\Omega_r^{(n)}\right]=-4\left(1-\nu\right)G\tilde{u}_j^{(0)}\left(\tilde{\mathbf{x}}_i^{(n)I}\right),$$
$$j=1,2,3,\quad i=1,2,\ldots,Q^{(n)I},\ n=1,2,\ldots,N^I,$$

$$\sum_{r=1}^{3}\sum_{q=1}^{Q^{(n)C}}g_{jirq}^{(n)C}\beta_{rq}^{(n)}+\sum_{k=1}^{N^C}\left(1-\delta_{kn}\right)\sum_{r=1}^{3}\sum_{q=1}^{Q^{(k)C}}e_{jirq}^{(kn)C}\beta_{rq}^{(k)}+\sum_{k=1}^{N^I}\sum_{r=1}^{3}\sum_{q=1}^{Q^{(k)I}}e_{jirq}^{(kn)IC}\tilde{\alpha}_{rq}^{(k)}=-\tfrac{1-\nu}{G}\sigma_{j3}^{0}\left(\mathbf{x}_i^{(n)C}\right),$$
$$j=1,2,3,\quad i=1,2,\ldots,Q^{(n)C},\quad n=1,2,\ldots,N^C,$$

$$\sum_{q=1}^{Q^{(n)I}}h_q^{(n)}\tilde{\alpha}_{jq}^{(n)}=0,\quad j=1,2,3,\quad n=1,2,\ldots,N^I,$$

$$\sum_{r=1}^{3}\sum_{q=1}^{Q^{(n)I}}h_{jrq}^{(n)}\tilde{\alpha}_{rq}^{(n)}=0,\quad j=1,2,3,\quad n=1,2,\ldots,N^I. \tag{24}$$

The formulas for the determination of coefficients $g_{jirq}^{(n)I}, g_{jirq}^{(n)C}, e_{jirq}^{(kn)I}, e_{jirq}^{(kn)C}, e_{jirq}^{(kn)CI}, e_{jirq}^{(kn)IC}, c_{jir}^{(n)}, h_q^{(n)}, h_{jrq}^{(n)}$

in terms of shape functions $\theta_q^{(n)I}$ and $\theta_q^{(n)C}$ and their derivatives are given in Appendix D. Under subsequent calculations of these coefficients the constant shape functions are assumed, i.e.

$$\theta_q^{(n)I}\left(\tilde{\mathbf{y}}^{(n)I}\right) = \begin{cases} 1, \ \tilde{\mathbf{y}}^{(n)I} \in \tilde{S}_q^{(n)I}; \\ 0, \ \tilde{\mathbf{y}}^{(n)I} \notin \tilde{S}_q^{(n)I} \end{cases}$$
$$\theta_q^{(n)C}\left(\mathbf{y}^{(n)C}\right) = \begin{cases} 1, \ \mathbf{y}^{(n)C} \in \tilde{S}_q^{(n)C}; \\ 0, \ \mathbf{y}^{(n)C} \notin \tilde{S}_q^{(n)C} \end{cases} \tag{25}$$

and the finite difference schemes to approximate the derivatives are used.

4 Numerical analysis of stress intensity factors

Once functions $\tilde{\alpha}_j^{(n)}$ and $\beta_j^{(n)}$ are defined, the displacements, strains and stresses at any arbitrary internal point of the solid can be estimated by use of the representation formulas (5), (6), (14). The main interest should be directed to the I-, II-, III-mode stress intensity factors $K_I^{(n)C}$, $K_{II}^{(n)C}$ and $K_{III}^{(n)C}$ $\quad (n = 1, 2, \ldots, N^C)$ as fracture parameters. The preservation of accuracy under calculation of these quantities is provided by their simple dependencies on the solutions of resulting BIE (see Aliabadi and Rooke, 1991):

$$K_I^{(n)C}(\varphi) = -\frac{2\pi G\sqrt{\pi a^{(n)C}}}{1-\nu}\beta_3^{(n)}\left(\mathbf{x}^{(n)C}\right)\Big|_{\left|\mathbf{x}^{(n)C}\right| = a^{(n)C}},$$

$$K_I^{(n)C}(\varphi) = -\frac{2\pi G\sqrt{\pi a^{(n)C}}}{1-\nu}\left[\beta_1^{(n)}\left(\mathbf{x}^{(n)C}\right)\Big|_{\left|\mathbf{x}^{(n)C}\right| = a^{(n)C}}\cos\varphi + \beta_2^{(n)}\left(\mathbf{x}^{(n)C}\right)\Big|_{\left|\mathbf{x}^{(n)C}\right| = a^{(n)C}}\sin\varphi\right],$$

$$K_I^{(n)C}(\varphi) = -\frac{2\pi G\sqrt{\pi a^{(n)C}}}{1-\nu}\left[\beta_1^{(n)}\left(\mathbf{x}^{(n)C}\right)\Big|_{\left|\mathbf{x}^{(n)C}\right| = a^{(n)C}}\sin\varphi - \beta_2^{(n)}\left(\mathbf{x}^{(n)C}\right)\Big|_{\left|\mathbf{x}^{(n)C}\right| = a^{(n)C}}\cos\varphi\right], \tag{26}$$

where φ is the angular coordinate of the nth crack front point.

Under calculation 161 boundary elements are used on each involved inhomogeneities. The Poisson's ratio was chosen as $\nu = 0.3$.

Testing accuracy and stability of proposed numerical algorithm is fulfilled for the single inclusion and single crack in the homogeneous field of tension and shear stresses, when the exact analytical solutions can be taken from Kassir and Sih, 1976; Kit and Khaj, 1989. An excellent agreement is fixed in these cases (the deviations of numerical and analytical values of functions $\tilde{\alpha}_j^{(1)}$ and $\beta_j^{(1)}$ are less than 0.1%).

Novel numerical results have been obtained for the inclined circular inclusion of radius a^I and crack of radius a^C in a matrix subjected to uniaxial tension $\sigma^0 = \text{const}$ at infinity normal to the crack plane. The disposition of inhomogeneities is characterized by that the center of inclusion lies in the crack plane on the distance d from the center of crack, the section line of inclusion and crack planes is parallel to the tangent to the crack front, then the inclination of inhomogeneities can be given by the angle γ between these planes. With the purpose of comparative analysis the following normalized mixed mode stress intensity factors in the vicinity of crack are introduced:

$$\overline{K}_I^C = K_I^{(1)C} \Big/ K_*$$
$$\overline{K}_{\mathrm{II}}^C = K_{\mathrm{II}}^{(1)C} \Big/ K_*$$
$$\overline{K}_{\mathrm{III}}^C = K_{\mathrm{III}}^{(1)C} \Big/ K_*, \tag{27}$$

where $K_* = 2\sigma^0\sqrt{a^C/\pi}$ is the I-mode stress intensity factor for a single crack under the same loading of solid.

The Figs. 2-4 show, that the influence of inclusion on the crack depends essentially on their mutual location. So, the interaction of coplanar inclusion and crack ($\gamma = 0$) causes the increase of the stress concentration in comparison with a single crack case ($\overline{K}_I^C > 1$, Fig. 2). The inclined inclusion ($\gamma \neq 0$) plays the role of reinforcement as to the I-mode stress intensity factor near the crack ($\overline{K}_I^C < 1$, Fig. 2). The II-mode and III-mode stress intensity factors demonstrate the complex distribution along the front of defect with change of their sign and front points with peak values of these quantities (Figs. 3, 4). The maximal reinforcing effect is reached in the case

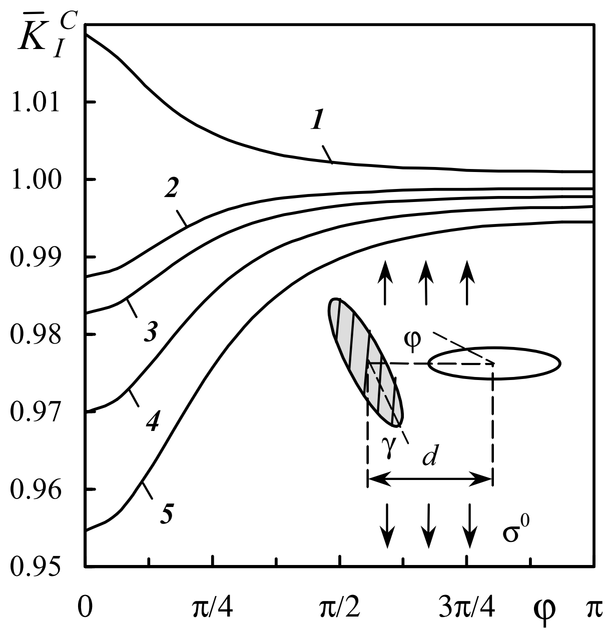

Figure 2 : I-mode stress intensity factor $\overline{K}_I^C$ versus angular coordinate φ for the inclined interacting inclusion and crack ($a^I = a^C$, $d = 2.1a^C$, $1 - \gamma = 0$, $2 - \gamma = \pi/6$, $3 - \gamma = \pi/4$, $4 - \gamma = \pi/3$, $5 - \gamma = \pi/2$)

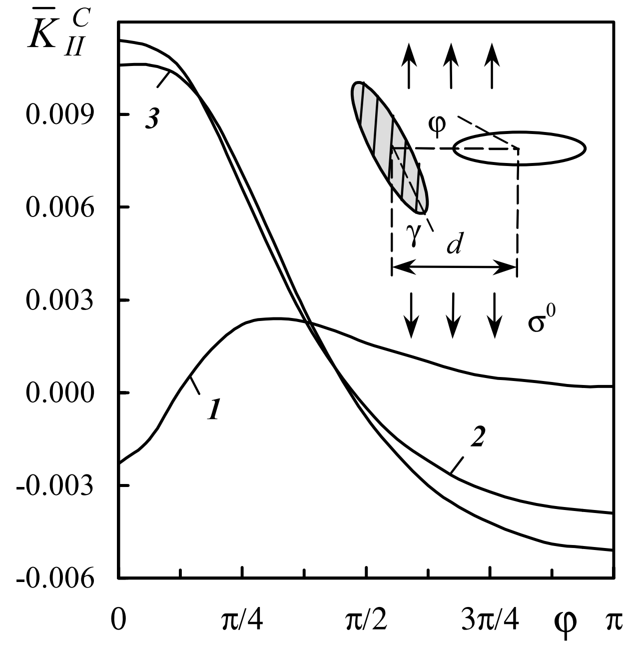

Figure 3 : II-mode stress intensity factor $\overline{K}_{II}^C$ versus angular coordinate φ for the inclined interacting inclusion and crack ($a^I = a^C$, $d = 2.1a^C$, $1 - \gamma = \pi/6$, $2 - \gamma = \pi/4$, $3 - \gamma = \pi/3$)

of perpendicular inclusion and crack ($\gamma = \pi/2$), when $\overline{K}_I^C < 1$, $\overline{K}_{\text{II}}^C = \overline{K}_{\text{III}}^C = 0$. This effect is more expressed in the cases of close inhomogeneities (Fig. 5) or the neighborhood of inhomogeneities with contrast sizes (Fig. 6).

5 Conclusions

The numerical tool, based on the advanced BIE method formulations, is developed and applied for the solution of three-dimensional elastostatic problems for remotely loaded matrix with the finite number interacting disk-inclusions and cracks. Proposed approach is not sensitive to the geometry parameters of inhomogeneities mutual orientations and sizes, also to the loading conditions. The above investigations open the possibilities for macrodescription of large number of thin-walled inclusions and cracks by the BIE method in conjunction with the effective field method, developed for the multiple inclusion model by Buryachenko, 2000 and for the multiple crack model by Kachanov, 1991.

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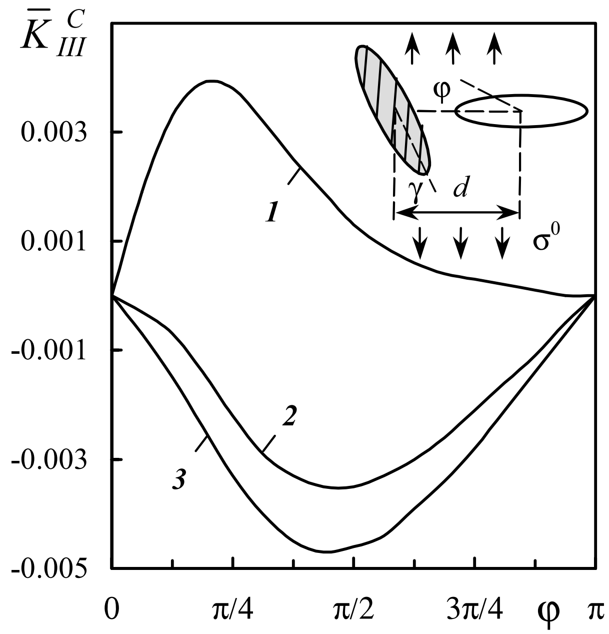

Figure 4 : III-mode stress intensity factor $\overline{K}_{III}^{C}$ versus angular coordinate φ for the inclined interacting inclusion and crack ($a^I = a^C$, $d = 2.1a^C$, $1-\gamma = \pi/6$, $2-\gamma = \pi/4$, $3-\gamma = \pi/3$)

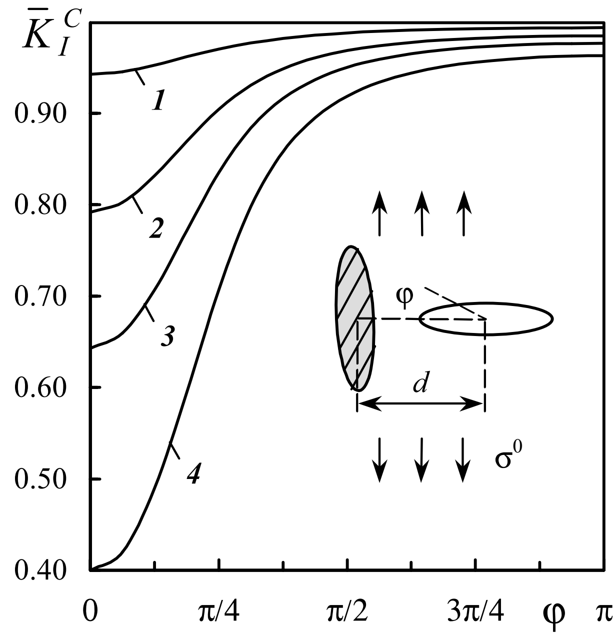

Figure 5 : I-mode stress intensity factor $\overline{K}_{I}^{C}$ versus angular coordinate φ for the perpendicular interacting inclusion and crack ($a^I = a^C$, $1-d = 2a^C$, $2-d = 1.5a^C$, $3-d = 1.3a^C$, $4-d = 1.1a^C$)

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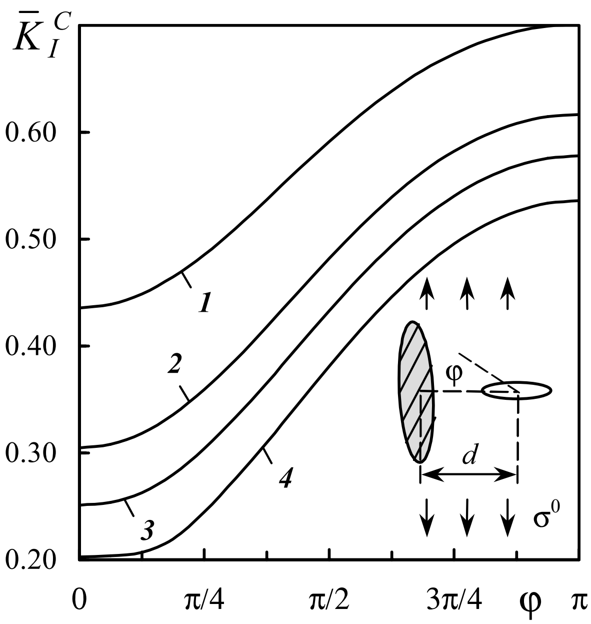

Figure 6 : I-mode stress intensity factor $\overline{K}_I^C$ versus angular coordinate φ for the perpendicular interacting macroinclusion and crack ($a^I = 10a^C,\ 1-d = 2a^C,\ 2-d = 1.5a^C,\ 3-d = 1.3a^C,\ 4-d = 1.1a^C$)

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Appendix A: The direct cosines between the axes of coordinate systems

	$O^{(n)I}x_1^{(n)I}$	$O^{(n)I}x_2^{(n)I}$	$O^{(n)I}x_3^{(n)I}$
$O^{(k)I}x_1^{(k)I}$	$l_1^{(kn)I}$	$l_2^{(kn)I}$	$l_3^{(kn)I}$
$O^{(k)I}x_2^{(k)I}$	$m_1^{(kn)I}$	$m_2^{(kn)I}$	$m_3^{(kn)I}$
$O^{(k)I}x_3^{(k)I}$	$p_1^{(kn)I}$	$p_2^{(kn)I}$	$p_3^{(kn)I}$

	$O^{(n)C}x_1^{(n)C}$	$O^{(n)C}x_2^{(n)C}$	$O^{(n)C}x_3^{(n)C}$
$O^{(k)I}x_1^{(k)I}$	$l_1^{(kn)IC}$	$l_2^{(kn)IC}$	$l_3^{(kn)IC}$
$O^{(k)I}x_2^{(k)I}$	$m_1^{(kn)IC}$	$m_2^{(kn)IC}$	$m_3^{(kn)IC}$
$O^{(k)I}x_3^{(k)I}$	$p_1^{(kn)IC}$	$p_2^{(kn)IC}$	$p_3^{(kn)IC}$

	$O^{(n)I}x_1^{(n)I}$	$O^{(n)I}x_2^{(n)I}$	$O^{(n)I}x_3^{(n)I}$
$O^{(k)C}x_1^{(k)C}$	$l_1^{(kn)CI}$	$l_2^{(kn)CI}$	$l_3^{(kn)CI}$
$O^{(k)C}x_2^{(k)C}$	$m_1^{(kn)CI}$	$m_2^{(kn)CI}$	$m_3^{(kn)CI}$
$O^{(k)C}x_3^{(k)C}$	$p_1^{(kn)CI}$	$p_2^{(kn)CI}$	$p_3^{(kn)CI}$

	$O^{(n)C}x_1^{(n)C}$	$O^{(n)C}x_2^{(n)C}$	$O^{(n)C}x_3^{(n)C}$
$O^{(k)C}x_1^{(k)C}$	$l_1^{(kn)C}$	$l_2^{(kn)C}$	$l_3^{(kn)C}$
$O^{(k)C}x_2^{(k)C}$	$m_1^{(kn)C}$	$m_2^{(kn)C}$	$m_3^{(kn)C}$
$O^{(k)C}x_3^{(k)C}$	$p_1^{(kn)C}$	$p_2^{(kn)C}$	$p_3^{(kn)C}$

Appendix B: Regular kernels of the BIE for the description of inclusion-crack interaction

$$R_{jr}^{(kn)I}\left(\mathbf{x}^{(kn)I},\mathbf{y}^{(k)I}\right) = (3-4\nu)$$

$$\left(l_j^{(kn)I}\delta_{1r} + m_j^{(kn)I}\delta_{2r} + p_j^{(kn)I}\delta_{3r}\right)\frac{1}{\left|\mathbf{x}^{(kn)I}-\mathbf{y}^{(k)I}\right|}$$

$$+\left[l_j^{(kn)I}\left(x_1^{(kn)I}-y_1^{(k)I}\right) + m_j^{(kn)I}\left(x_2^{(kn)I}-y_2^{(k)I}\right)\right.$$

$$\left.+p_j^{(kn)I}x_3^{(kn)I}\right]$$

$$\times\left[\delta_{1r}\left(x_1^{(kn)I}-y_1^{(k)I}\right)+\delta_{2r}\left(x_2^{(kn)I}-y_2^{(k)I}\right)\right.$$
$$\left.+\delta_{3r}x_3^{(kn)I}\right]\frac{1}{\left|\mathbf{x}^{(kn)I}-\mathbf{y}^{(k)I}\right|^3},$$

$$L_{jr}^{(kn)CI}\left(\mathbf{x}^{(kn)CI},\mathbf{y}^{(k)C}\right)=$$
$$-2(1-2\nu)\left[\left(p_j^{(kn)CI}\delta_{1r}-l_j^{(kn)CI}\delta_{3r}\right)\left(x_1^{(kn)CI}-y_1^{(k)C}\right)\right.$$
$$+\left(p_j^{(kn)CI}\delta_{2r}-m_j^{(kn)CI}\delta_{3r}\right)\left(x_2^{(kn)CI}-y_2^{(k)C}\right)$$
$$\left.+\left(l_j^{(kn)CI}\delta_{1r}+m_j^{(kn)CI}\delta_{2r}+p_j^{(kn)CI}\delta_{3r}\right)x_3^{(kn)CI}\right]$$
$$\times\frac{1}{\left|\mathbf{x}^{(kn)CI}-\mathbf{y}^{(k)C}\right|^3}$$
$$-6x_3^{(kn)CI}\left[l_j^{(kn)CI}\left(x_1^{(kn)CI}-y_1^{(k)C}\right)\right.$$
$$\left.+m_j^{(kn)CI}\left(x_2^{(kn)CI}-y_2^{(k)C}\right)+p_j^{(kn)CI}x_3^{(kn)CI}\right]$$
$$\times\left[\delta_{1r}\left(x_1^{(kn)CI}-y_1^{(k)C}\right)+\delta_{2r}\left(x_2^{(kn)CI}-y_2^{(k)C}\right)\right.$$
$$\left.+\delta_{3r}x_3^{(kn)CI}\right]\frac{1}{\left|\mathbf{x}^{(kn)CI}-\mathbf{y}^{(k)C}\right|^5},$$

$$R_{jr}^{(kn)IC}\left(\mathbf{x}^{(kn)C},\mathbf{y}^{(k)C}\right)$$
$$=\left\{\left[(1-4\nu)\left(l_{j3}^{(kn)C}+m_{j3}^{(kn)C}\right)-p_{j3}^{(kn)C}\right]\delta_{3r}\right.$$
$$\left.-(1+\nu)\left(p_{j3*}^{(kn)C}\delta_{1r}+m_{j3*}^{(kn)C}\delta_{2r}\right)\right\}$$
$$\times\frac{1}{\left|\mathbf{x}^{(kn)C}-\mathbf{y}^{(k)C}\right|^3}$$
$$+3\left\{\left[\nu m_{j3*}^{(kn)C}\delta_{2r}-(1-2\nu)l_{j3}^{(kn)C}\delta_{3r}\right]\left(x_1^{(kn)C}-y_1^{(k)C}\right)^2\right.$$
$$+\left[\nu p_{j3*}^{(kn)C}\delta_{1r}-(1-2\nu)m_{j3}^{(kn)C}\delta_{3r}\right]\left(x_2^{(kn)C}-y_2^{(k)C}\right)^2$$
$$-\left[(1-2\nu)\left(l_{j3}^{(kn)C}+m_{j3}^{(kn)C}\right)+2p_{j3}^{(kn)C}\right]\delta_{3r}x_3^{(kn)C^2}$$
$$-\left[\nu\left(m_{j3*}^{(kn)C}\delta_{1r}+p_{j3*}^{(kn)C}\delta_{2r}\right)+(1-2\nu)l_{j3*}^{(kn)C}\delta_{3r}\right]$$
$$\times\left(x_1^{(kn)C}-y_1^{(k)C}\right)\left(x_2^{(kn)C}-y_2^{(k)C}\right)$$
$$-\left[\nu l_{j3*}^{(kn)C}\delta_{1r}+\left((1-2\nu)l_{j3}^{(kn)C}+m_{j3}^{(kn)C}+p_{j3}^{(kn)C}\right)\delta_{2r}\right.$$
$$\left.+m_{j3*}^{(kn)C}\delta_{3r}\right]\times\left(x_2^{(kn)C}-y_2^{(k)C}\right)x_3^{(kn)C}$$
$$-\left[\left(l_{j3}^{(kn)C}+(1-2\nu)m_{j3}^{(kn)C}+p_{j3}^{(kn)C}\right)\delta_{1r}+\nu l_{j3*}^{(kn)C}\delta_{2r}\right.$$
$$\left.\left.+p_{j3*}^{(kn)C}\delta_{3r}\right]\times\left(x_1^{(kn)C}-y_1^{(k)C}\right)x_3^{(kn)C}\right\}\frac{1}{\left|\mathbf{x}^{(kn)C}-\mathbf{y}^{(k)C}\right|^5}$$
$$+15\left[l_{j3}^{(kn)C}\left(x_1^{(kn)C}-y_1^{(k)C}\right)^2\right.$$
$$+m_{j3}^{(kn)C}\left(x_2^{(kn)C}-y_2^{(k)C}\right)^2+p_{j3}^{(kn)C}\left(x_3^{(kn)C}\right)^2$$
$$+l_{j3*}^{(kn)C}\left(x_1^{(kn)C}-y_1^{(k)C}\right)\left(x_2^{(kn)C}-y_2^{(k)C}\right)$$
$$+m_{j3*}^{(kn)C}\left(x_2^{(kn)C}-y_2^{(k)C}\right)x_3^{(kn)C}$$
$$\left.+p_{j3*}^{(kn)C}\left(x_1^{(kn)C}-y_1^{(k)C}\right)x_3^{(kn)C}\right]\frac{x_3^{(kn)C}}{\left|\mathbf{x}^{(kn)C}-\mathbf{y}^{(k)C}\right|^7},$$

$$L_{jr}^{(kn)IC}\left(\mathbf{x}^{(kn)IC},\mathbf{y}^{(k)I}\right)=$$
$$-2\left\{\left[\left(2\nu\delta_{3j}+(1-4\nu)l_{j3}^{(kn)IC}\right.\right.\right.$$
$$\left.-m_{j3}^{(kn)IC}-p_{j3}^{(kn)IC}\right)\delta_{1r}$$
$$\left.+(1-2\nu)\left(l_{j3*}^{(kn)IC}\delta_{2r}+p_{j3*}^{(kn)IC}\delta_{3r}\right)\right]$$
$$\times\left(x_1^{(kn)IC}-y_1^{(k)I}\right)$$
$$+\left[\left(2\nu\delta_{3j}+(1-4\nu)m_{j3}^{(kn)IC}-l_{j3}^{(kn)IC}-p_{j3}^{(kn)IC}\right)\delta_{2r}\right.$$
$$\left.+(1-2\nu)\left(l_{j3*}^{(kn)IC}\delta_{1r}+m_{j3*}^{(kn)IC}\delta_{3r}\right)\right]\left(x_2^{(kn)IC}-y_2^{(k)I}\right)$$
$$+\left[\left(2\nu\delta_{3j}+(1-4\nu)p_{j3}^{(kn)IC}\right.\right.$$
$$\left.-l_{j3}^{(kn)IC}-m_{j3}^{(kn)IC}\right)\delta_{3r}$$
$$\left.\left.+(1-2\nu)\left(p_{j3*}^{(kn)IC}\delta_{1r}+m_{j3*}^{(kn)IC}\delta_{2r}\right)\right]x_3^{(kn)IC}\right\}$$

$$\times \frac{1}{\left|\mathbf{x}^{(kn)IC}-\mathbf{y}^{(k)I}\right|^3}$$

$$-6\left[l_{j3}^{(kn)IC}\left(x_1^{(kn)IC}-y_1^{(k)I}\right)^2\right.$$

$$+m_{j3}^{(kn)IC}\left(x_2^{(kn)IC}-y_2^{(k)I}\right)^2$$

$$+p_{j3}^{(kn)IC}x_3^{(kn)IC2}$$

$$+l_{j3*}^{(kn)IC}\left(x_1^{(kn)IC}-y_1^{(k)I}\right)\left(x_2^{(kn)IC}-y_2^{(k)I}\right)$$

$$+m_{j3*}^{(kn)IC}\left(x_2^{(kn)IC}-y_2^{(k)I}\right)x_3^{(kn)IC}$$

$$\left.+p_{j3*}^{(kn)IC}\left(x_1^{(kn)IC}-y_1^{(k)I}\right)x_3^{(kn)IC}\right]$$

$$\times\left[\delta_{1r}\left(x_1^{(kn)IC}-y_1^{(k)I}\right)\right.$$

$$\left.+\delta_{2r}\left(x_2^{(kn)IC}-y_2^{(k)I}\right)+\delta_{3r}x_3^{(kn)IC}\right]$$

$$\times \frac{1}{\left|\mathbf{x}^{(kn)IC}-\mathbf{y}^{(k)I}\right|^5}$$

$$j,r=1,2,3. \tag{28}$$

Appendix C: Values of the regularizing integrals

$$M^{(n)}\left(\mathbf{x}^{(n)I}\right)=\pi^2,$$

$$I_0^{(n)}\left(\mathbf{x}^{(n)I}\right)=\frac{\pi^2}{2},$$

$$I_1^{(n)}\left(\mathbf{x}^{(n)I}\right)=0,$$

$$I_2^{(n)}\left(\mathbf{x}^{(n)I}\right)=\frac{\pi^2}{2},$$

$$P_{00}^{(n)}\left(\mathbf{x}^{(n)C}\right)=-\pi^2,$$

$$P_{10}^{(n)}\left(\mathbf{x}^{(n)C}\right)=-\frac{\pi^2}{2}x_1^{(n)C},$$

$$P_{01}^{(n)}\left(\mathbf{x}^{(n)C}\right)=-\frac{\pi^2}{2}x_2^{(n)C},$$

$$P_{20}^{(n)}\left(\mathbf{x}^{(n)C}\right)=\frac{\pi^2}{16}\left(\left(4a^{(n)}\right)^2-\left(x_1^{(n)C}\right)^2-3\left(x_2^{(n)C}\right)^2\right),$$

$$P_{02}^{(n)}\left(\mathbf{x}^{(n)C}\right)=\frac{\pi^2}{16}\left(\left(4a^{(n)}\right)^2-3\left(x_1^{(n)C}\right)^2-\left(x_2^{(n)C}\right)^2\right),$$

$$P_{20}^{(n)}\left(\mathbf{x}^{(n)C}\right)=\frac{\pi^2}{8}x_1^{(n)C}x_2^{(n)C},$$

$$J_{002}^{(n)}\left(\mathbf{x}^{(n)C}\right)=-\frac{\pi^2}{2},$$

$$J_{000}^{(n)}\left(\mathbf{x}^{(n)C}\right)=-\frac{\pi^2}{2},$$

$$J_{001}^{(n)}\left(\mathbf{x}^{(n)C}\right)=0,$$

$$J_{102}^{(n)}\left(\mathbf{x}^{(n)C}\right)=-\frac{3\pi^2}{8}x_1^{(n)C},$$

$$J_{010}^{(n)}\left(\mathbf{x}^{(n)C}\right)=-\frac{3\pi^2}{8}x_2^{(n)C},$$

$$J_{101}^{(n)}\left(\mathbf{x}^{(n)C}\right)=-\frac{\pi^2}{8}x_2^{(n)C},$$

$$J_{100}^{(n)}\left(\mathbf{x}^{(n)C}\right)=-\frac{\pi^2}{8}x_1^{(n)C},$$

$$J_{202}^{(n)}\left(\mathbf{x}^{(n)C}\right)=\frac{\pi^2}{32}\left(6\left(a^{(n)}\right)^2-\left(x_1^{(n)C}\right)^2-5\left(x_2^{(n)C}\right)^2\right),$$

$$J_{020}^{(n)}\left(\mathbf{x}^{(n)C}\right)=\frac{\pi^2}{32}\left(6\left(a^{(n)}\right)^2-5\left(x_1^{(n)C}\right)^2-\left(x_2^{(n)C}\right)^2\right),$$

$$J_{201}^{(n)}\left(\mathbf{x}^{(n)C}\right)=\frac{\pi^2}{16}x_1^{(n)C}x_2^{(n)C},$$

$$J_{110}^{(n)}\left(\mathbf{x}^{(n)C}\right)=\frac{\pi^2}{16}x_1^{(n)C}x_2^{(n)C},$$

$$J_{200}^{(n)}\left(\mathbf{x}^{(n)C}\right)=\frac{\pi^2}{32}\left(2\left(a^{(n)}\right)^2-\left(x_1^{(n)C}\right)^2-\left(x_2^{(n)C}\right)^2\right), \tag{29}$$

Appendix D: Coefficients of system of linear algebraic equations

$$g_{jirq}^{(n)I}=\delta_{iq}\left\{\left[(3-4\nu)M^{(n)*}\left(\tilde{\mathbf{x}}_i^{(n)I}\right)\right.\right.$$

$$\left.+\delta_{1j}I_2^{(n)*}\left(\tilde{\mathbf{x}}_i^{(n)I}\right)+\delta_{2j}I_0^{(n)*}\left(\tilde{\mathbf{x}}_i^{(n)I}\right)\right]\delta_{jr}$$

$$\left.+\left(\delta_{1j}+\delta_{2j}\right)\delta_{(3-j)r}I_1^{(n)*}\left(\tilde{\mathbf{x}}_i^{(n)I}\right)\right\}$$

$$+(3-4\nu)\,\delta_{jr}\int\int_{\tilde{S}_0^{(n)I}}\frac{\theta_q^{(n)I}\left(\tilde{\mathbf{y}}^{(n)I}\right)\sin\tilde{y}_1^{(n)I}}{d\left(\tilde{\mathbf{x}}_i^{(n)I},\tilde{\mathbf{y}}^{(n)I}\right)}dS_{\tilde{\mathbf{y}}}$$

$$+\left(\delta_{1j}+\delta_{2j}\right)\times$$

$$\times \int\int_{\tilde{S}_0^{(n)I}} \left\{ \left[f_j\left(\tilde{\mathbf{x}}_i^{(n)I}, \tilde{y}^{(n)I}\right)\right]^2 \delta_{jr} + f_1\left(\tilde{\mathbf{x}}_i^{(n)I}, \tilde{\mathbf{y}}^{(n)I}\right) f_2\left(\tilde{\mathbf{x}}_i^{(n)I}, \tilde{\mathbf{y}}^{(n)I}\right) \delta_{(3-j)r} \right\} \frac{\theta_q^{(n)I}\left(\tilde{\mathbf{y}}^{(n)I}\right)\sin\tilde{y}_1^{(n)I}}{\left[d\left(\tilde{\mathbf{x}}_i^{(n)I}, \tilde{\mathbf{y}}^{(n)I}\right)\right]^3} dS_{\tilde{\mathbf{y}}},$$

$$j,r = 1,2,3, \quad i,q = 1,2,\ldots,Q^{(n)I}, \quad n = 1,2,\ldots N^I,$$

$$\begin{aligned}
g_{jirq}^{(n)C} = {} & \delta_{iq} \left\langle \left\{ \left[\delta_{3j} + (1-2\nu)(\delta_{1j}+\delta_{2j})\right] P_{00}^{(n)*}\left(\mathbf{x}_i^{(n)C}\right) + 3\nu\left[\delta_{1j} J_{002}^{(n)*}\left(\mathbf{x}_i^{(n)C}\right) + \delta_{2j} J_{000}^{(n)*}\left(\mathbf{x}_i^{(n)C}\right)\right] \right\} \delta_{jr} + 3\nu(\delta_{1j}+\delta_{2j})\delta_{(3-j)r} J_{001}^{(n)*}\left(\mathbf{x}_i^{(n)C}\right) \right\rangle \\
& + \left\langle \left\{ \left[\delta_{3j} + (1-2\nu)(\delta_{1j}+\delta_{2j})\right] P_{10}^{(n)*}\left(\mathbf{x}_i^{(n)C}\right) + 3\nu\left[\delta_{1j} J_{102}^{(n)*}\left(\mathbf{x}_i^{(n)C}\right) + \delta_{2j} J_{100}^{(n)*}\left(\mathbf{x}_i^{(n)C}\right)\right] \right\} \delta_{jr} + 3\nu(\delta_{1j}+\delta_{2j})\delta_{(3-j)r} J_{101}^{(n)*}\left(\mathbf{x}_i^{(n)C}\right) \right\rangle \\
& \frac{\partial\theta_q^{(n)C}\left(\mathbf{x}^{(n)C}\right)}{\partial x_1^{(n)C}}\Bigg|_{\mathbf{x}^{(n)C}=\mathbf{x}_i^{(n)C}} \\
& + \left\langle \left\{ \left[\delta_{3j} + (1-2\nu)(\delta_{1j}+\delta_{2j})\right] P_{01}^{(n)*}\left(\mathbf{x}_i^{(n)C}\right) + 3\nu\left[\delta_{1j} J_{012}^{(n)*}\left(\mathbf{x}_i^{(n)C}\right) + \delta_{2j} J_{010}^{(n)*}\left(\mathbf{x}_i^{(n)C}\right)\right] \right\} \delta_{jr} + 3\nu(\delta_{1j}+\delta_{2j})\delta_{(3-j)r} J_{011}^{(n)*}\left(\mathbf{x}_i^{(n)C}\right) \right\rangle \\
& \times \frac{\partial\theta_q^{(n)C}\left(\mathbf{x}^{(n)C}\right)}{\partial x_2^{(n)C}}\Bigg|_{\mathbf{x}^{(n)C}=\mathbf{x}_i^{(n)C}} \\
& + \frac{1}{2} \left\langle \left\{ \left[\delta_{3j} + (1-2\nu)(\delta_{1j}+\delta_{2j})\right] P_{20}^{(n)*}\left(\mathbf{x}_i^{(n)C}\right) + 3\nu\left[\delta_{1j} J_{202}^{(n)*}\left(\mathbf{x}_i^{(n)C}\right) + \delta_{2j} J_{200}^{(n)*}\left(\mathbf{x}_i^{(n)C}\right)\right] \right\} \delta_{jr} + 3\nu(\delta_{1j}+\delta_{2j})\delta_{(3-j)r} J_{201}^{(n)*}\left(\mathbf{x}_i^{(n)C}\right) \right\rangle \\
& \frac{\partial^2\theta_q^{(n)C}\left(\mathbf{x}^{(n)C}\right)}{\partial x_1^{(n)C^2}}\Bigg|_{\mathbf{x}^{(n)C}=\mathbf{x}_i^{(n)C}} \\
& + \frac{1}{2} \left\langle \left\{ \left[\delta_{3j} + (1-2\nu)(\delta_{1j}+\delta_{2j})\right] P_{02}^{(n)*}\left(\mathbf{x}_i^{(n)C}\right) + 3\nu\left[\delta_{1j} J_{022}^{(n)*}\left(\mathbf{x}_i^{(n)C}\right) + \delta_{2j} J_{020}^{(n)*}\left(\mathbf{x}_i^{(n)C}\right)\right] \right\} \delta_{jr} + 3\nu(\delta_{1j}+\delta_{2j})\delta_{(3-j)r} J_{021}^{(n)*}\left(\mathbf{x}_i^{(n)C}\right) \right\rangle \\
& \frac{\partial^2\theta_q^{(n)C}\left(\mathbf{x}^{(n)C}\right)}{\partial x_2^{(n)C^2}}\Bigg|_{\mathbf{x}^{(n)C}=\mathbf{x}_i^{(n)C}} + \\
& + \left\langle \left\{ \left[\delta_{3j} + (1-2\nu)(\delta_{1j}+\delta_{2j})\right] P_{11}^{(n)*}\left(\mathbf{x}_i^{(n)C}\right) + 3\nu\left[\delta_{1j} J_{112}^{(n)*}\left(\mathbf{x}_i^{(n)C}\right) + \delta_{2j} J_{110}^{(n)*}\left(\mathbf{x}_i^{(n)C}\right)\right] \right\} \delta_{jr} + 3\nu(\delta_{1j}+\delta_{2j})\delta_{(3-j)r} J_{111}^{(n)*}\left(\mathbf{x}_i^{(n)C}\right) \right\rangle \\
& \frac{\partial^2\theta_q^{(n)C}\left(\mathbf{x}^{(n)C}\right)}{\partial x_1^{(n)C}\partial x_2^{(n)C}}\Bigg|_{\mathbf{x}^{(n)C}=\mathbf{x}_i^{(n)C}} \\
& + \left[\delta_{3j} + (1-2\nu)(\delta_{1j}+\delta_{2j})\right]\delta_{jr} \times \int\int_{S_0^{(n)C}} \frac{\sqrt{\left(b^{(n)}\right)^2 - \left(y_1^{(n)C}\right)^2 - \left(y_2^{(n)C}\right)^2}\,\theta_q^{(n)C}\left(\mathbf{y}^{(n)C}\right)}{\left|\mathbf{x}_i^{(n)C} - \mathbf{y}^{(n)C}\right|^3} dS_{\mathbf{y}} \\
& + 3\nu(\delta_{1j}+\delta_{2j}) \int\int_{S_0^{(n)C}} \sqrt{\left(b^{(n)}\right)^2 - \left(y_1^{(n)C}\right)^2 - \left(y_2^{(n)C}\right)^2} \\
& \times \left[\frac{\left(x_{ji}^{(n)C} - y_j^{(n)C}\right)^2}{\left|\mathbf{x}_i^{(n)C} - \mathbf{y}^{(n)C}\right|^2}\delta_{jr} + \frac{\left(x_{1i}^{(n)C} - y_1^{(n)C}\right)\left(x_{2i}^{(n)C} - y_2^{(n)C}\right)}{\left|\mathbf{x}_i^{(n)C} - \mathbf{y}^{(n)C}\right|^2}\delta_{(3-j)r} \right] \\
& \frac{\theta_q^{(n)C}\left(\mathbf{y}^{(n)C}\right)}{\left|\mathbf{x}_i^{(n)C} - \mathbf{y}^{(n)C}\right|^3} dS_{\mathbf{y}},
\end{aligned}$$

$$j,r = 1,2,3, \quad i,q = 1,2,\ldots,Q^{(n)C}, \quad n = 1,2,\ldots N^C,$$

$$e_{jirq}^{(kn)I} = \int\int_{\tilde{S}^{(k)I}} \theta_q^{(k)I}\left(\tilde{\mathbf{y}}^{(k)I}\right) \tilde{R}_{jr}^{(kn)I}\left(\tilde{\mathbf{x}}^{(kn)I}, \tilde{\mathbf{y}}^{(k)I}\right)\Big|_{\tilde{\mathbf{x}}^{(n)I} = \tilde{\mathbf{x}}_i^{(n)I}} dS_{\mathbf{y}},$$

$$j,r=1,2,3,\quad i,q=1,2,\ldots,Q^{(n)I},$$
$$k\neq n,\quad k,n=1,2,\ldots N^{I},$$

$$e_{jirq}^{(kn)C}=\iint\limits_{S^{(k)C}}\sqrt{\left(b^{(k)}\right)^2-\left(y_1^{(k)C}\right)^2-\left(y_2^{(k)C}\right)^2}$$
$$\times\theta_q^{(k)C}\left(\mathbf{y}^{(k)C}\right)R_{jr}^{(kn)C}\left(\mathbf{x}^{(kn)C},\mathbf{y}^{(k)C}\right)\Big|_{\mathbf{x}^{(n)C}=\mathbf{x}_i^{(n)C}}dS_{\mathbf{y}},$$

$$j,r=1,2,3,\quad i=1,2,\ldots,Q^{(n)C},$$
$$q=1,2,\ldots,Q^{(n)I},$$
$$k\neq n,\quad k,n=1,2,\ldots N^{C},$$

$$e_{jirq}^{(kn)CI}=\iint\limits_{S^{(k)C}}\sqrt{\left(b^{(k)}\right)^2-\left(y_1^{(k)C}\right)^2-\left(y_2^{(k)C}\right)^2}$$
$$\times\theta_q^{(k)C}\left(\mathbf{y}^{(k)C}\right)\tilde{L}_{jr}^{(kn)CI}\left(\tilde{\mathbf{x}}^{(kn)CI},\mathbf{y}^{(k)C}\right)\Big|_{\mathbf{x}^{(n)C}=\mathbf{x}_i^{(n)C}}dS_{\mathbf{y}},$$

$$j,r=1,2,3,\quad i=1,2,\ldots,Q^{(n)I},$$
$$q=1,2,\ldots,Q^{(n)C},\quad k=1,2,\ldots N^{C},$$
$$n=1,2,\ldots N^{I},$$

$$e_{jirq}^{(kn)IC}=$$
$$\iint\limits_{\tilde{S}^{(k)I}}\theta_q^{(k)I}\left(\tilde{\mathbf{y}}^{(k)I}\right)L_{jr}^{(kn)IC}\left(\mathbf{x}^{(kn)IC},\tilde{\mathbf{y}}^{(k)I}\right)\Big|_{\mathbf{x}^{(n)C}=\mathbf{x}_i^{(n)C}}dS_{\tilde{\mathbf{y}}},$$

$$j,r=1,2,3,\quad i=1,2,\ldots,Q^{(n)C},$$
$$q=1,2,\ldots,Q^{(n)I},\quad k=1,2,\ldots N^{I},$$
$$n=1,2,\ldots N^{C},$$

$$c_{jir}^{(n)}=a^{(n)}\sin\tilde{x}_{1i}^{(n)I}\Big[(\delta_{3j}\delta_{1r}-\delta_{3r}\delta_{1j})\sin\tilde{x}_{2i}^{(n)I}$$
$$+(\delta_{3r}\delta_{2j}-\delta_{3j}\delta_{2r})\cos\tilde{x}_{2i}^{(n)I}\Big],$$

$$j,r=1,2,3,\quad i=1,2,\ldots,Q^{(n)I},$$
$$n=1,2,\ldots N^{I},$$

$$h_q^{(n)}=\iint\limits_{\tilde{S}^{(n)I}}\theta_q^{(n)I}\left(\tilde{\mathbf{y}}^{(n)I}\right)\sin\tilde{y}_1^{(n)I}dS_{\tilde{\mathbf{y}}},$$

$$q=1,2,\ldots,Q^{(n)I},$$
$$n=1,2,\ldots,N^{I},$$

$$h_{jrq}^{(n)}=\iint\limits_{\tilde{S}^{(n)I}}\theta_q^{(n)I}\left(\tilde{y}^{(n)I}\right)\sin^2\tilde{y}_1^{(n)I}$$
$$\times\Big[(\delta_{j1}\delta_{r3}+\delta_{j3}\delta_{r1})\sin\tilde{y}_2^{(n)I}$$
$$+(\delta_{j2}\delta_{r3}+\delta_{j3}\delta_{r1})\cos\tilde{y}_2^{(n)I}\Big]dS_{\tilde{\mathbf{y}}},$$

$$j,r=1,2,3,\quad q=1,2,\ldots,Q^{(n)I},\qquad(30)$$
$$n=1,2,\ldots N^{I}.$$