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A Meshless Method for the Laplace and Biharmonic Equations Subjected to Noisy Boundary Data

B. Jin[1,2]

Abstract: In this paper, we propose a new numerical scheme for the solution of the Laplace and biharmonic equations subjected to noisy boundary data. The equations are discretized by the method of fundamental solutions. Since the resulting matrix equation is highly ill-conditioned, a regularized solution is obtained using the truncated singular value decomposition, with the regularization parameter given by the L-curve method. Numerical experiments show that the method is stable with respect to the noise in the data, highly accurate and computationally very efficient.

keyword: The method of fundamental solutions, truncated singular value decomposition, L-curve method, Laplace equation, biharmonic equation.

1 Introduction

The Laplace and biharmonic equations arise naturally in many areas of science and engineering. For example, the Laplace equation is widely used to model potential problems and steady state heat conduction. The biharmonic equation, on the other hand, is frequently used to describe the Stokes flow of fluid.

For the case that exact boundary conditions are specified on the complete boundary, this has been extensively studied in the past. For example, Smyrlis and Karageorghis (2003) proposed a fast scheme for biharmonic Dirichlet problems on a disk, while Tsai, Young and Cheng (2002) investigated the mshless BEM for the velocity-vorticity formulation of Stokes flow. Unfortunately, in many practical situations, the boundary data are measured, which are unavoidably contaminated by inherent measurement errors, thus the stability of the numerical method with respect to the noise in the data is of vital importance for obtaining physically meaningful results. The only studies that investigate the stability problems for the Laplace and biharmonic equations are due to Cannon (1964), Cannon and Cecchi (1966), and Lesnic, Elliott and Ingham (1998). The first two employed mathematical programming techniques. However, their theory did not address the issue of finding higher order derivatives, and no numerical results were given to justify the theory. In [Lesnic, Elliott and Ingham (1998)], the authors proposed a direct method based on the boundary element method (BEM), and it could yield stable and accurate results for higher order derivative. Despite the popularity of the BEM in recent years, however, there are still problems hampering its efficient implementation. Among these are difficulty of meshing a surface, especially in higher dimensions, the requirement of evaluation of singular integrals, and slow convergence due to the use of lower-order polynomial approximations.

The purpose of this study is to develop a fast and stable numerical scheme for the solution of the Laplace and biharmonic equations subjected to noisy boundary data. The new scheme employs the method of fundamental solutions to discretize the differential equations, and uses truncated singular value decomposition to solve the resulting matrix equation.

2 The method of fundamental solutions

The method of fundamental solutions (MFS) is an inherently meshless, exponentially convergent, boundary-type method for the solution of elliptic partial differential equations. For details of the method, we refer to the comprehensive survey [Golberg and Chen (1998)] and references therein. The basic idea of the method is to approximate the solution of the problem by a linear combination of fundamental solutions of the governing differential operator.

Let Ω be a simply-connected bounded domain in the two-dimensional space, and $\partial\Omega$ its boundary. In this paper,

[1] Center for Engineering and Scientific Computation, Zhejiang University, Hangzhou, Zhejiang, 310027, P.R. China.
[2] Department of Mathematics, Zhejiang University, Hangzhou, Zhejiang, 310027, P.R. China.

we consider the Dirichlet problem for the Laplace equation

$$\Delta u(\mathbf{x}) = 0, \ \mathbf{x} \in \Omega, \tag{1}$$

$$u(\mathbf{x}) = h(\mathbf{x}), \ \mathbf{x} \in \partial\Omega, \tag{2}$$

where Δ is the Laplace operator, $h(\mathbf{x})$ is a given function, and the biharmonic equation

$$\Delta^2 u(\mathbf{x}) = 0, \ \mathbf{x} \in \Omega, \tag{3}$$

subjected to the following boundary conditions

$$u(\mathbf{x}) = f(\mathbf{x}), \ \mathbf{x} \in \partial\Omega, \tag{4}$$

$$\frac{\partial u(\mathbf{x})}{\partial n} = g(\mathbf{x}), \ \mathbf{x} \in \partial\Omega, \tag{5}$$

where $n(\mathbf{x})$ is the unit outward normal on $\partial\Omega$, and $f(\mathbf{x})$ and $g(\mathbf{x})$ are known functions. Note that there is no boundary condition on the vorticity $\phi(\mathbf{x}) = \Delta u(\mathbf{x})$ or its derivative. The problem is to determine the normal derivative of the potential $\partial_n u(\mathbf{x})$ for the Laplace equation, and the vorticity $\phi(\mathbf{x})$ and the normal derivative of the vorticity $\partial_n \phi(\mathbf{x})$ for the biharmonic equation.

In the MFS, n_s source points $\{\mathbf{y}_j, \ j = 1,2,\ldots,n_s\}$ are distributed on a fictitious boundary outside of the solution domain Ω, and n_b fixed points $\{\mathbf{x}_i, i = 1,2,\ldots,n_b\}$ are chosen along the boundary $\partial\Omega$. The approximation of the solution $\overline{u}(\mathbf{x})$ to the Laplace equation can be written as

$$\overline{u}(\mathbf{x}) = \sum_{j=1}^{n_s} c_j G_j(\mathbf{x}), \ \mathbf{x} \in \Omega, \tag{6}$$

where $G_j(\mathbf{x}) = u^*(\mathbf{x} - \mathbf{y}_j)$, $u^*(\mathbf{x})$ is the fundamental solution to the Laplace operator, and $\{c_j, j = 1,2,\ldots,n_s\}$ are coefficients to be determined. Following [Karageorghis and Fairweather (1987), Smyrlis and Karageorghis (2003)], the solution to the biharmonic equation can be approximated by a linear combination of fundamental solutions to both the Laplace and biharmonic equations

$$\overline{u}(\mathbf{x}) = \sum_{j=1}^{n_s} c_j G_j(\mathbf{x}) + \sum_{j=1}^{n_s} d_j H_j(\mathbf{x}), \mathbf{x} \in \Omega \tag{7}$$

where $H_j(\mathbf{x}) = u^\#(\mathbf{x} - \mathbf{y}_j)$, $u^\#(\mathbf{x})$ is the fundamental solution to the biharmonic equation, and $\{d_j, \ j = 1,2,\ldots,n_s\}$ are coefficients to be determined.

The fundamental solutions $u^*(\mathbf{x})$ and $u^\#(\mathbf{x})$ to the Laplace and biharmonic equations in $\mathbf{R}^2$ are given by

$$u^*(\mathbf{x}) = \ln(r), \ \mathbf{x} \in \mathbf{R}^2, \tag{8}$$

and

$$u^\#(\mathbf{x}) = r^2 \ln(r), \ \mathbf{x} \in \mathbf{R}^2, \tag{9}$$

respectively, where $r = \|\mathbf{x}\|_2$, $\|\cdot\|_2$ is the Euclidean norm on $\mathbf{R}^d$.

The approximate solution $\overline{u}(\mathbf{x})$ satisfies the differential equation, and the coefficients must be chosen such that the boundary conditions are satisfied. By collocating the boundary conditions into the approximate solution, we arrive at the following system of linear equations

$$h(\mathbf{x}_i) = \sum_{j=1}^{n_s} a_j G_j(\mathbf{x}_i), \ i = 1,2,\ldots,n_b, \tag{10}$$

for the Laplace equation, and for the biharmonic equation we have

$$f(\mathbf{x}_i) = \sum_{j=1}^{n_s} c_j G_j(\mathbf{x}_i) + \sum_{j=1}^{n_s} d_j H_j(\mathbf{x}_i), \ i = 1,2,\ldots,n_b, \tag{11}$$

$$g(\mathbf{x}_i) = \sum_{j=1}^{n_s} c_j \partial_n G_j(\mathbf{x}_i) + \sum_{j=1}^{n_s} d_j \partial_n H_j(\mathbf{x}_i), \ i = 1,2,\ldots,n_b. \tag{12}$$

In brevity, we have the following matrix equation

$$\mathbf{Ac} = \mathbf{b}, \tag{13}$$

where $\mathbf{A} = (A_{ij})$ is an interpolation matrix, $\mathbf{c}$ is the coefficient vector to be determined, $\mathbf{b}$ is the known data vector. For the Laplace equation, $A_{ij} = G_j(\mathbf{x}_i)$, and $\mathbf{b} = (h(\mathbf{x}_1), h(\mathbf{x}_2), \ldots, h(\mathbf{x}_{n_b}))^T$. Similar formulae can be obtained for the biharmonic equation.

If n_b is taken to be equal to n_s, then we have a square interpolation matrix $\mathbf{A}$. Otherwise, n_b is taken to be greater than n_s, then the matrix equation must be solved in a least squares sense. The solution of the matrix equation will be discussed in more details in the next section.

To implement the method, there remains one thing to be determined, i.e. the distribution of the source points. Source points can either be pre-assigned and kept fixed through the solution process or be determined simultaneously with the coefficients during the solution process.

For details of the latter, we refer to the comprehensive survey by Fairweather and Karageorghis (1998). In fact, the distribution of the source points are not of great importance, provided that some minor conditions are satisfied [Mitic and Rashed (2004)].

3 Regularization techniques

One difficulty with the method of fundamental solutions is that the condition number of the interpolation matrix is extremely large, as observed by Kitagawa (1991) and Golberg and Chen (1998). The cause for this is that the method of fundamental solutions can be regarded as a Fredholm integral equation of the first kind, which is notorious for its ill-posedness. For the solution of problems with exact data, this does not pose great challenges, since no noise is present in the data. Standard methods for solving matrix equations are able to produce accurate results. However, for problems with noisy data, the large condition number of the matrix can be disastrous, and standard methods may fail to yield satisfactory results. In order to obtain stable and accurate results, more advanced computational methods must be applied to solve the resulting matrix equation. Regularization methods are the most powerful and efficient methods for solving ill-posed problems. In our computation we use the truncated singular value decomposition [Hansen (1987)] to solve the matrix equation. Other regularization methods, such as the Tikhonov regularization method [Tikhonov and Arsenin (1977)], may be considered, and similar results can be obtained, but these will not further pursued in this paper.

In the singular value decomposition (SVD), the matrix $\mathbf{A}$ is decomposed into

$$\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^T, \tag{14}$$

where $\mathbf{U} = [\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n]$ and $\mathbf{V} = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m]$ are column orthonormal matrices, with column vectors called left and right singular vectors, respectively, T denotes matrix transposition, and $\Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_m)$ is a diagonal matrix with nonnegative diagonal elements in nonincreasing order, which are the singular values of $\mathbf{A}$.

A convenient measure of the conditioning of the matrix $\mathbf{A}$ is the condition number *Cond* defined as

$$Cond = \frac{\sigma_1}{\sigma_m}, \tag{15}$$

i.e. the ratio between the largest singular value and the smallest singular value. By means of the SVD, the solution $\mathbf{a}^0$ can be written as

$$\mathbf{a}^0 = \sum_{i=1}^{k} \frac{\mathbf{u}_i^T \mathbf{b}}{\sigma_i} \mathbf{v}_i, \tag{16}$$

where k is the rank of $\mathbf{A}$. For an ill-conditioned matrix equation, there are small singular values, therefore the solution is dominated by contributions from small singular values when noise is present in the data. One simple remedy to the difficulty is to leave out contributions from small singular values, i.e. taking $\mathbf{a}^p$ as an approximate solution, where $\mathbf{a}^p$ is defined as:

$$\mathbf{a}^p = \sum_{i=1}^{p} \frac{\mathbf{u}_i^T \mathbf{b}}{\sigma_i} \mathbf{v}_i, \tag{17}$$

where $p \leq k$ is the regularization parameter, which determines when one starts to leave out small singular values. Note that if $p = k$, the approximate solution is exactly the least squares solution. This method is known as truncated singular value decomposition (TSVD) in the inverse problem community.

The singular value decomposition has been applied to analyze the matrix equation arising from the method of fundamental solutions previously [Ramachandran (2002)]. But it was not discussed in the context of regularization methods, and the results presented there were limited to exact data.

The performance of regularization methods depends to a great deal on the suitable choice of the regularization parameter. One extensively studied criterion is discrepancy principle [Morozov (1984)], however, it requires a reliable estimation of the amount of noise in the data, which may be unavailable in practical problems. Heuristic approaches are more preferable in case that no *a priori* information about the noise is available. For TSVD, several heuristic approaches have been proposed in the literature, including L-curve method [Hansen (1992); Hansen and O'Leary (1993)] and generalized cross validation [Golub, Heath and Wahba (1979)]. In this paper, we use the L-curve method to provide appropriate regularization parameters.

4 Numerical experiments and discussions

In this section, we present the results obtained by the general numerical scheme, MFS+TSVD, described in the previous two sections, to demonstrate its efficacy.

4.1 Numerical examples

The solution domain under consideration is one simple two-dimensional smooth geometry, i.e. a circular domain $\Omega = \{\mathbf{x} = (x_1, x_2) | x_1^2 + x_2^2 < 4\}$. To illustrate the accuracy of the method, we consider following analytical solutions, which are taken from [Lesnic, Elliott and Ingham (1998)].

The analytical potential $u(\mathbf{x})$ is given by

$$u(\mathbf{x}) = \cos x_1 \cosh x_2 + \sin x_1 \sinh x_2, \ \mathbf{x} \in \Omega, \tag{18}$$

and the biharmonic stream function $u(\mathbf{x})$ and the vorticity $\phi(\mathbf{x})$ are given by

$$u(\mathbf{x}) = \frac{x_1 \sin x_1 \cosh x_2 - x_1 \cos x_1 \sinh x_2}{2}, \ \mathbf{x} \in \Omega, \tag{19}$$

and

$$\phi(\mathbf{x}) = \cos x_1 \cosh x_2 + \sin x_1 \sinh x_2, \ \mathbf{x} \in \Omega, \tag{20}$$

respectively.

The exact boundary data can be easily derived from the analytical solution. In order to investigate the stability of the method, we use simulated noisy data generated by

$$\tilde{h}(\mathbf{x}_i) = h(\mathbf{x}_i) + \varepsilon\zeta, \ i = 1, 2, \ldots, n_b, \tag{21}$$

for $h(\mathbf{x})$, where ζ is a normally distributed random variable with zero mean and unit standard deviation, and ε indicates the level of noise in the data, and is defined by

$$\varepsilon = \max |h| \times \frac{r}{100} = \frac{r}{20}, \tag{22}$$

where r is the percentage of additive noise included in the data. For the biharmonic equation, we have similar formulae for $f(\mathbf{x})$ and $g(\mathbf{x})$.

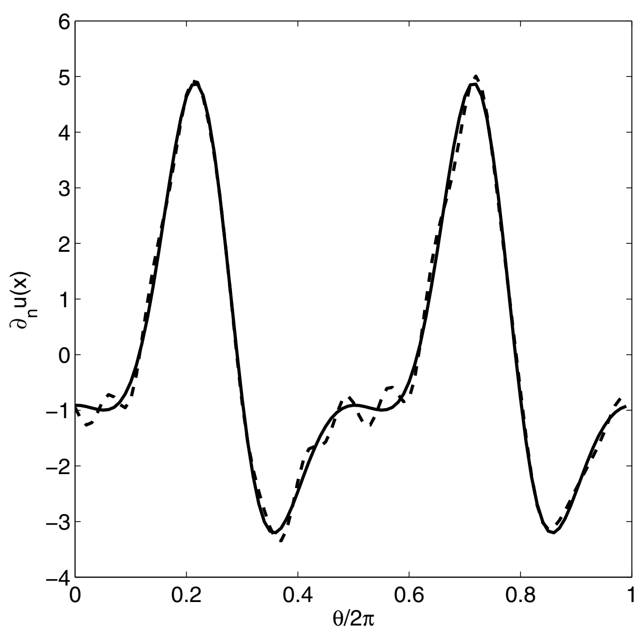
4.2 Effect of regularization method

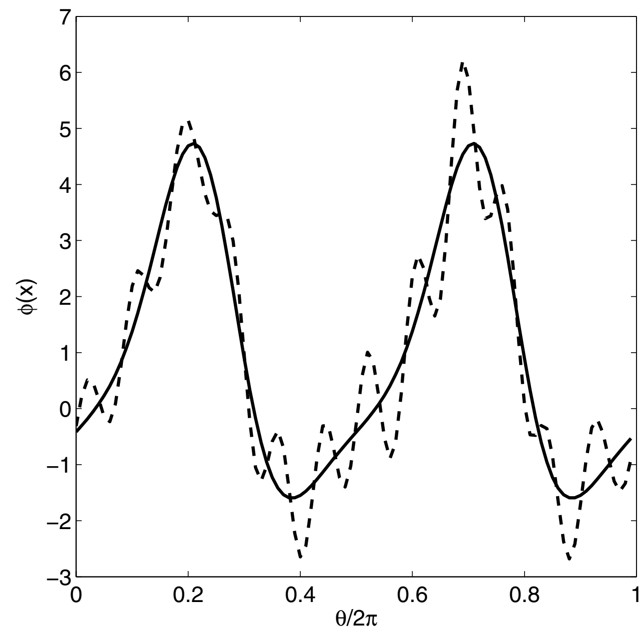
In this subsection, we investigate how the regularization method improves the accuracy of the numerical results. For the results presented in this subsection, the number of collocation points on $\partial\Omega = \{\mathbf{x} = (x_1, x_2) | x_1^2 + x_2^2 = 4\}$ is 40, and the number of source points is 30. The source points are distributed evenly on a circle centered at the origin with radius 8.

For exact data, the large condition number does not pose great challenges, and standard methods, such as the least square method or Gaussian elimination are able to yield accurate results. This is clearly shown by numerous numerical examples presented in the literature. Due to the exponential convergence property of the MFS [Bolgomolny (1985); Golberg and Chen (1998)], the numerical solution for exact data is extremely accurate. For example, the maximum error is less than 1.5×10^{-7} for the reconstructed normal of potential $\partial_n u(\mathbf{x})$ by the least squares method. Thus a few collocation points are sufficient to yield very accurate results, and the size of resulting matrix equation is quite small. The computational effort for the singular value decomposition is negligible. Therefore the method is computationally very efficient. To achieve the same accuracy by the BEM or FEM, the corresponding mesh must be very fine, which undoubtedly would increase the computation time considerably.

For noisy data, the least squares method cannot yield stable results. The results for the normal derivative of the potential $\partial_n u$, the boundary vorticity $\phi(\mathbf{x})$ and its normal derivative $\partial_n \phi(\mathbf{x})$ for noisy data of level 1% ($r = 1$) are shown in Fig. 1. In the figure, the boundary $\partial\Omega$ is parameterized using plane polar coordinates (r, θ). The numerical solution for $\partial_n u(\mathbf{x})$ is graphically reasonable, however, for $\phi(\mathbf{x})$ and $\partial_n \phi(\mathbf{x})$, it is highly oscillatory and cannot be used as an approximate solution.

The cause for the large oscillations in the results by the least squares method is that there are many very small singular values for the interpolation matrix $\mathbf{A}$, as we can see from Equation (16). The distribution of the singular values for the interpolation matrices arising from the Laplace and biharmonic equations is given in Fig. 2. From the figure, the singular values of the interpolation matrix decay gradually to zero without any obvious gap and eventually cluster at zero, which is typically the case for matrix equations derived from Fredholm integral equations of the first kind. The condition number of the interpolation matrix is 4.74×10^{10} and 4.44×10^{14} for the Laplace and biharmonic equations, respectively, which is enormous compared with the size of the matrix equation. The L-curve for the corresponding matrix equation is shown in Fig. 3, which is a log-log plot of the solution norm $\|\mathbf{c}\|_2$ against the residual norm $\|\mathbf{A}\mathbf{c} - \mathbf{b}\|_2$, for various regularization parameters. The regularization parameter corresponding to the corner of the L-curve is taken as a final regularization parameter, since at the corner a good tradeoff between the residual and solution norm is achieved, and we can expect reasonable results when tak-





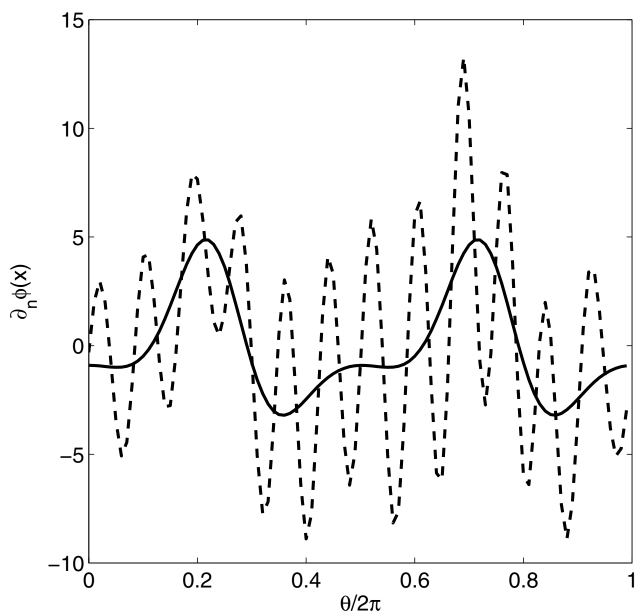


Figure 1 : The reconstructed $\partial_n u(\mathbf{x})$, $\phi(\mathbf{x})$ and $\partial_n \phi(\mathbf{x})$ by the least squares method, with 1% noise in the data, where the solid and dashed curves represent the analytical and numerical solutions, respectively.

ing it as the final regularization parameter. The corners of the L-curves for the Laplace and biharmonic equations correspond to $p = 15$ and $p = 28$, respectively.

The numerical results by TSVD are shown in Fig. 4, where the regularization parameters used are indicated as above. Comparing Fig. 4 with Fig. 1, it is easy to see that the results by the regularization method are far more accurate than that by the least squares method. The error in the numerical results is maintained at a level comparable with the error in the data. Thus the regularization method is indispensable to obtain stable and accurate results. TSVD restores the stability of the method by filtering out the noise in the data effectively. The results also show that the L-curve method provides an appropriate regularization parameter for TSVD. The accuracy of the numerical results presented here is comparable with that in [Lesnic, Elliott and Ingham (1998)]. However, the method of fundamental solutions is far more accurate for exact data, mathematically much simpler, and thus much easier to implement than the BEM. Thus the proposed

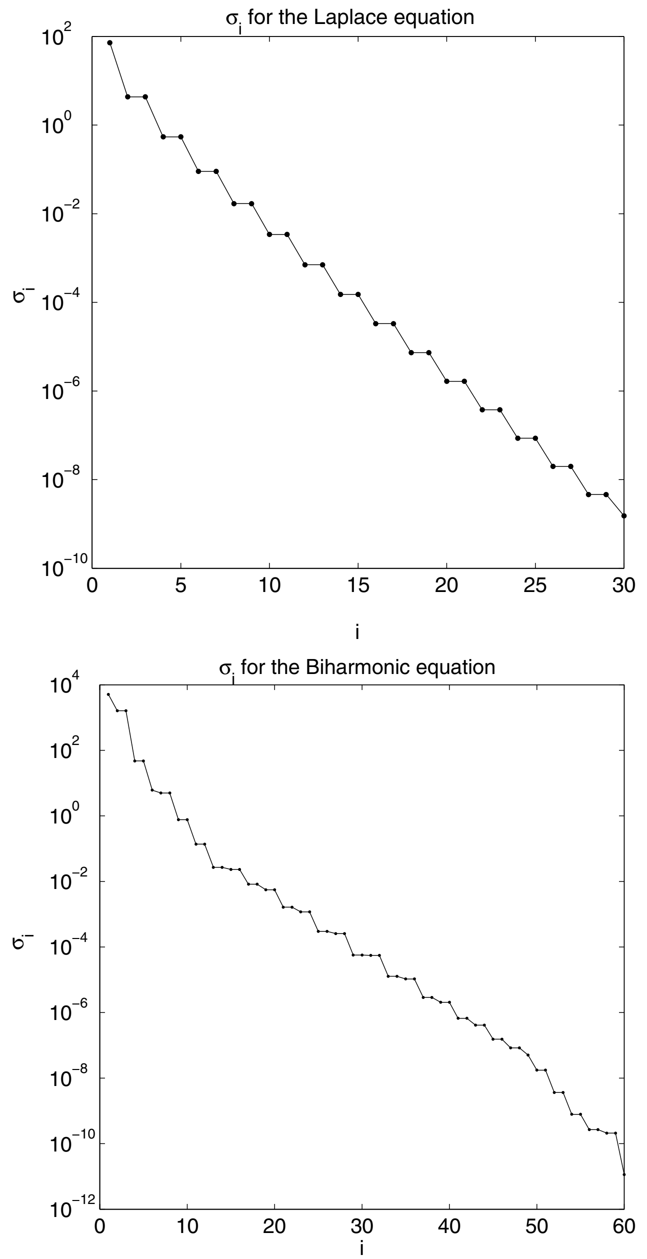


Figure 2 : The distribution of the singular values for the interpolation matrix **A** for the Laplace and biharmnic equations.

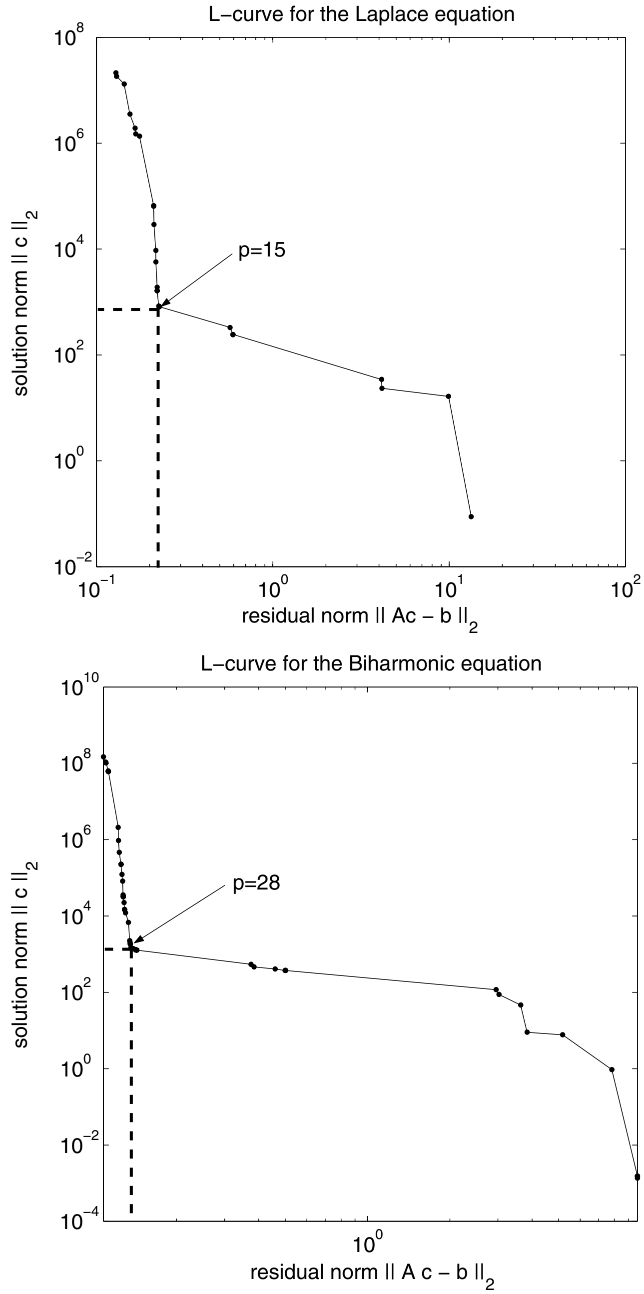
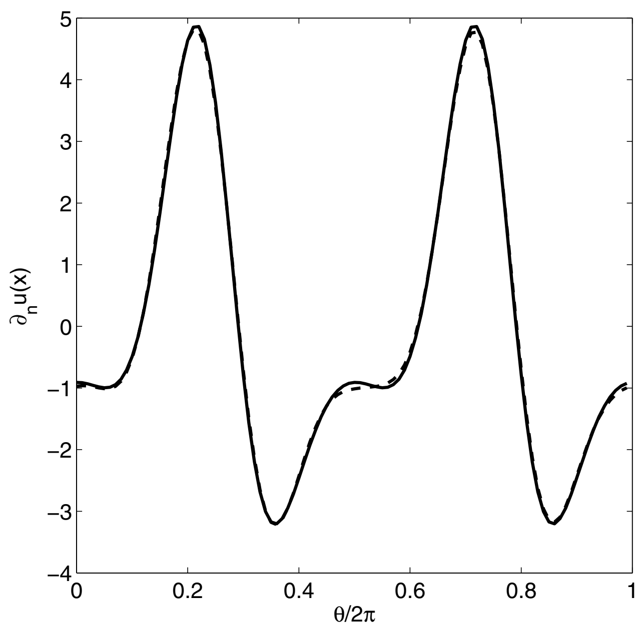


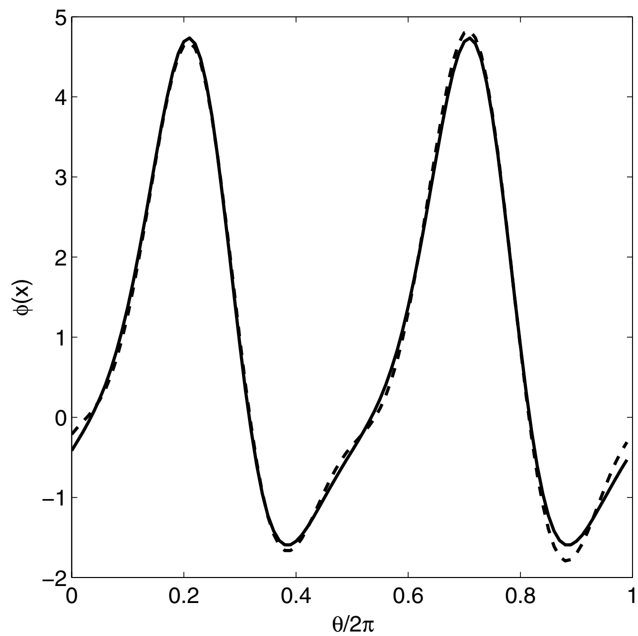
Figure 3 : The L-curve for the Laplace and biharmonic equations with 1% noise in the data.

numerical scheme is advantageous over methods based on the BEM.

4.3 Effect of source location

In this subsection, we investigate the effect of the source location on the accuracy of the numerical results. The number of collocation points on the boundary is fixed as 40, and the number of source points is taken to be 30. The source points are distributed evenly on a circle of various radii to examine the effect of source location. To measure the accuracy of the numerical solution $\overline{u}$, we use





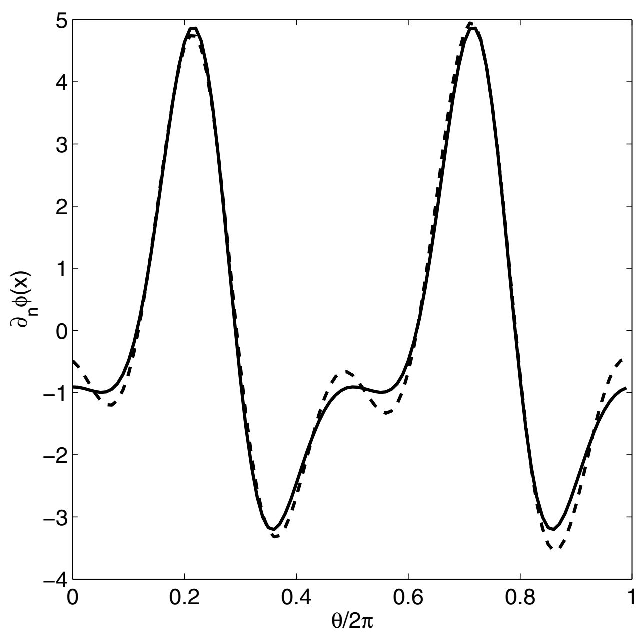


Figure 4 : The reconstructed $\partial_n u(\mathbf{x})$, $\phi(\mathbf{x})$ and $\partial_n \phi(\mathbf{x})$ by TSVD, with 1% noise in the data, where the solid and dashed curves represent the analytical and numerical solutions, respectively.

relative error rel_u defined as

$$rel_u = \frac{\sqrt{\sum_{j=1}^{N}(u_j - \overline{u}_j)^2}}{\sqrt{\sum_{j=1}^{N} u_j^2}}, \tag{23}$$

where u_j and $\overline{u}_j$ are the analytical and numerical results, respectively, and N is the number of points on the boundary on which the solution is evaluated. In this paper, N is taken to be 100, and the points are distributed evenly along the boundary.

The accuracy of the numerical results for the Laplace equation with the source points located on a circle with various radii is shown in Tab. 1. In the table, R is the radius of the source circle, *Cond* is the condition number of the interpolation matrix, p is the regularization parameter determined by the L-curve method, and *rel* is the relative error. The number in the parenthesis indicates a decimal exponent.

From the table, the conditioning of the interpolation ma-

Table 1 : Results for the Laplace equation with 1% noise in the data.

R	$Cond$	p	$rel_{\partial_n u(\mathbf{x})}$
4	6.63(5)	17	0.021
6	5.45(8)	15	0.023
8	4.74(10)	15	0.023
10	1.49(12)	15	0.023

Table 2 : Results for the biharmonic equation with 1% noise in the data.

R	$Cond$	p	$rel_{\phi(\mathbf{x})}$	$rel_{\partial_n \phi(\mathbf{x})}$
4	9.20(8)	30	0.048	0.115
6	1.76(12)	29	0.037	0.065
8	4.44(14)	29	0.042	0.087
10	3.28(16)	29	0.045	0.101

trix deteriorates steadily with the increase of the radius of the source circle. However, the regularization parameter given by the L-curve method is almost the same for all radii concerned. The accuracy of the numerical results is relatively independent of the location of the source points. For problems with exact data, this has been previously established by Ramachandran (2002). Similar conclusions can be drawn for the biharmonic equation, the results for which are presented in Tab. 2.

For non-smooth geometry, the method works equally well, as long as the solution to the problem is smooth, which is usually sufficient to guarantee exponential convergence of MFS [Golberg and Chen (1998)]. From the numerical verification demonstrated above, it can be observed that the proposed method is computationally efficient, stable with respect to the noise in the data. The method is also feasible in handling informal boundary conditions such as the oblique boundary conditions. Furthermore, the approximation of higher order derivatives of the solution is readily available by simple and direct function evaluation. In comparison with existing methods for this problem, the method could be a competitive alternative.

There are several extensions of the proposed method, which further highlights its advantages. Firstly, although this paper considers only problems in the two-dimensional space, the method is readily extended to higher dimensional problems. Secondly, the general scheme, MFS + TSVD, applies also to other elliptic partial differential equations subjected to noisy boundary data, as long as the fundamental solution to the corresponding differential operator is known. Thirdly, combined with the matured numerical technique – the dual reciprocity method [Nardini and Brebbia (1982)], the numerical scheme can easily accommodate non-homogeneous equations with simple and minor modifications.

Last of all, for large-scale problems, or problems on more complex domains or for multiply connected regions, a large number of collocation points may be needed, and the size of resulting matrix equation may be large. This time, the numerical scheme, MFS+TSVD, is of limited use, since the computation of singular value decomposition of large matrix is prohibitive to use. One promising alternative for TSVD is to use the iterative regularization methods, such as conjugate gradient type methods [Hanke and Hansen (1993)], where the computation-intensive step in each iteration, i.e. the matrix-vector multiplication, can be greatly accelerated using the fast multipole method [Saavedra and Power (2003)].

5 Conclusions

In this paper, we have developed a new numerical technique for the solution of the Laplace and biharmonic equations subjected to noisy boundary data. It has been established that standard methods for solving the resulting matrix equation produce unstable results. However, stable results can be obtained using truncated singular value decomposition to solve the resulting matrix equation, with the regularization parameter given by the L-curve method. It's also shown that the accuracy of the numerical results is relatively independent of the locations of the source points. The numerical results show that the method is accurate, stable with respect to the noise in the data, and computationally efficient. Several possible extensions of the scheme are also briefly discussed.

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