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Computation of Short Wave Equation Pulses Using Nonlinear Solitary Waves

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Abstract: A new method is described that has the potential to greatly extend the range of application of current Eulerian time domain electromagnetic or acoustic computational methods for certain problems.

The method involves adding a simple, nonlinear term to the discretized wave equation. As such, it does not require major restructuring of methods or codes that have already been developed. Researchers and engineers who are solving problems for scattering or propagation of short pulses should be able to use the new technique, in many cases as a simple "add on" or callable subroutine, to allow the propagation of short pulses over long distances, even if their solver is low order and the grid is coarse compared to the pulse width (which it must be if the distances are large). The method has many of the advantages of Green's Function based integral equation schemes for long distance propagation. However, unlike these schemes, since it is an Eulerian finite difference technique, and allows short pulses to automatically propagate through regions of varying index of refraction and undergo multiple scattering.

The new method, "Confinement", is based on an earlier, very successful technique, "Vorticity Confinement", that can also be thought of an "add on", which allows the propagation of thin, concentrated vortices over arbitrarily long distances, yet keeps the Eulerian finite difference property of the original fluid dynamic solution method.

In the paper the application of Confinement to the scalar wave equation in 1, 2 & 3 dimensions, including scattering will be described.

keyword: Numerical analysis, wave equation, computational acoustics, computational electromagnetics, vorticity confinement.

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1 Introduction

There are many important problems where thin, concentrated pulses must be numerically convected over long distances. Examples include acoustic and EM pulses scattered or produced by aircraft, rotorcraft and submarines. Often, for these cases, the main interest is in the far field, where the integrated amplitude through the pulse at each point along the pulse surface and the motion of the centroid surface are important, rather than the details of the internal structure. In general, these pulse surfaces can originate in many places, multiply scatter, propagate through varying index of refraction, and have complex topology. Accordingly, we consider Eulerian methods where very general topologies can be treated.

Within this scope, there have been many efforts over decades to discretize and solve the time dependent wave equations. Elaborate codes have been developed to treat complex geometries, such as entire aircraft (we have in mind codes developed by M. Visbal of WPAFB, V. Shankar of Hypercomp Inc. and others). The application of these is, of course, limited by the requirement that a sufficient number of grid cells must span the pulse to accurately solve the equations.

A new method has been developed that has the potential to greatly extend the range of application of these computational methods for certain problems. The goal of this effort is that researchers and engineers who are solving problems for scattering or propagation of pulses should be able to use the new technique, in many cases as a simple "add on" or callable subroutine, to allow the propagation of short pulses over long distances, even if their solver is low order and the grid is coarse compared to the pulse width (which it must be if the distances are large). The new method has many of the advantages of Green's Function based integral equation schemes for long distance propagation. However, unlike these schemes, since it is an Eulerian finite difference technique, and allows short pulses to automatically propagate through regions of varying index of refraction and undergo multiple scattering.

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Confinement involves treating a thin feature, such as a pulse, as a type of weak solution of the governing partial differential equation (pde). Within the feature, a nonlinear *difference* equation, as opposed to *finite difference* equation, is solved that does not necessarily represent a Taylor expansion discretization of the pde. The approach is similar to shock capturing [Lax(1957)], where conservation laws are satisfied, so that integral quantities such as total amplitude and centroid motion are accurately computed for the feature. A more general approach is needed, however, than for shocks, as discussed below. Basically, we treat the features as multi-dimensional nonlinear discrete solitary waves that "live" on the computational lattice. These obey a "confinement" relation that is a generalization to multiple dimensions of some earlier 1-D contact discontinuity capturing schemes.

Differences between Confinement and conventional 1-D shock capturing, are that:

First, unlike shocks, characteristics do not point into the feature, and extra terms must be designed to prevent it from spreading due to numerical effects in the convection. (Harten [Harten(1978)] developed such a scheme, but for contact discontinuities in 1-D compressible flow.)

Second, thin wave equation pulses, vortex filaments or thin streams of passive scalars, are intrinsically multidimensional: A concatenation of 1-D "capturing" operators along separate grid axes will not, generally, give smooth solutions. Due to the multidimensional nature, it seems necessary to pay some attention to the (modeled) structure within the feature, even though it is sampled on only a few grid cells in the cross-section.

First, a short critique of conventional methods for these problems will be given. The basic new method will then be described. Initial results in 1, 2 & 3D will finally be presented.

The method presented has a similar goal to that of [Bleszynski, Bleszynski and Jaroszewicz (2004)] in that they propagate a continuous wave surface in the high frequency limit. The main difference is that they use a system of coupled rays and we use an Eulerian approach. Also, they are already treating diffractive effects, which we are now starting to do.

2 Current Methods

Conventional Eulerian approaches to the wave equation problem involve, of course, formulating governing pde's, discretizing them and solving them as accurately as possible on feasible computational grids, assuming smooth enough solutions. For smooth, non-thin pulses, these methods are well known to converge to the correct solution as the number of points across the pulse, N, becomes large: Error estimates are asymptotic in N. For accurate solutions, even higher order, complex discretization methods typically require N to be at least $\sim$8 or 10 so that the error obeys the large N estimate and is small [Visbal and Gaitonde (1998)]. Even then, solutions degrade over long convection distances (thousands of pulse widths). As a result, conventional methods will be inefficient (or not even feasible) for thin pulses convecting over long distances. Further, adaptive, unstructured grids cannot improve the resolution significantly for realistic problems with many thin, time dependent pulse surfaces.

3 Confinement Approach

3.1 Basic Features

For the above reasons, for the problems considered, it is important to have only very few (2 or 3) grid points to represent the cross section of a pulse surface at each point along the surface and to propagate it with no numerical spreading. This small number of grid points is consistent with the desire to only compute a few integral quantities across the pulse, such as total amplitude and centroid position, and perhaps width or a small number of moments. Then, the difference scheme can, effectively, serve as a simple, implicit "solitary wave" model that *represents* the wave.

An important point is that both the solitary wave pulse thickness and the physical pulse thickness (they may be different) are assumed to be small compared to the other scales in the region where the method is used. Thus, the pulse will propagate according to geometrical optics (high frequency limit) in the region.

The basic idea is that we want to propagate the minimum amount of information necessary to describe the pulse. When it is thick, compared to other dimensions, such as the nearby details of the scatterer, we may choose to use a fine grid and represent the full physical pulse profile. As it propagates away, we may just be interested, as explained, in integral quantities at each point along the pulse surface, i.e., along a ray normal to the surface. As the pulse propagates away, we may have to use a coarser grid that may even have cells larger than the physical pulse thickness, while retaining this information in our "representative" solitary wave.

An important point is that, when the pulse thickness is much less than the radius of curvature of the pulse surface, it is more efficient to describe the pulse profile by a number of "moment fields". The resolution of the thickness profile then depends linearly on the number of these moment fields, which only increases linearly with the resolution. This is then also true of computational storage and work requirements. This should be contrasted with conventional discrete Eulerian schemes, where the cell size is determined by the required resolution. There, for general configurations of surfaces, the number of grid nodes (and computer storage) in 3-D increases like the third power of the resolution and, (including time step changes), the work increases like the fourth power.

As explained in the next section, when the grid is coarse, the Confinement method allows pulse surfaces to propagate over arbitrarily long distances while treating them as nonlinear solitary waves, spread over ~ 2 grid cells, thus allowing information to be accurately propagated. On the other hand, when the grid is fine and details need to be resolved, the Confinement term automatically become small and the method can automatically become to conventional computational acoustics (or electromagnetics). Further, if a pulse propagates through a smooth medium as a solitary wave and then encounters a scatterer where details must be resolved, the pulse can be "reconstituted" on the (new) fine grid, if necessary, using additional moments. This reconstitution will require a "pulse shaping" step. This can easily be effected since, in addition to the common positive numerical diffusion, with confinement, we can have a stable, *total* negative diffusion, as explained below. Thus, the fine grid pulse can be expanded or contracted until its moments agree with the correct values (this is a subject of current work). This feature will be important in many cases, for example, when a pulse scatters from an aircraft wing, propagates many pulse widths, and scatters again from a tail where fine details must be resolved. Further, multiple scattering in inlets for short pulses should provide an important application.

3.2 Approach

The governing equation discussed here is the discretized scalar wave equation, with an added Confinement term: (the approach also works for vectors or tensors, such as Maxwell's equations.)

$$\partial_t^2 \phi = \sigma^2 \nabla^2 \phi + \frac{h^2}{\Delta t} \nabla^2 \partial_t [\mu\phi - \varepsilon\Phi] \tag{1}$$

where ϕ is the scalar amplitude, σ is the index of refraction, μ is a diffusion coefficient that includes numerical effects (we assume physical diffusion is much smaller), and the discretized grid cell size is h and time step, Δt. For the last term, $\varepsilon\Phi$, ε is a numerical coefficient that, together with μ, controls the thickness and time scales of the propagating pulse. Φ will be defined below. For this reason, we refer to the two terms in the brackets as "Confinement terms". We assume conventional, not necessarily high order discretizations are used for the differential operators.

We have found that, at least in tests involving propagation through regions of constant index of refraction, the results are similar to plottable accuracy, whether or not the time derivative is included on the RHS of Eq. (1). However, since the time derivative enforces a relaxation to the desired pulse shape (as explained below), we believe it should be included in general.

The basic idea is that we want the computed thin pulses to maintain their profile and total amplitude as their centroid surfaces are propagated through the field. (We want the same for separate pulse fields representing moments.) The requirement that they relax to their profile in a small number of time steps and have a support of a small number of grid cells determines the two parameters, ε and μ. Also, we assume that the index of refraction field in which the pulse is propagating is slowly varying in time and space compared to these scales (this is required anyway if the grid cell size and time step are to resolve this field). We then have a two-scale problem with the thin pulse obeying a "fast" dynamics.

$$\nabla^2(\mu\phi - \varepsilon\Phi) \approx 0, \tag{2}$$

Thin pulses are then propagated through the field by the "slow" variable, σ. Exactly the same type of discussion applies to the convection of passive scalars, as described in [Steinhoff, Fan, Wang and Dietz (2003)].

In general, the integrals that we are interested in are not sensitive to the parameters ε and μ over a wide range of values, as long as the computed pulses are thin.

An important feature of the Confinement method is that, since it is a second derivative in space and first in time, the total amplitude and centroid of the surface are not changed by the added confinement terms, even under discretization.

3.3 Formulation

The formulation for Confinement will first be described for a stationary pulse ($\sigma = 0$), for clarity. The scalar formulation presented here is related to that presented in [Steinhoff, Wenren, Underhill and Puskas (1995), and Steinhoff, Puskas, Babu, Wenren and Underhill (1997)] in 1-D. This "fast" dynamics will be realized in a wave equation computation in the limit of small time step, or if a separate "Confinement" iteration is done each time step. Excellent results are found with convection and are shown in [Steinhoff, Fan, Wang and Dietz (2003)] for vorticity as well as convecting passive scalars.

For this case, we have an iteration for a non-negative scalar, ϕ:

$$\phi^{n+1} = \phi^n + \mu h^2 \nabla^2 \phi^n - \varepsilon h^2 \nabla^2 \Phi^n \tag{3}$$

where

$$\Phi^n = \left[\frac{\sum_l C_l (\tilde{\phi}^n)^{-1}}{\sum_l C_l} \right]^{-1} \tag{4}$$

$$\tilde{\phi}^n = |\phi|^n + \delta \tag{5}$$

where the sum is over a set of grid nodes near and including the node where Φ is computed, the absolute value is taken and δ, a small positive constant ($\sim 10^{-8}$) is added to prevent problems due to finite precision. The coefficients, C_l, can depend on l, but good results are obtained by simply setting them all to 1 for the wave equation (different values are used for passive scalar convection to avoid using downwind values [Steinhoff, Fan, Wang and Dietz (2003)]). Eq. (4) is related to the harmonic mean.

For example, in 2-D, except for convecting scalars, the form used in this study is

$$\Phi^n_{ij} = \left[\frac{\sum\limits_{\alpha=-1}^{+1} \sum\limits_{\beta=-1}^{+1} \left(\tilde{\phi}^n_{i+\alpha, j+\beta} \right)^{-1}}{N} \right]^{-1} \tag{6}$$

where the number of terms in the sum is N=9. Here, we assume $\phi^n \geq 0$. Negative values can also be accommodated with a small extension. Both μ and ε are positive.

An important feature is that all terms are homogeneous of degree 1 in Eq. (4). This is important because the confinement should not depend on the scale of the quantity being confined. Another important feature is the nonlinearity. It is easy to show that a linear combination of terms, for example of second and fourth order, cannot lead to a stable confinement for any finite range of coefficients.

For smooth ϕ fields (long wavelengths), the last term in Eq. (1) represents a diffusion. If $\mu \leq \varepsilon$, the <u>total</u> diffusion (in the long wavelength limit) is negative. However, the iteration of Eq. (3) is still not divergent and has been observed to converge for values of ε several times that of μ (depending on value of μ).

3.4 Analysis of Small Time Step Form

(Sections 3.4 and 3.5 are close to part of [Steinhoff, Fan, Wang and Dietz (2003)]).

Stability of the iteration as $n \to \infty$ can easily be shown [Steinhoff and Lynn (2002)] for a range of values of μ and ε, including $\mu \leq \varepsilon$. We only have to start with a non-negative initial (ϕ^0) field and show that, for the μ and ε values, ϕ^n remains non-negative. Since the sum of ϕ values is conserved, there is thus an upper bound.

Assuming convergence as $n \to \infty$, we have

$$\nabla^2 (\mu\phi - \varepsilon\Phi) = 0 \tag{7}$$

If ϕ (and hence Φ) vanishes in the far field, away from the pulse, we have $\mu\phi = \varepsilon\Phi$,

If the point (i, j) is given the label $l = 0$, we then have

$$\phi_0^{-1} - \frac{\mu}{\varepsilon N} \sum_l \phi_l^{-1} = 0 \tag{8}$$

There are many solutions of this equation. The ones of importance to us are of the form

$$\phi_{ij} = A\,\mathrm{sec}h[\alpha(z-z_0)] \tag{9}$$

$$z = x_i\cos\theta + y_j\sin\theta \tag{10}$$

A, α, z_0, θ constant, and where $x_i = ih$, $y_j = jh$, h is the grid cell size, and we use the form corresponding to $C_l = 1$ in Eq. (4). This converges to a straight pulse (in 2-D) concentrated about a line at angle θ. It is easy to see that α satisfies

$$\frac{\varepsilon}{\mu} = [1 + 2ch(\alpha h\cos\theta) + 2ch(\alpha h\sin\theta)]/5 \tag{11}$$

for $N = 5$.

An important point is that we obtain close to the same invariance properties as the original pde: The solution is translationally invariant (z_0 is arbitrary) and close to rotationally invariant (θ is arbitrary with a width, given by α in Eq.(11), having some dependence on θ).

3.5 Convection of Passive Scalar

Since the wave equation is, of course, closely related to the convection equation, we present some analysis for the latter, since it is simpler. This analysis shows that the pulse convects with the weighted mean velocity, where the pulse amplitude is the weight. This "Ehrenfest" type of relation should extend to the full wave equation.

The following argument assumes, for each convection step (n), there is at least one confinement step so that the feature remains compact. If ϕ represents a confined passive scalar, then, using a conservative convection routine, we have the following relationships for the dynamics of the convecting solitary wave (we describe the 2-D case for simplicity):

We have a discretization of

$$\partial_t\phi = -\vec{\nabla}\cdot(\vec{q}\phi) + h^2\nabla^2(\mu\phi - \varepsilon\Phi)/\Delta t \tag{12}$$

assuming $\vec{\nabla}\cdot\vec{q} = 0$. Then,

$$\phi^{n+1} = \phi^n - \Delta t\vec{\nabla}_{disc.}\cdot(\vec{q}\phi) + h^2\nabla^2_{disc.}(\mu\phi - \varepsilon\Phi) \tag{13}$$

where discrete operators are labeled.

For conservative discretization, the total amplitude

$$\Omega \equiv \sum_{ij}\phi^n_{ij} \tag{14}$$

is independent of n. If we define the centroid

$$<\vec{X}>^n \equiv \sum_{ij}\vec{x}_{ij}\,\phi^n_{ij}/\Omega \tag{15}$$

and the weighted mean velocity

$$<\vec{q}>^n \equiv \sum_{ij}\vec{q}^{\,n}_{ij}\phi^n_{ij}/\Omega \tag{16}$$

where $\vec{x}_{ij}$ is the (fixed) position vector of node (i, j), and ϕ_{ij} and $\vec{q}_{ij}$ are the scalar value and the velocity at that node, then the centroid evolves according to:

$$<\vec{X}>^{n+1} = <\vec{X}>^n + \Delta t <\vec{q}>^n \tag{17}$$

Since we are, at this point, only interested in the "expectation values" for thin pulses and that the pulses remain compact, spread over only a few cells, this Ehrenfest-type relation is exactly what we need. Only the variables of importance are, effectively, solved for. This shows that the pulses, when isolated, evolve as surfaces with essentially no internal dynamics (assuming they remain confined as thin surfaces). However, we keep the very important Eulerian feature that the number of pulses is not fixed. We could, for example, create additional solitary waves by inserting a source: No additional computational markers need be created, as in Lagrangian schemes. For this study, we show that pulses, for example, reflect and thereby increase automatically in number. This will be seen in the results of Sec. 4.

4 Results

For the scalar wave equation, a simple second-order centered difference method was used for the discretization of Eq. (1). We solve it through two steps: the first is a conventional wave equation solver step, and the second is the confinement step

$$\phi^* = 2\phi^n - \phi^{n-1} + \sigma^2(\Delta t)^2\nabla^2_{disc.}\phi^n \tag{18}$$

$$\phi^{n+1} = \phi^* + h^2\nabla^2_{disc.}\delta^-_n(\mu\phi^* - \varepsilon\Phi^*) \tag{19}$$

where $\nabla^2_{disc.}$ and δ^-_n are the second-order centered difference approximation for the Laplace ∇^2 operator and the backward difference operator in n.

For the 1-D and 2-D result plots, axes are labeled with grid node location. For the 2-D and 3-D results, plots of amplitude are made using dense contours.

4.1 1-D Wave Equation

We start a single pulse in the center of a 256 cell grid with periodic boundary conditions. This pulse has an initial amplitude of 2 at the single, central grid point and is zero at all other points. The resulting two opposite moving waves are shown in Fig. 1 for no Confinement (1a) and Confinement (1b). A minimal diffusion necessary for stability, $\mu = 0.025$, with $\varepsilon = 0$ was used in 1a while in 1b $\mu = 0.1$, $\varepsilon = 0.8$. The rapid diffusion can be seen in 1a,b, while in 1c,d the bulk of the pulses remain confined to ∼3 grid cells and have no diffusion. Other tests show that they continue unaltered for up to about 10^6 time steps, and beyond if double precision is used. This was also shown with an earlier Confinement form in [Steinhoff, Wenren, Underhill and Puskas (1995)].

It should be noted that even though the Confinement is nonlinear, there is virtually no interaction when the waves pass through each other. This was shown in detail in [Steinhoff, Wenren, Underhill and Puskas (1995)].

4.2 2-D Wave Propagation

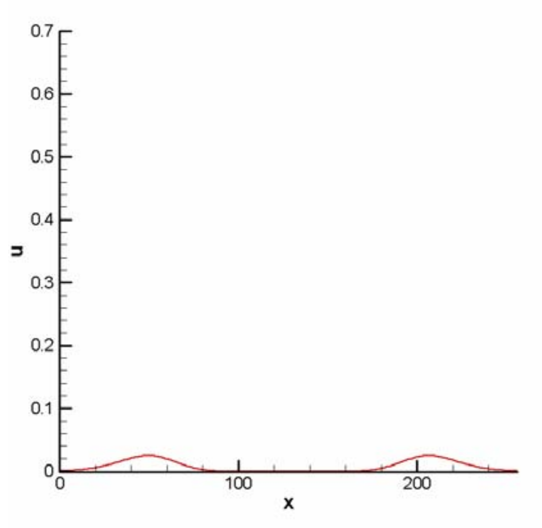
Waves were propagated on a $(128)^2$ cell grid with reflecting boundary conditions. Confinement values used were $\mu = 0.08$, $\varepsilon = 0.6$. Of course, the actual wave equation exhibits a "tail" behind a pulse in 2-D. This can be seen to be suppressed by the Confinement, and, effectively, only the steep pulse front is accurately computed. The tail, since it is smooth, could be computed with no Confinement. The main interest, however, is in 3-D and this was not done.

4.2.1 Convex Wave

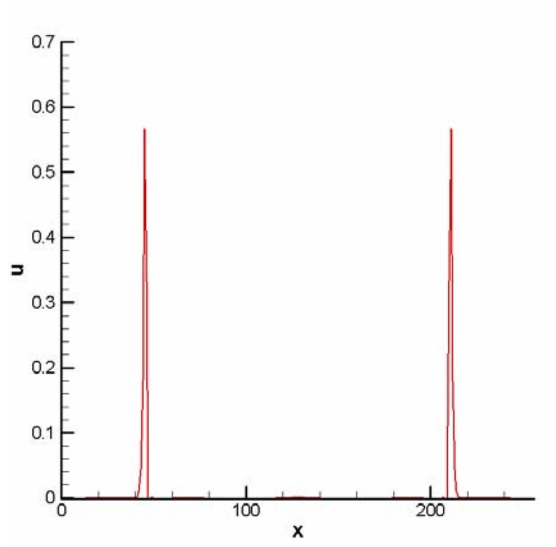
An outward propagating, initially circular pulse surface (diameter 64 cells) was computed. It can be seen in Fig.2 that it remains sharply confined, even after many reflections. Again, as in 1-D, there is no discernable interaction between intersecting waves.

4.2.2 Concave and Convex Waves

The same computation was done as in 4.2.1, but with an initially 2:1 elliptical surface, with 64 cell major axis. Both inward and outward moving waves were formed.



(a) fifteenth pass
(no Confinement)



(b) fifteenth pass
(with Confinement)

Figure 1 : 1D pulse propagation

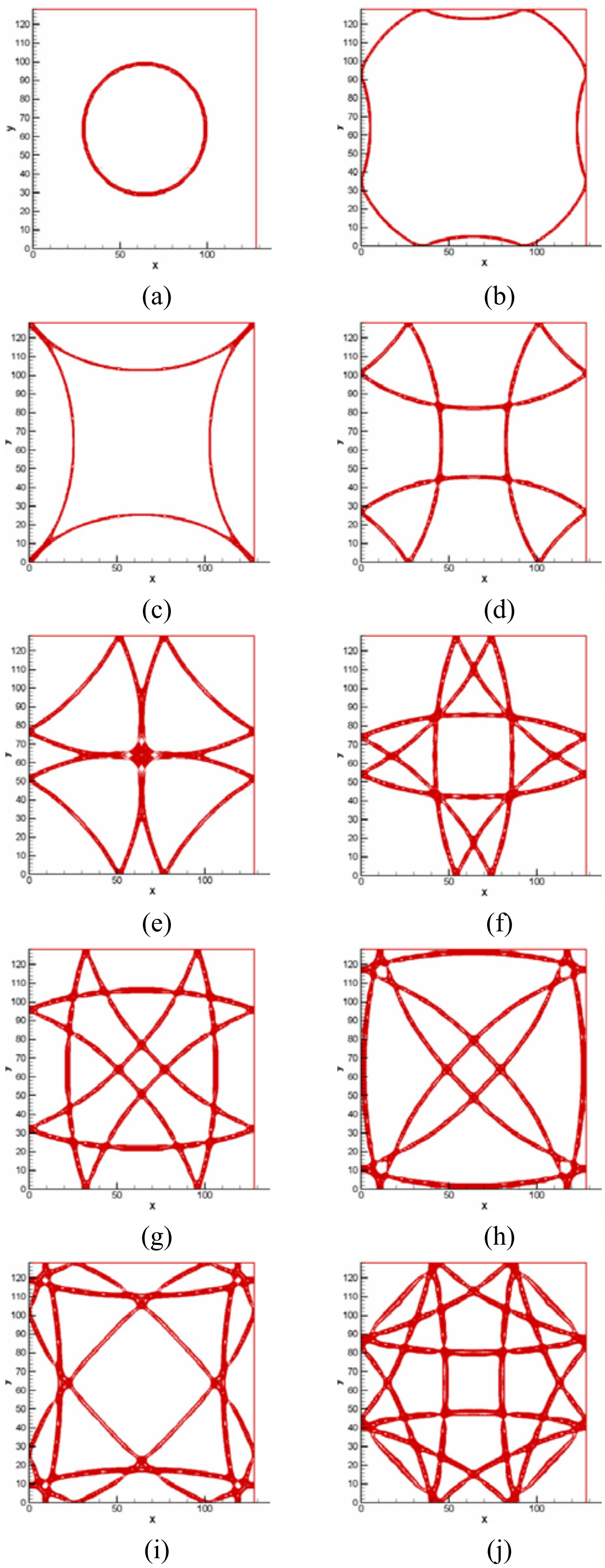


Figure 2 : 2D circle wave propagation

The inward moving wave can be seen to form cusps and "swallowtails". These are only initially resolved, because of the coarseness of the grid. The basic discrete wave equation method, without Confinement, would, of course, not do better. It is well known that refinement is needed in such regions for direct application of finite difference schemes [Benamou and Solliec (2000)].

4.3 3-D Convex Wave Propagation

An expanding, initially spherical pulse surface was computed on a coarse, $(64)^3$ cell grid with reflecting boundary conditions. The initial diameter was 32 cells. Confinement values used were $\mu = 0.05$ and $\varepsilon = 0.4$. As in the 2-D case, the pulse remains completely confined, even after many reflections.

5 Conclusion

A new Eulerian technique is introduced for solving the wave equation for short pulses. The method, "Confinement", reduces to standard Eulerian ones for smooth, long wavelength pulses. However, unlike conventional schemes, it does not diffuse short pulses. Instead, they are "Confined" and propagate as nonlinear solitary waves that "live" on the computational lattice. As such, they can be propagated over indefinitely long distances, while remaining only 2$\sim$3 cells thick. These pulses *represent* the short physical pulses and accurately propagate integral quantities at each point along the pulse surface, such as total amplitude, centroid position, pulse width and other desired moments. It is argued that, for thin pulses, the method can easily be implemented in existing codes, allowing them to be extended to treat much higher wavelengths/shorter pulses, without extensive reformulation. Examples are shown in 1-D, 2-D and 3-D.

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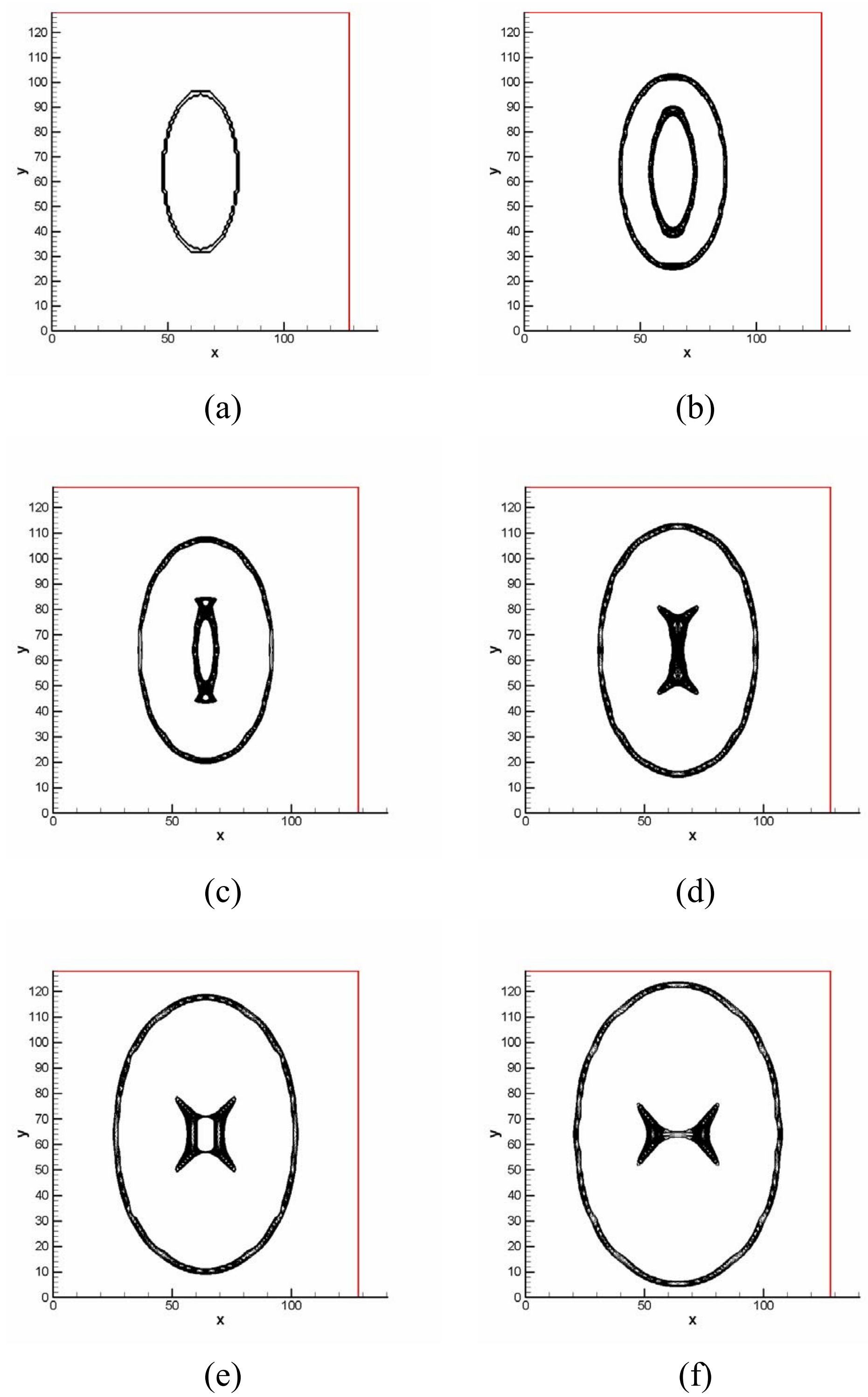

Figure 3 : 2D convex and concave wave propagation: Cusp Formulation

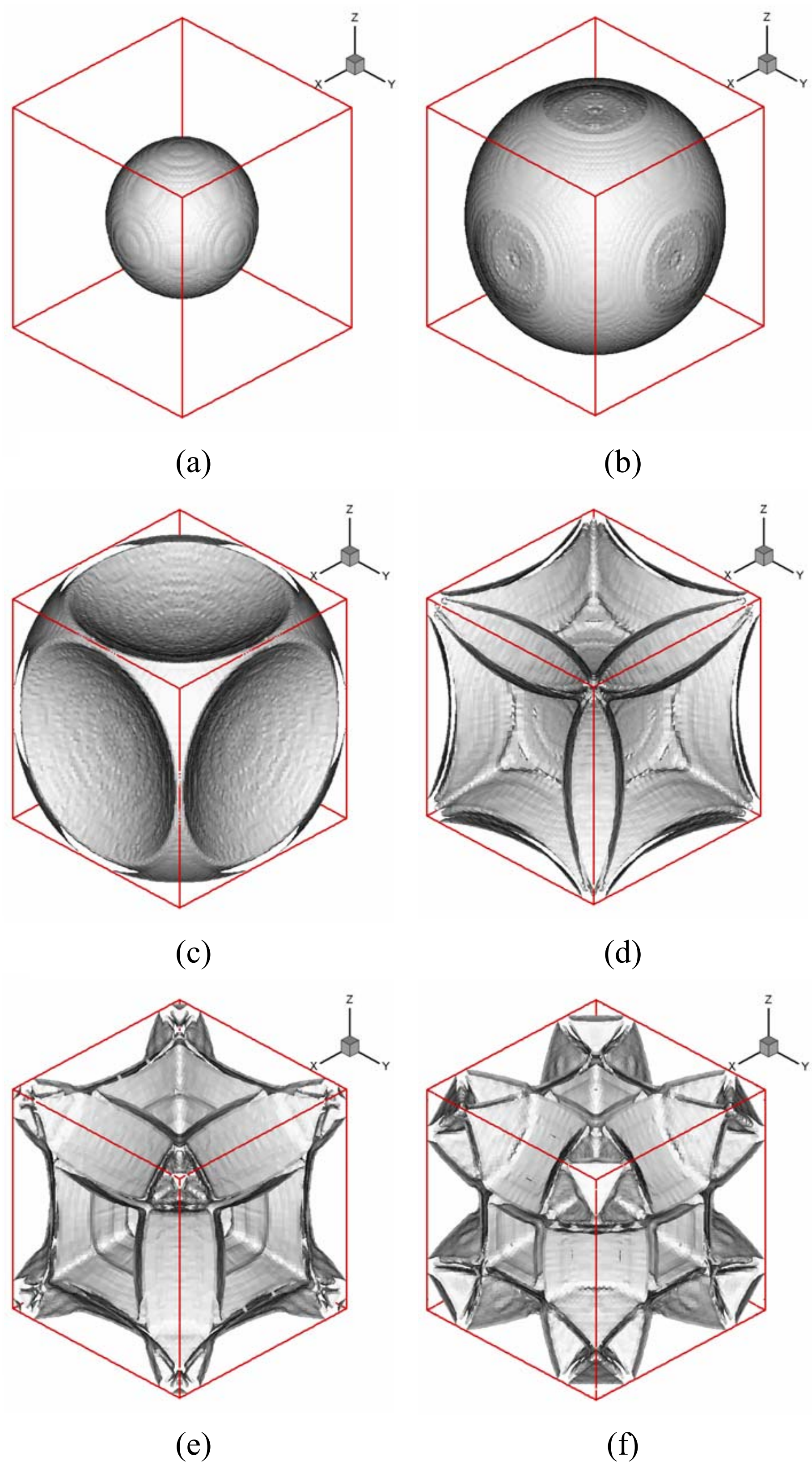

Figure 4 : 3D convex wave propagation

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