A New Application of the Panel Clustering Method for 3D SGBEM

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Abstract: This paper is devoted to the study of a new application of the Panel Clustering Method [Hackbusch and Sauter (1993); Hackbusch and Nowak (1989)]. By considering a classical 3D Neumann screen problem in its boundary integral formulation discretized with the Galerkin BEM, which requires the evaluation of double integrals with hypersingular kernel, we recall and use some recent results of analytical evaluation of the inner hypersingular integrals. Then we apply the Panel Clustering Method (PCM) for the evaluation of the outer integral. For this approach error estimate is shown. Numerical examples and comparisons with classical PCM technique are presented.

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1 Introduction

The Boundary Element Methods (BEM) have become an important technique for solving linear elliptic partial differential equations appearing in many relevant engineering applications (e.g. acustics, elastostatic, plasticity, elastodynamics, etc.) [Bonnet, Maier and Polizzotto (1998); Bonnet (1995); Ervin and Stefan (1990); Maier, Miccoli, Novati and Perego (1995); Maier, Diligenti and Carini (1991); Sirtori, Maier, Novati and Miccoli (1992)]. By means of the foundamental solution of the considered differential equation a large class of both exterior and interior elliptic boundary value problems can be formulated as a linear integral equation on the boundary of the given domain.

The BEM can offer substantial computational advantages over other numerical techniques. However, in order to achieve an efficient numerical implementation of general validity, a number of issues have to be dealt with special attention. One of the most significant and important issue of the practical application of the Symmetric Galerkin BEM is the evaluation of weakly singular, Cauchy singular and even hypersingular integrals over boundary elements.

Moreover, difficulties increase from 1D to 2D boundaries, but while literature is nowadays wide for 2D problems (see for instance [Aimi, Diligenti and Monegato (1999); Balakrishna, Gray and Kane (1994); Carini, Diligenti, Maranesi and Zanella (1999)]) few methods have been proposed for 3D problems (see [Andra and Schnak (1997); Sauter and Lage (2001)]). In recent works, to overcome the difficulties of singular double integrations, regularization of singular and hypersingular BIEs to weakly singular integral equations was proposed. In [Kieser, Schwab and Wendland (1992)] for hypersingular kernels a regularization that depends on the explicit knowledge of a kernel expansion in local polar-coordinates is introduced; the coefficient functions in this expansion are computed automatically by using Maple symbolic manipulation procedures. In [Sauter and Lage (2001)] a direct approach for evaluating hypersingular integrals is presented and a development of transformation techniques is introduced. After these regularizing transformations Gauss-Legendre cubature rules are applied for the approximation of the derived weakly singular integrals. In [Aimi and Diligenti (2002)] we have performed analytically the inner integration without any sort of regularization procedure, giving the explicit result with a significant simplification. Then, we studied the type of singularity of this result as a function of the outer variable of integration, in order to give some indications about the numerical quadrature schemes needed for the remaining outer integral. To compute these integrals we proposed in [Aimi and Diligenti (2002)] efficient formulas which only require to define a regular quasi uniform triangulation of the boundary of the problem, and specify the local degrees of the approximant.

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At any rate, the numerical solution of boundary integral equations by the SGBEM leads to a dense linear system which becomes very large when dealing with boundaries in 3D: in this case computational costs become very high, especially for the evaluation of the *extra-diagonal* terms of the system matrix.

Many different techniques have been recently proposed to reduce the computational cost of the Galerkin matrix evaluation and of the linear system solution. We recall for instance the wavelets-based methods [Dorobantu (1984); Dahmen, Prössdorf and Schneider (1994)] that yield impressive compression rates: nevertheless wavelets are in general only applicable for structured surface triangulation and for simple geometries. An algebraic approach to the data-sparse realization of non-local operators is constituted by the class of algebraic methods like H-matrices [Hackbusch (1999)]: the idea is to approximate blocks of the discrete operator by low-rank matrices. The choice of the approximation method can be based on a singular value decomposition or a convenient expansion of the integral kernel.

The Panel Clustering Method was introduced firstly in [Hackbusch and Nowak (1989)] and further developed by various authors [Giebermann (to appear); Sauter (2000)]. It is based on a fast evaluation of the extra diagonal elements of the matrix and it makes use of two basic ingredients: a hierarchical structuring of the surface triangulation called *cluster tree* and a convenient truncated expansion of the kernel functions.

In this paper we introduce the use of PCM for the evaluation of the outer Galerkin integrals after the analytical inner integration. In particular our attention is focused on a classical 3D Neumann screen problem. The use of results of analytical integration allows the exact calculation of the inner integrals and the introduction of the truncated expansion (and of the related error) only for the evaluation of the outer integral. For this approach error estimate is shown. Numerical examples and comparisons with classical PCM technique are presented.

2 The model problem

Let Γ be a polygonal surface piece in $\mathbb{R}^3$, with a piecewise analytic boundary γ, referred to a Cartesian orthogonal coordinate system $\mathbf{x} = (x_1, x_2, x_3)$. As model problem we consider the *Neumann screen* problem: *given* $f \in H^{-\frac{1}{2}}(\Gamma)$, *find* $u(\mathbf{x})$ *in* $\mathbb{R}^3 \setminus \overline{\Gamma}$ *such that*

$$\Delta u = 0 \qquad \text{in } \mathbb{R}^3 \setminus \overline{\Gamma} \tag{1}$$

$$\frac{\partial u}{\partial \mathbf{n}} = f \qquad \text{on } \Gamma \tag{2}$$

$$u = O(\|\mathbf{x}\|^{-1}) \ \text{ as } \|\mathbf{x}\| \to \infty \tag{3}$$

where $\frac{\partial u}{\partial \mathbf{n}}$ denotes the derivative with respect to the outer normal $\mathbf{n}$ to Γ, which exists outdside the common sides of polygons not lying on a same plane. Let $\tilde{\Gamma}$ be a piecewise smooth, closed surface containing Γ. The definition of Sobolev spaces is as usual [Lions and Magenes (1972)]: for real s

$$H^s(\tilde{\Gamma}) = \begin{cases} \{u_{|\tilde{\Gamma}} : u \in H^{s+1/2}(\mathbb{R}^3)\} & ,s > 0, \\ L^2(\tilde{\Gamma}) & ,s = 0, \\ (H^{-s}(\tilde{\Gamma}))' \text{ (dual space)} & ,s < 0. \end{cases}$$

These spaces are used to define the corresponding spaces of distributions on Γ, namely

$$\tilde{H}^s(\Gamma) = \{u \in H^s(\tilde{\Gamma}) : \operatorname{supp} u \subset \overline{\Gamma}\}$$

$$H^s(\Gamma) = \{u_{|\Gamma} : u \in H^s(\tilde{\Gamma})\}.$$

In [Ervin and Stefan (1990)] the above problem, which appears in linear elasticity when an interior crack opens under normal loading and whose corresponding problem for the Helmholtz equation describes the scattering of acoustic fields by a hard screen, is converted into the integral equation

$$Dv(\mathbf{x}) := \neq\!\!\!\!\!\!\int_\Gamma S(\mathbf{x}, \mathbf{y}-\mathbf{x}) v(\mathbf{y})\, d\Gamma_{\mathbf{y}} = f(\mathbf{x}), \qquad \mathbf{x} \in \Gamma, \tag{4}$$

where S is the hypersingular fundamental solution of the Laplace operator

$$S(\mathbf{x}, \mathbf{y}-\mathbf{x}) := -\frac{1}{4\pi}\left\{\frac{\mathbf{n_x}\cdot\mathbf{n_y}}{r^3} - 3\frac{\mathbf{n_x}\cdot\mathbf{r}\ \mathbf{n_y}\cdot\mathbf{r}}{r^5}\right\} \tag{5}$$

with $\mathbf{r} = \mathbf{y} - \mathbf{x}$ and $r = \|\mathbf{y} - \mathbf{x}\|$, and $v = [u]_{|\Gamma} = u_{|\Gamma^+} - u_{|\Gamma^-}$ gives the jump of u across the screen Γ, with Γ^+ (Γ^-) denoting the upper (lower) side of Γ according to the normal vector $\mathbf{n}$. The operator D is defined by a hypersingular finite part integral in the sense of Hadamard (see [Kieser, ScHwab and Wendland (1992)]).

It is shown in [Costabel (1988)] that even for Lipschitz surfaces Γ the operator D in (4) is positive definite on $\tilde{H}^{\frac{1}{2}}(\Gamma)$, that is there exists a constant $\alpha > 0$ such that for all $v \in \tilde{H}^{\frac{1}{2}}(\Gamma)$

$$< Dv, v >_{L^2(\Gamma)} \geq \alpha \|v\|^2_{\tilde{H}^{\frac{1}{2}}(\Gamma)}. \tag{6}$$

Equation (4) will be solved in a weak sense. If we set $V_\Gamma^\circ := \tilde{H}^{\frac{1}{2}}(\Gamma)$ the weak formulation of our integral problem is: *given* $f \in H^{-\frac{1}{2}}(\Gamma)$, *find a boundary function* $v \in V_\Gamma^\circ$ *satisfying*

$$< Dv, u >_{L^2(\Gamma)} = < f, u >_{L^2(\Gamma)} \qquad \forall u \in V_\Gamma^\circ. \tag{7}$$

Under the assumptions made, problem (7) has a unique solution [Costabel (1988)].

In order to perform the Galerkin method for the equation (7), we need a family of finite-dimensional subspaces $\{V_h\}$ defined on Γ. Let $T_h = \{T_i\}_{i=1}^{M_h}$ be a regular quasi-uniform triangulation of Γ, where T_i is a triangle (also called *panel*), $\overline{\Gamma} = \bigcup_{i=1}^{M_h} T_i$, $T_i \bigcap T_j, i \neq j$, is empty, a side or a vertex and each T_i belongs to one and only one face of the polygonal boundary Γ; further, $\text{diam}(T_i) \leq h, i = 1, \ldots, M_h$. All boundary elements $T_i, i = 1, \ldots, M_h$, can be obtained by a linear mapping A_i applied to a reference element

$$T = \left\{(\xi_1, \xi_2) \in \mathbb{R}^2 : 0 < \xi_1 < 1, 0 < \xi_2 < 1 - \xi_1\right\}. \tag{8}$$

Then $T_i = A_i(T)$ defines a triangular element of T_h.

If $\{\varphi_k\}_{k=1}^{K_p}$ denotes the standard local finite element basis of degree $p \geq 0$ defined on T, the corresponding local basis on T_i is defined by "lifting" finite element functions φ_k from T to T_i, i.e.

$$\varphi_k^{(i)}(\mathbf{x}) = \varphi_k \circ A_i^{-1}(\mathbf{x}), \;\; k = 1, \ldots, K_p, \; \mathbf{x} \in T_i.$$

Piecewise polynomials shape functions of degree $p > 0$ are defined through the standard assembling of the local basis functions defined on each T_i. More precisely, if N_Γ is the total number of nodes of T_h fixed in Γ, then with the above procedure we will define N_Γ shape functions $\psi_1, \ldots, \psi_{N_\Gamma}$ of degree p for the approximation of v on Γ, i.e. we define $V_h = \text{span}\left\{\psi_1, \ldots, \psi_{N_\Gamma}\right\}$. In particular, for our problem, as approximating subspace V_h of V_Γ° we can take the space of piecewise linear functions.

The Galerkin boundary element scheme for (7) is the following: find $v_h \in V_h$, such that

$$< Dv_h, u_h >_{L^2(\Gamma)} = < f, u_h >_{L^2(\Gamma)}, \qquad \forall u_h \in V_h. \tag{9}$$

Problem (9) admits a unique solution.

Next, we write down the Galerkin scheme (9) in the form of a linear symmetric system of algebraic equations for the unknown coefficients X in the boundary element basis

$$AX = F \tag{10}$$

The matrix elements are double integrals with hypersingular kernel of the following type

$$\begin{aligned} A_{lm} &= \int_\Gamma \psi_l(\mathbf{x}) \;{=\!\!\!\!\!\!\int}_\Gamma S(\mathbf{x}, \mathbf{y}-\mathbf{x}) \psi_m(\mathbf{y}) d\Gamma_\mathbf{y} d\Gamma_\mathbf{x} = \\ &= \int_{\text{supp}(\psi_l)} \psi_l(\mathbf{x}) \;{=\!\!\!\!\!\!\int}_{\text{supp}(\psi_m)} S(\mathbf{x}, \mathbf{y}-\mathbf{x}) \psi_m(\mathbf{y}) d\Gamma_\mathbf{y} d\Gamma_\mathbf{x}, \\ & l, m = 1, \ldots, N_\Gamma. \end{aligned} \tag{11}$$

3 Analytical evaluation of hypersingular inner integrals

Let us consider integral (11), where we have fixed ψ_l, ψ_m linear shape functions. We can write (11) in the form

$$\sum_{i=1}^{N_l} \int_{T_i^{(l)}} \varphi^{(l,i)}(\mathbf{x}) \sum_{j=1}^{N_m} {=\!\!\!\!\!\!\int}_{T_j^{(m)}} S(\mathbf{x}, \mathbf{y}-\mathbf{x}) \varphi^{(m,j)}(\mathbf{y}) d\Gamma_\mathbf{y} d\Gamma_\mathbf{x} \tag{12}$$

where we have denoted with $T_1^{(m)}, \ldots, T_{N_m}^{(m)}$ the triangles of T_h forming the support of $\psi_m(\mathbf{y})$, having common vertex in a node of T_h, let us say $P_0^{(m)}$, and with $\varphi^{(m,j)}(\mathbf{y})$ the local basis function on $T_j^{(m)}$ such that

$$\varphi^{(m,j)}(\mathbf{y}) = \psi_m(\mathbf{y}), \quad \text{for } \mathbf{y} \in T_j^{(m)}.$$

A similar notation holds for the outer integral.

We will perform the analytical evaluation of each

$${=\!\!\!\!\!\!\int}_{T_j} S(\mathbf{x}, \mathbf{y}-\mathbf{x}) \varphi^{(j)}(\mathbf{y}) d\Gamma_\mathbf{y}, \;\; j = 1, \ldots N_m \tag{13}$$

where, to simplify the notation, we have omitted the upper index m; we will do the same for the upper index l

in (12), and consider this simplified notation through the paper.

Single results (13), where, to fix the ideas, we can think $\mathbf{x} \in T_i$, will be summed up to form the result of the inner integration in (11).

The source point $\mathbf{x}$ will be given by integration knots of outer integration and we recall that, owing to the Galerkin discretization approach we are using, the outer variable of integration $\mathbf{x}$ has to be considered either in the interior of T_j (when $T_i \equiv T_j$) or outside T_j.

Let us start with the critical situation $\mathbf{x} \in \mathring{T}_j$ (where $\mathring{T}_j$ is the interior of T_j): only in this case in fact we have to deal effectively with the hypersingularity of the kernel $S(\mathbf{x}, \mathbf{y} - \mathbf{x})$. We write integral (13) in local coordinates, i.e. on the reference triangle T (see (8)), with the following changes of variables: $\mathbf{y} = A_j(\eta)$, $\mathbf{x} = A_j(\xi)$.

Let us denote with $\tilde{S}(\xi, \eta - \xi)\tilde{\varphi}^{(j)}(\eta)$ the integrand in (13) rewritten in local coordinates, with P_0, P_j, P_{j+1} the vertices of T_j (we suppose from now on that $P_0 \equiv O$; otherwise we substitute each vector $\mathbf{v} \in \mathbb{R}^3$ with $\mathbf{v} - P_0$), with $\mathbf{v}^\perp$ the clockwise $\frac{\pi}{2}-$rotated of $\mathbf{v}$ in the plane of T_j, and $\mathbf{x}(\xi) = A_j(\xi)$. The following result holds.

Theorem 1 [Aimi and Diligenti (2002)] . *When* $\mathbf{x} \in \mathring{T}_j$ *there holds*

$$\oint_T \tilde{S}(\xi, \eta - \xi)\tilde{\varphi}^{(j)}(\eta) d\sigma_\eta = -\frac{1}{4\pi}[f_1^{(j)}(\mathbf{x}(\xi)) + f_2^{(j)}(\mathbf{x}(\xi))] \tag{14}$$

where

$$f_1^{(j)}(\mathbf{x}(\xi)) = \frac{P_j \cdot P_{j+1}^\perp \|\mathbf{x}(\xi)\|}{P_j \cdot \mathbf{x}(\xi)^\perp \, P_{j+1} \cdot \mathbf{x}(\xi)^\perp} + \frac{\|\mathbf{x}(\xi)\| - \|P_j - \mathbf{x}(\xi)\|}{P_j \cdot \mathbf{x}(\xi)^\perp} + \frac{\|P_{j+1} - \mathbf{x}(\xi)\| - \|\mathbf{x}(\xi)\|}{P_{j+1} \cdot \mathbf{x}(\xi)^\perp} \tag{15}$$

and

$$\begin{aligned} f_2^{(j)}(\mathbf{x}(\xi)) = \\ c_1 \ln \frac{[(\mathbf{x}(\xi) - P_{j+1}) \cdot (P_j - P_{j+1}) + \|P_j - P_{j+1}\| \|\mathbf{x}(\xi) - P_{j+1}\|]}{[(\mathbf{x}(\xi) - P_j) \cdot (P_j - P_{j+1}) + \|P_j - P_{j+1}\| \|\mathbf{x}(\xi) - P_j\|]} \\ + c_2 \ln \frac{[\mathbf{x}(\xi) \cdot P_{j+1} + \|P_{j+1}\| \|\mathbf{x}(\xi)\|]}{[(\mathbf{x}(\xi) - P_{j+1}) \cdot P_{j+1} + \|P_{j+1}\| \|\mathbf{x}(\xi) - P_{j+1}\|]} \\ - c_3 \ln \frac{[\mathbf{x}(\xi) \cdot P_j + \|P_j\| \|\mathbf{x}(\xi)\|]}{[(\mathbf{x}(\xi) - P_j) \cdot P_j + \|P_j\| \|\mathbf{x}(\xi) - P_j\|]}, \end{aligned} \tag{16}$$

with $c_1 = \frac{\|P_j - P_{j+1}\|}{P_j \cdot P_{j+1}^\perp}$, $c_2 = \frac{P_{j+1} \cdot (P_j - P_{j+1})}{(P_j \cdot P_{j+1}^\perp)\|P_{j+1}\|}$ *and* $c_3 = \frac{P_j \cdot (P_j - P_{j+1})}{(P_j \cdot P_{j+1}^\perp)\|P_j\|}$.

Remark 1 This result is made of two parts: one containing logarithmic functions in ξ with at most boundary log-singularities, the other, the first one, containing functions which give rise, in the outer variable of integration ξ, again to hypersingularities when $\mathbf{x}(\xi)$ tends to the sides P_j and P_{j+1} of triangle T_j. In fact, since $\mathbf{x}(\xi) = \xi_1 P_j + \xi_2 P_{j+1}$, it is easy to see that when $\xi_1 \to 0$ (i.e. when $\mathbf{x}$ tends to the side P_{j+1}) the hypersingularity of (15) is, up to the coefficient $-\frac{1}{4\pi}$, and remembering that in this case $T_i \equiv T_j$, of the form

$$-\frac{2}{\xi_1}(1 - \xi_2)\|P_{i+1}\|; \tag{17}$$

analogously, when $\xi_2 \to 0$, we have a singularity of the form

$$-\frac{2}{\xi_2}(1 - \xi_1)\|P_i\|. \tag{18}$$

Anyway, these critical functions (15) in the variable ξ will disappear in the final inner sum over the support of ψ_m (see (12)).

When $\mathbf{x}$ belongs to the same plane of T_j but it is exterior, the hypersingularity of the kernel does not rise effectively; in fact the distance r between $\mathbf{x}$ and $\mathbf{y}$ cannot vanish and the inner integration (13) can be classically performed. Anyway, the result is the same as (14): we only have to consider $\mathbf{x}(\xi) = A_i(\xi)$ with mapping A_i obviously different from A_j.

When $\mathbf{x}$ and T_j don't lye on the same plane evidently, $\mathbf{x}$ is always outside T_j. Having set $\mathbf{x}(\xi) = A_i(\xi)$, $\mathbf{n}_\mathbf{y} = \frac{P_j \times P_{j+1}}{\|P_j \times P_{j+1}\|}$, $\mathbf{n}_\mathbf{x} = \mathbf{n}_{\mathbf{x}(\xi)}$, the following result holds:

Theorem 2 [Aimi and Diligenti (2002)]. *There holds*

$$\oint_T \tilde{S}(\xi, \eta - \xi)\tilde{\varphi}^{(j)}(\eta) d\sigma_\eta = -\frac{1}{4\pi}[g_1^{(j)}(\mathbf{x}(\xi)) + g_2^{(j)}(\mathbf{x}(\xi)) + g_3^{(j)}(\mathbf{x}(\xi))] \tag{19}$$

where

$$g_1^{(j)}(\mathbf{x}(\xi)) = \frac{\mathbf{n_x}\cdot(P_{j+1}\times\mathbf{x}(\xi))}{\|P_{j+1}\times\mathbf{x}(\xi)\|^2}\left(\frac{(P_{j+1}-\mathbf{x}(\xi))\cdot\mathbf{x}(\xi)}{\|\mathbf{x}(\xi)\|}+\|P_{j+1}-\mathbf{x}(\xi)\|\right) - \frac{\mathbf{n_x}\cdot(P_j\times\mathbf{x}(\xi))}{\|P_j\times\mathbf{x}(\xi)\|^2}\left(\frac{(P_j-\mathbf{x}(\xi))\cdot\mathbf{x}(\xi)}{\|\mathbf{x}(\xi)\|}+\|P_j-\mathbf{x}(\xi)\|\right), \tag{20}$$

$$g_2^{(j)}(\mathbf{x}(\xi)) = c_1\ln\frac{[(\mathbf{x}(\xi)-P_{j+1})\cdot(P_j-P_{j+1})+\|P_j-P_{j+1}\|\|\mathbf{x}(\xi)-P_{j+1}\|]}{[(\mathbf{x}(\xi)-P_j)\cdot(P_j-P_{j+1})+\|P_j-P_{j+1}\|\|\mathbf{x}(\xi)-P_j\|]} + c_2\ln\frac{[\mathbf{x}(\xi)\cdot P_{j+1}+\|P_{j+1}\|\|\mathbf{x}(\xi)\|]}{[(\mathbf{x}(\xi)-P_{j+1})\cdot P_{j+1}+\|P_{j+1}\|\|\mathbf{x}(\xi)-P_{j+1}\|]} \\ c_3\ln\frac{[\mathbf{x}(\xi)\cdot P_j+\|P_j\|\|\mathbf{x}(\xi)\|]}{[(\mathbf{x}(\xi)-P_j)\cdot P_j+\|P_j\|\|\mathbf{x}(\xi)-P_j\|]}, \tag{21}$$

and

$$g_3^{(j)}(\mathbf{x}(\xi)) = -c_4\Big\{ \mathrm{Arctan}\frac{(P_j\times(P_{j+1}-\mathbf{x}(\xi)))\cdot((P_{j+1}-P_j)\times(\mathbf{x}(\xi)-P_j))\|P_j\|}{\|(P_j\times P_{j+1})\times(P_j\times\mathbf{x}(\xi))\|\|P_{j+1}-\mathbf{x}(\xi)\|} - \mathrm{Arctan}\frac{(P_j\times\mathbf{x}(\xi))\cdot((P_j-\mathbf{x}(\xi))\times(P_j-P_{j+1}))\|P_j\|}{\|(P_j\times P_{j+1})\times(P_j\times\mathbf{x}(\xi))\|\|P_j-\mathbf{x}(\xi)\|} + \mathrm{Arctan}\frac{(P_{j+1}\times\mathbf{x}(\xi))\cdot(P_j\times(\mathbf{x}(\xi)-P_{j+1}))\|P_j\|}{\|(P_j\times P_{j+1})\times(P_j\times\mathbf{x}(\xi))\|\|P_{j+1}-\mathbf{x}(\xi)\|} - \mathrm{Arctan}\frac{(P_{j+1}\times\mathbf{x}(\xi))\cdot(P_j\times\mathbf{x}(\xi))\|P_j\|}{\|(P_j\times P_{j+1})\times(P_j\times\mathbf{x}(\xi))\|\|\mathbf{x}(\xi)\|}\Big\}. \tag{22}$$

with $c_1 = \frac{(\mathbf{n_x}\cdot\mathbf{n_y})\|P_j-P_{j+1}\|}{\|P_j\times P_{j+1}\|}$, $c_2 = \frac{(\mathbf{n_x}\cdot\mathbf{n_y})P_{j+1}\cdot(P_j-P_{j+1})}{\|P_j\times P_{j+1}\|\|P_{j+1}\|}$, $c_3 = \frac{(\mathbf{n_x}\cdot\mathbf{n_y})P_j\cdot(P_j-P_{j+1})}{\|P_j\times P_{j+1}\|\|P_j\|}$ *and* $c_4 = \frac{P_j\cdot(P_j-P_{j+1})\,\mathbf{n_x}\cdot P_{j+1}-P_{j+1}\cdot(P_j-P_{j+1})\,\mathbf{n_x}\cdot P_j}{\|P_j\times P_{j+1}\|^2}\,\mathrm{sign}(\mathbf{n_y}\cdot\mathbf{x}(\xi))$.

This result generalizes what we have given in (14) for the case $\mathbf{x}$ and T_j belonging to the same plane and, in particular contains also the case $\mathbf{x}\in\mathring{T}_j$. Evidently, (19), as function of the outer variable of integration ξ, is made up of three parts: one containing regular trigonometric functions, one containing logarithmic functions which produce at most boundary log-singularities and one, the first part, of functions which could give rise to hypersingularities (as it is the case when the outer triangle T_i coincides with the inner one T_j).

Nevertheless, it has been shown in [Aimi and Diligenti (2002)] that for a fixed outer triangle T_i, the whole inner integral in (12), rewritten using local coordinates on the reference triangle T, as function of the outer variable of integration ξ, is made up only by logarithmic functions with at most boundary singularities and smooth trigonometric terms, since the terms of type (20) cancel each other in the whole inner sum.

Remark 2 When the external triangle T_i coincides with one of the inner triangles, T_j, we have already seen the exact singularities which come out, after the inner integration over T_j, in the variables ξ_1 and ξ_2 (see Remark 1). In the final inner sum in (12), these singularities, to be more exact, cancel only with the contribute of the results of the inner integrations over T_{j-1} and T_{j+1}. It can be shown from (20) that, as function of ξ, the integral $⨎_T \tilde{S}(\xi,\eta-\xi)\tilde{\varphi}^{(j-1)}(\eta)d\sigma_\eta$ presents a singularity of the form (up to the coefficient $-\frac{1}{4\pi}$) $\frac{2}{\xi_2}(1-\xi_1)\|P_i\|$ when $\xi_2\to 0$, and $⨎_T \tilde{S}(\xi,\eta-\xi)\tilde{\varphi}^{(j+1)}(\eta)d\sigma_\eta$ presents a singularity of the form $\frac{2}{\xi_1}(1-\xi_2)\|P_{i+1}\|$ when $\xi_1\to 0$. These singularities are exactly opposite to those found after the integration over $T_j\equiv T_i$ (see (17), (18)).

4 The panel clustering technique for the outer integration

The aim of the present paper is to couple the Panel Clustering Method (PCM) [Hackbusch and Nowak (1989)] with the inner analytical integration for the Galerkin BEM. The PCM makes use of two basic ingredients: a hierarchical structuring of the surface triangulation $\mathcal{T}_h$ called *cluster tree* and a convenient truncated expansion of the integrand. The definition of cluster tree and of η-admissibility for our approach are the following.

Definition 1 *A cluster* τ *is a non empty union of panels* T_i*, equipped with a center* $\mathbf{x}_\tau$ *and a radius* ρ_τ *such that the ball* $B_{\rho_\tau}(\mathbf{x}_\tau)=\{\mathbf{x}\in\mathbb{R}^3\ s.t.\ \|\mathbf{x}-\mathbf{x}_\tau\|\le\rho_\tau\}$ *is the smallest ball containing* τ.

Definition 2 *A cluster tree* C *is a hierarchy of clusters ordered with respect to the relation of inclusion. Any two* *clusters* τ, $\tau' \in C$ *satisfy the following*

$$\tau \neq \tau' \Rightarrow \begin{cases} \tau \subset \tau' \ \ or \\ \tau' \subset \tau \ \ or \\ \tau \cap \tau' = \emptyset, \end{cases}$$

provided that $T_h \in C$ *and* $T_i \in C$ *for all* i (fig. 1).

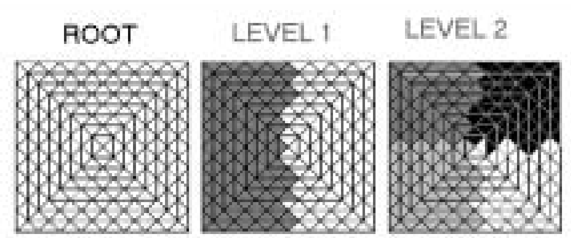


Figure 1 : Example of cluster tree constructed on a square domain equipped with a uniform triangulation (only the first three levels are represented).

Different algorithms for the construction of cluster tree have been proposed, leading to a balanced binary tree [Giebermann (to appear)] or to an oct-tree [Sauter (2000)].

Definition 3 *Let* $\eta \in (0,1)$ *be; a panel* T_j *and a cluster* τ *are said* η*-admissible if*

$$\rho_\tau < \eta\, dist(T_j, \mathbf{x}_\tau). \tag{23}$$

Here, $dist(T_j, \mathbf{x}_\tau)$ denotes the distance of the center $\mathbf{x}_\tau$ from the panel T_j.

Definition 4 *The far field of a panel* T_j *is the set of all clusters* η*-admissible to it. The near field is the far field complementary part of the triangulation* T_h.

Consider the coefficient matrix entries A_{lm} (see (11)): given a panel T_j in the support of the shape function $\psi_m(\mathbf{y})$, panels T_i in the support of the test function $\psi_l(\mathbf{x})$ may belong to the near field or to the far field of T_j. The matrix A is therefore decomposed in the sum of two matrices: $A = A^{near} + A^{far}$, the first due to the contribution of all near-fields and the second due to the contribution of all far-fields. The near field matrix elements

$$A_{lm}^{near} = \int_\Gamma \psi_l(\mathbf{x}) \ \not\!\!\int_\Gamma S(\mathbf{x}, \mathbf{y}-\mathbf{x})\psi_m(\mathbf{y}) d\Gamma_\mathbf{y} d\Gamma_\mathbf{x}$$

are evaluated analytically for the inner integration and by using recent quadrature rules for the outer weakly singular integral [Aimi and Diligenti (2002); Monegato and Scuderi (1999)]. The matrix A^{far} will be approximated.

The PCM acts on the far field matrix A^{far}: in its original form for Galerkin BEM [Hackbusch and Nowak (1989)], it considers an expansion of the kernel in both variables.

In the present work the panel clustering technique is applied after the inner analytical integration process, performing a local approximation in $\mathbf{x}$ of the function (14) (or (19) in the case that T_i and T_j don't lye on the same plane).

Owing to Remark 1, in the far-field we consider the partial result (16) of the inner analytical integration. Then, in each cluster τ belonging to the far-field of T_j, we make a local approximation of $f_2^{(j)}$ (see (16)) if T_i and T_j are on the same plane using a truncated Taylor expansion

$$f_2^{(j)}(\mathbf{x}) \cong \sum_{|\alpha|<n} G_\alpha(\mathbf{x}-\mathbf{x}_\tau) \tag{24}$$

and in order to evaluate A_{lm}^{far} we compute integrals of the form

$$\sum_{|\alpha|<n} \int_{T_i} G_\alpha(\mathbf{x}-\mathbf{x}_\tau)\varphi^{(i)}(\mathbf{x}) d\Gamma_\mathbf{x} \qquad T_i \in \tau. \tag{25}$$

Analogously we proceed on the function $g_2^{(j)} + g_3^{(j)}$ of the result (19) in the case that T_i and T_j are not on the same plane.

The difference betweeen the original PCM and (25) is that the use of analytical integration results allows the exact calculation of the inner integrals and the introduction of the truncated expansion (and of the related error) only for the evaluation of the outer integral.

5 Error estimate

In this section we consider the case of T_i and T_j lying on the same plane. In the Appendix the general case (T_i and T_j not on the same plane) is presented too. Our aim is to analyze the error due to the substitution of $f_2^{(j)}$ with its truncated Taylor expansion. Let us consider a fixed triangle T_j with vertices P_j, P_{j+1}, O where we have performed the inner analytical integration. For the outer integral, now we have to consider the function $f_2^{(j)}$, that in this section we simply indicate with f. We observe that the function f is constituted by sum of functions of the same kind, that is

$$f(\mathbf{x}) = \sum_{i=1}^{3} c_i h_i(\mathbf{x}) \tag{26}$$

where, for instance

$$h_1(\mathbf{x}) = \tag{27}$$

$$\ln \frac{[(\mathbf{x}-P_{j+1})\cdot(P_j-P_{j+1}) + \|P_j-P_{j+1}\|\|\mathbf{x}-P_{j+1}\|]}{[(\mathbf{x}-P_j)\cdot(P_j-P_{j+1}) + \|P_j-P_{j+1}\|\|\mathbf{x}-P_j\|]}.$$

Now we will work on h_1 and we will finally extend the results to h_2 and h_3. If we consider a Taylor expansion of order $n-1$ ($n \geq 1$) of h_1, the remainder $R_n(\mathbf{x}_\tau, \mathbf{x}_\tau - \mathbf{x})$ has the form

$$R_n(\mathbf{x}_\tau, \mathbf{x}_\tau - \mathbf{x}) = \frac{1}{n!}\left(\frac{d}{dt}\right)^n \Phi_1(t)$$

where

$$\Phi_1(t) := h_1(\mathbf{x}_\tau - t(\mathbf{x}_\tau - \mathbf{x})), \qquad t \in (0,1).$$

If we use the Cauchy integral formula, we can rewrite $R_n(\mathbf{x}_\tau, \mathbf{x}_\tau - \mathbf{x})$ in the form:

$$R_n(\mathbf{x}_\tau, \mathbf{x}_\tau - \mathbf{x}) = \frac{1}{2\pi i}\oint_{|z|=r} \frac{\Phi_1(z)}{(z-t)^{n+1}} \mathrm{d}z \quad \text{ for } |t| < r < R$$

where $\Phi_1(z)$ is the analytical continuation of Φ_1 in the complex plane and $|z| < R$ is a region where $\Phi_1(z)$ is holomorphic. In the following Theorem 3 we solve the problem of finding a radius R such that $\Phi_1(z)$ is holomorphic for $|z| < R$. We observe that, having set

$$\xi := \mathbf{x}_\tau - P_{j+1}, \quad \tilde{\xi} := \mathbf{x}_\tau - P_j,$$

$$\beta := P_j - P_{j+1}, \quad \mu := \mathbf{x}_\tau - \mathbf{x}, \tag{28}$$

we have

$$\Phi_1(t) = \ln \frac{[(\xi - t\mu)\cdot\beta + \|\beta\|\|\xi - t\mu\|]}{[(\tilde{\xi} - t\mu)\cdot\beta + \|\beta\|\|\tilde{\xi} - t\mu\|]}$$
$$= \ln \frac{\left[|\xi - t\mu| + \frac{(\xi - t\mu)\cdot\beta}{\|\beta\|}\right]}{\left[|\tilde{\xi} - t\mu| + \frac{(\tilde{\xi} - t\mu)\cdot\beta}{\|\beta\|}\right]}. \tag{29}$$

The analytical continuation in $\mathbb{C}$ of (29) is:

$$\Phi_1(z) = \ln \frac{\left[\left(\sum_{i=1}^{2}(\xi_i - z\mu_i)^2\right)^{1/2} + \sum_{i=1}^{2}\frac{(\xi_i - z\mu_i)\beta_i}{\|\beta\|}\right]}{\left[\left(\sum_{i=1}^{2}(\tilde{\xi}_i - z\mu_i)^2\right)^{1/2} + \sum_{i=1}^{2}\frac{(\tilde{\xi}_i - z\mu_i)\beta_i}{\|\beta\|}\right]}. \tag{30}$$

The following fundamental result holds.

Theorem 3 *The function $\Phi_1(z)$, $z \in \mathbb{C}$, defined in* (30), *is holomorphic for*

$$|z| < \min\left\{\frac{\|\xi\|}{\|\mu\|}, \frac{\|\tilde{\xi}\|}{\|\mu\|}\right\}.$$

Proof. We are looking for a circle with center in the origin of the complex plane in which the function $\Phi_1(z)$ is analytical. Let us start considering $z = 0$. In this case the logarithmic function in (30) is well defined if

$$0 < \frac{\|\xi\|\|\beta\| + \xi\cdot\beta}{\|\tilde{\xi}\|\|\beta\| + \tilde{\xi}\cdot\beta} < \infty,$$

that is for $\xi \neq -\alpha_1\beta$ and $\tilde{\xi} \neq -\alpha_2\beta$, with $\alpha_1, \alpha_2 \in \mathbb{R}^+$ (note that, from definitions (28), $\tilde{\xi} = \xi - \beta$). But even if this happens, we observe that ξ, $\tilde{\xi}$ and β have all the same direction and $\exists\gamma \in \mathbb{R}^+$, such that $\tilde{\xi} = \gamma\xi$. With a limiting process we obtain

$$\Phi_1(0) = \ln(\gamma),$$

and the logarithm is well defined in $z = 0$ also in this geometrical case.

Now our aim is to determine the radius of the circle centered in $z = 0$ where $\Phi_1(z)$ is holomorphic. Hence we must find the points in the complex plane where the numerator, the denominator, the arguments of the square roots in (30) vanish. Considering the numerator, we have

$$\left(\sum_{i=1}^{2}(\xi_i - z\mu_i)^2\right)^{1/2} = 0 \quad \text{for } \hat{z} = \frac{(\xi\cdot\mu) \pm i(\xi\cdot\mu^\perp)}{\|\mu\|^2},$$
$$\text{with } |\hat{z}| = \frac{\|\xi\|}{\|\mu\|}. \tag{31}$$

$$\left[\left(\sum_{i=1}^{2}(\xi_i - z\mu_i)^2\right)^{1/2} + \sum_{i=1}^{2}\frac{(\xi_i - z\mu_i)\beta_i}{\|\beta\|}\right] = 0$$

$$\text{for } \hat{z}^* = \frac{(\xi\cdot\beta^\perp)}{(\mu\cdot\beta^\perp)} \quad \text{with } (\mu\cdot\beta^\perp)\neq 0. \tag{32}$$

Observe that $\hat{z}^* \neq 0$ if and only if $\xi\cdot\beta^\perp \neq 0$, that implies $\mu\cdot\beta^\perp\neq 0$ (in fact if $\mu\cdot\beta^\perp = 0$, then from the definitions (28), $\xi\cdot\beta^\perp = 0$ too). Analogous results hold for the denominator of (30): $\tilde{z} = \frac{(\tilde{\xi}\cdot\mu)\pm i(\tilde{\xi}\cdot\mu^\perp)}{\|\mu\|^2}$, $|\tilde{z}| = \frac{\|\tilde{\xi}\|}{\|\mu\|}$ and $\tilde{z}^* = \frac{(\tilde{\xi}\cdot\beta^\perp)}{(\mu\cdot\beta^\perp)}$ with $(\mu\cdot\beta^\perp)\neq 0$. Since $\hat{z}^* = \tilde{z}^* =: z^*$, the numerator and the denominator in (30) vanish for the same value of the variable z, hence we have an indeterminate form. Using L'Hôspital theorem iteratively, we have

$$\lim_{z\to z^*}\Phi_1(z) = \frac{|\mu\cdot\tilde{\xi}^\perp|}{|\mu\cdot\xi^\perp|}. \tag{33}$$

The result in (33) is finite and different from zero if $\mu\cdot\xi^\perp\neq 0$ and $\mu\cdot\tilde{\xi}^\perp\neq 0$, respectively. In these cases $\hat{z}^* = \tilde{z}^*$ gives no problem. Otherwise if $\xi\cdot\mu^\perp = 0$, then $\exists\tilde{\gamma}\in\mathbb{R}$ such that $\mu = \tilde{\gamma}\xi$, hence, substituting in $\frac{(\xi\cdot\beta^\perp)}{(\mu\cdot\beta^\perp)}$, we have that $\Phi_1(z)$ is holomorphic in $|z| < \frac{1}{|\tilde{\gamma}|} = \frac{\|\xi\|}{\|\mu\|}$ (analogously, for $\mu\cdot\tilde{\xi}^\perp = 0$, we obtain the circle $|z| < \frac{\|\tilde{\xi}\|}{\|\mu\|}$).
We can conclude that, considering all the possible situations, the function $\Phi_1(z)$ is holomorphic for $|z| < \min\left\{\frac{\|\xi\|}{\|\mu\|}, \frac{\|\tilde{\xi}\|}{\|\mu\|}\right\}$ and the theorem is proved.

Observing that analogous results hold starting from $h_j,\ j = 2,3$ (see (26)), we can prove the following

Theorem 4 *The function $\Phi(z)$, $z\in\mathbb{C}$, analytical continuation of $f(\mathbf{x}_\tau - t(\mathbf{x}_\tau - \mathbf{x}))$, $t\in(0,1)$, defined in (26), is holomorphic for*

$$|z| < R, \qquad \textit{where} \ \ R := \frac{dist(\mathbf{x}_\tau, T_j)}{\|\mu\|}. \tag{34}$$

Proof. Observing that $\Phi(z) = \sum_{k=1}^{3}\Phi_j(z)$ and using the previously obtained results of Theorem 3, we have that $\Phi(z)$ is analytical for

$$|z| < \min\left\{\frac{\|\mathbf{x}_\tau\|}{\|\mu\|}, \frac{\|\mathbf{x}_\tau - P_j\|}{\|\mu\|}, \frac{\|\mathbf{x}_\tau - P_{j+1}\|}{\|\mu\|}\right\}.$$

Since

$$dist(\mathbf{x}_\tau, T_j) \leq \min\left\{\|\mathbf{x}_\tau\|, \|\mathbf{x}_\tau - P_j\|, \|\mathbf{x}_\tau - P_{j+1}\|\right\}$$

we have the thesis.

Theorem 5 *The remainder of the Taylor expansion of order $n-1$ of the function f defined in (26) satisfies the inequality*

$$|R_n(\mathbf{x}_\tau, \mathbf{x}_\tau - \mathbf{x})| \leq M 2^{n+1}\eta^n$$

$$\textit{where } M := \max_{|z|\leq r}|\Phi(z)|, \quad r < R, \tag{35}$$

where R is defined in (34).

Proof. Having set $\alpha := \frac{\|\mu\|}{dist(\mathbf{x}_\tau, T_j)} \leq \eta < \eta_0 < 1$, we use the Cauchy integral formula

$$R_n(\mathbf{x}_\tau, \mathbf{x}_\tau - \mathbf{x}) = \frac{1}{2\pi i}\oint_{|z|=r}\frac{\Phi(z)}{(z-t)^{n+1}}\,\mathrm{d}z \quad \text{for } |t| < r < \frac{1}{\alpha}.$$

If we consider $0 < t < 1 < r$, then $|z-t| \geq r-1$. Hence if we choose $\eta_0 = \frac{1}{2}$, $r = 1 + \frac{1}{2\alpha} \in (1, \frac{1}{\alpha})$ and using the η-admissibility (23), we can conclude

$$\begin{aligned} |R_n(\mathbf{x}_\tau, \mathbf{x}_\tau - \mathbf{x})| &\leq rM(r-1)^{-n-1} \leq 2^{n+1}M\alpha^n \\ &\leq 2^{n+1}M\frac{\|\mu\|^n}{dist(\mathbf{x}_\tau, T_j)^n} \leq 2^{n+1}M\eta^n. \end{aligned} \tag{36}$$

6 Numerical results

In this section, we show the effectiveness of the method proposed in Section 4 with three examples.

6.1 A plane square domain

Consider a square domain $\Omega = [-2,2]^2$, equipped with a uniform triangulation (256 triangles) and we choose a triangle T_0 and a cluster τ_0 η-admissible to it. In fig. 2 the black triangle represents T_0, and the circle $B_{\mathbf{x}_{\tau_0}}$ determines the cluster τ_0 (painted in gray) with $\eta = 0.2$.

As a test-problem we considered the evaluation of the integral

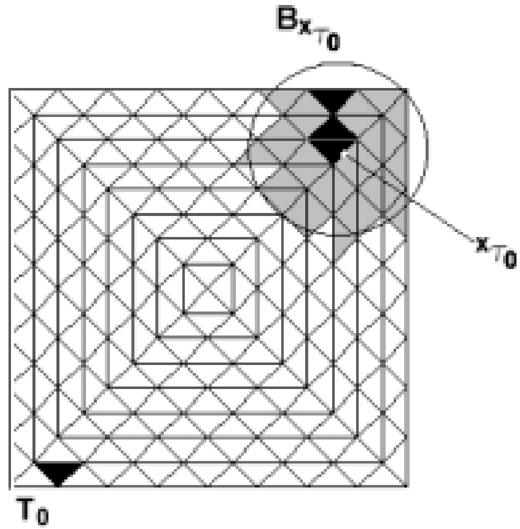


Figure 2 : The domain considered for the numerical test: the triangle T_0 (black) and a circle $B_{\mathbf{x}_{\tau_0}}$ that defines a cluster (in gray) η-admissible to T_0 (η=0.2). Inside of the cluster there are T_1, T_2, T_3 (in black).

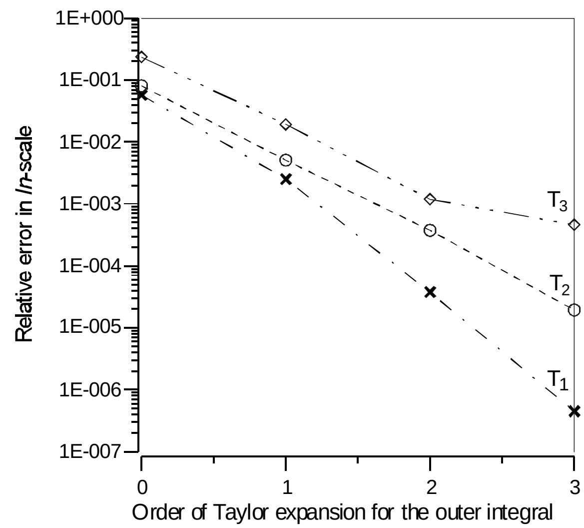


Figure 3 : Convergence of the *PCMa* method for different triangles in the cluster ($\eta = 0.2$).

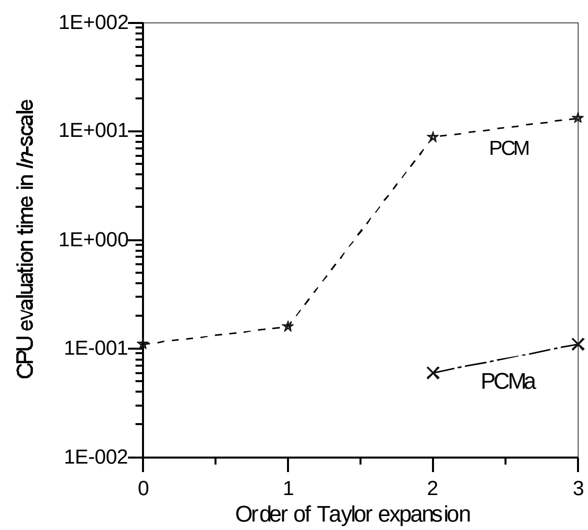


Figure 4 : Comparison between CPU-evaluation times for *PCM* and *PCMa* methods calculated for the triangle T_3 ($\eta = 0.2$).

$$I = \int_{T_i} \varphi^{(i)}(\mathbf{x}) \int_{T_0} \frac{1}{\|\mathbf{x}-\mathbf{y}\|^3} \varphi^{(0)}(\mathbf{y}) d\Gamma_{\mathbf{y}}\, d\Gamma_{\mathbf{x}}, \quad i = 1,2,3 \tag{37}$$

where $T_i \in \tau_0$ are three triangles ordered with respect to the distance $d_i := \|\mathbf{b}_{T_i} - \mathbf{x}_{\tau_0}\|$ with $\mathbf{b}_{T_i}$ the baricenter of T_i, $\varphi^{(i)}, \varphi^{(0)}$ are linear shape functions. In fig.2 $T_1, T_2, T_3 \in \tau_0$ are painted in black. In fig. 3 the behavior of the relative error for different orders of Taylor expansion for the outer integral after the inner analytical integration (with respect to a reference value obtained with a double Gaussian numerical quadrature with 16 digits precision) is shown. As we expected, the error grows as far as d_i ($i = 1,2,3$) grows. On the other hand the relative error decreases when we increase the order $n-1$ of Taylor expansion ($n = 1,2,3,4$). In table 1 we show a comparison between results obtained with our technique (*PCMa*) for integral (37) and the Panel Clustering Method (here indicated as *PCM*) for triangles T_1, T_2 and T_3. In the first case (triangle T_1) the center of expansion $\mathbf{x}_{\tau_0} \in T_1$ and we can observe that the relative error decreases more rapidly with *PCMa* approach. If we consider T_3 the behavior of *PCMa* and *PCM* method are similar (in fact the speed of convergence of the two methods is substantially the same). Nevertheless a comparison between employed CPU-times shows immediately that the *PCMa* approach has a very smaller computational cost (fig. 4). This is true for all the situations described. The calculations were made by using the software MATHEMATICA 4.0 [Wolfram (1999)]. Note that the symbol "$-$" introduced in the tables means that the CPU-evaluation time is less than the time unit measurable by MATHEMATICA. We used

$T_1, d_1 = 0.1318$				
	Relative error		*CPU − time(sec)*	
	PCM	*PCMa*	*PCM*	*PCMa*
$n = 1$	$9.589E-2$	$5.752E-2$	0.11	–
$n = 2$	$7.345E-3$	$2.513E-3$	0.16	–
$n = 3$	$3.957E-4$	$3.767E-5$	8.90	0.05
$n = 4$	$1.252E-5$	$4.494E-7$	17.75	0.06

$T_2, d_2 = 0.2430$				
	Relative error		*CPU − time(sec)*	
	PCM	*PCMa*	*PCM*	*PCMa*
$n = 1$	$3.666E-2$	$8.066E-2$	0.11	–
$n = 2$	$3.788E-3$	$5.081E-3$	0.16	–
$n = 3$	$2.508E-4$	$3.744E-4$	9.23	0.06
$n = 4$	$1.305E-5$	$1.934E-5$	21.97	0.11

$T_3, d_3 = 0.5560$				
	Relative error		*CPU − time(sec)*	
	PCM	*PCMa*	*PCM*	*PCMa*
$n = 1$	$1.860E-1$	$2.362E-1$	0.11	–
$n = 2$	$5.440E-3$	$1.928E-2$	0.16	–
$n = 3$	$2.036E-3$	$1.187E-3$	8.84	0.06
$n = 4$	$2.270E-4$	$4.611E-4$	13.24	0.11

Table 1 : Comparison between the results for T_1, T_2 and T_3 ($\eta = 0.2$).

$T_1, d_1 = 0.1318$				
	Relative error		*CPU − time(sec)*	
	PCM	*PCMa*	*PCM*	*PCMa*
$n = 1$	$1.130E-1$	$6.775E-2$	0.16	–
$n = 2$	$1.013E-2$	$3.445E-3$	0.22	–
$n = 3$	$6.266E-4$	$5.489E-5$	9.89	0.05
$n = 4$	$2.077E-5$	$8.958E-7$	21.59	0.17

$T_3, d_3 = 0.5560$				
	Relative error		*CPU − time(sec)*	
	PCM	*PCMa*	*PCM*	*PCMa*
$n = 1$	$2.329E-1$	$2.958E-1$	0.11	–
$n = 2$	$1.003E-2$	$3.137E-2$	0.11	–
$n = 3$	$3.400E-3$	$1.594E-3$	13.18	0.06
$n = 4$	$5.068E-4$	$9.684E-4$	21.70	0.17

$T_5, d_5 = 1.0491$				
	Relative error		*CPU − time(sec)*	
	PCM	*PCMa*	*PCM*	*PCMa*
$n = 1$	$6.831E-1$	$7.690E-1$	0.05	–
$n = 2$	$1.755E-1$	$2.464E-1$	0.10	–
$n = 3$	$2.222E-2$	$2.759E-2$	14.50	0.05
$n = 4$	$1.454E-2$	$1.248E-2$	23.62	0.17

Table 2 : Comparison between the results for T_1, T_3 and T_5 ($\eta = 0.5$).

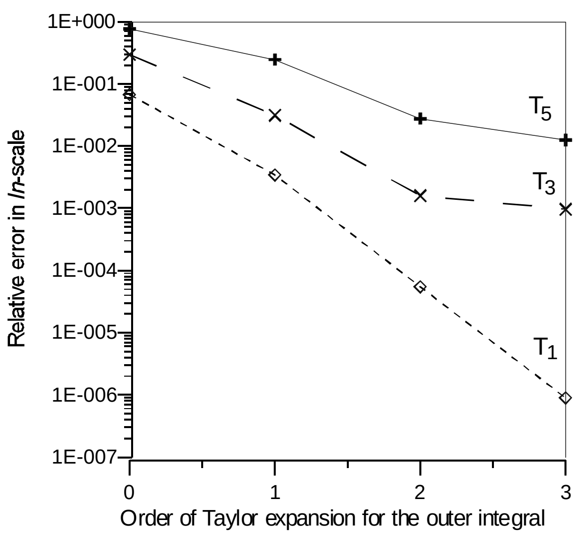


Figure 5 : Convergence of the *PCMa* method for different triangles (η=0.5)

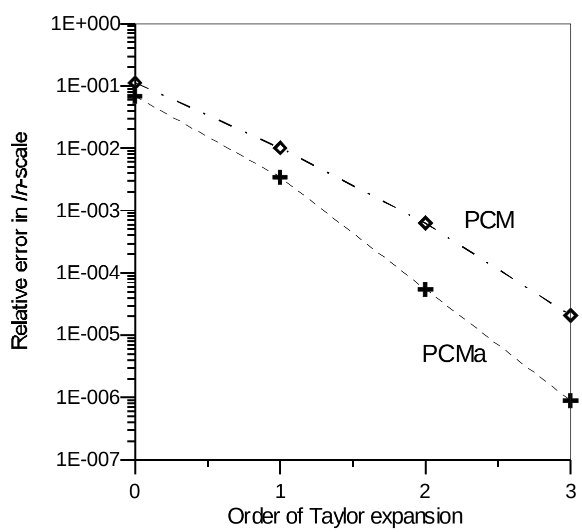


Figure 6 : Comparison: speed of convergence of *PCM* and *PCMa* methods for the triangle T_1, that includes the center of expansion (η=0.5)

a personal computer with a Pentium III processor and in this case the time unit is $1/1000$ sec.

In table 2 and in figures 5-6 analogous results are presented using $\eta = 0.5$. In this case the cluster becomes greater and for the numerical tests we chose three triangles (T_1, T_3, T_5) with distances form the center of the cluster respectively $d_1 = 0.131762, d_3 = 0.555903, d_5 = 1.04914$.

6.2 A polygonal surface in $\mathbb{R}^3$

In this case we considered a surface piece Γ lying on the boundary of a cube $\Omega = [-2,2]^3$, equipped with a uniform triangulation. As in the previous case, we choose a triangle T_0 and a cluster τ η-admissible to it and, as a test-problem, we considered the evaluation of the integral

$$I = \int_{T_i} \varphi^{(i)}(\mathbf{x}) \int_{T_0} \left[\frac{\mathbf{n_x} \cdot \mathbf{n_y}}{\|\mathbf{x}-\mathbf{y}\|^3} - 3 \frac{\mathbf{r} \cdot \mathbf{n_x} \, \mathbf{r} \cdot \mathbf{n_y}}{\|\mathbf{x}-\mathbf{y}\|^5} \right] \varphi^{(0)}(\mathbf{y}) d\Gamma_{\mathbf{y}} \, d\Gamma_{\mathbf{x}},$$
$$i = 1, 2, \ldots 5, \quad (38)$$

where $T_i \in \tau$ are triangles in the cluster. In fig.7 the triangle T_0 and the cluster η-admissible to it (with $\eta = 0.2$) considered in the numerical tests are shown. In table 3 and in figures 8-11 the relative errors and the CPU-times calculated for different triangles in the η-admissible cluster and for different orders of expansion are presented, in the case of *PCM* and *PCMa* methods. One can observe that the speed of convergence of the two methods is similar but the CPU-time used for *PCMa* is much smaller than that of *PCM* method.

6.3 A square plane crack in an infinite elastic medium

We consider a pressurized square crack Γ lying in the plane $\mathbf{e}_2 \times \mathbf{e}_3$, embedded in an infinite isotropic elastic medium with negligible volume forces and subject to remote uniform loading $\sigma_\infty \mathbf{e}_1$ (fig. 12a). The two crack faces Γ^+, Γ^- are geometrically identical surfaces having opposite unit normal vectors $\mathbf{n}^+, \mathbf{n}^-$.

The formulation of the problem reads

$$\oint_\Gamma \mathbf{G}_{pp}(\mathbf{x}-\mathbf{y}; -\mathbf{e}_1; -\mathbf{e}_1) \, \mathbf{w}(\mathbf{y}) \, d\Gamma_{\mathbf{y}} = -\sigma_\infty \mathbf{e}_1 \,, \qquad \mathbf{x} \in \Gamma \quad (39)$$

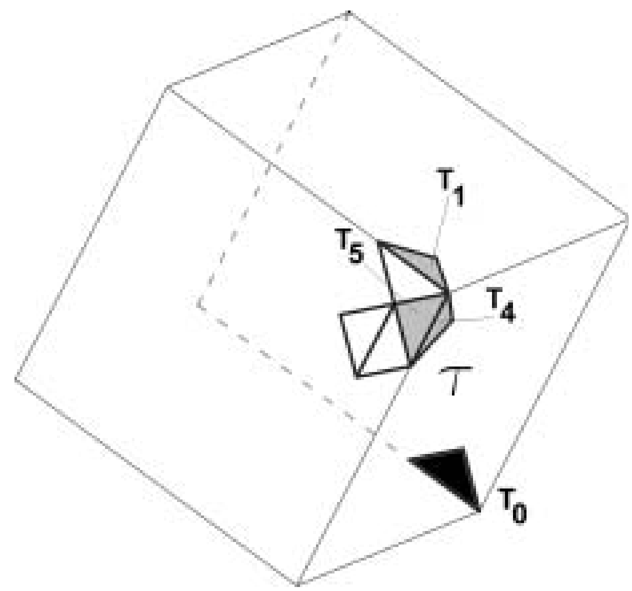


Figure 7 : Second example: the triangle T_0 (black) and a cluster τ η-admissible to T_0 (η=0.2).

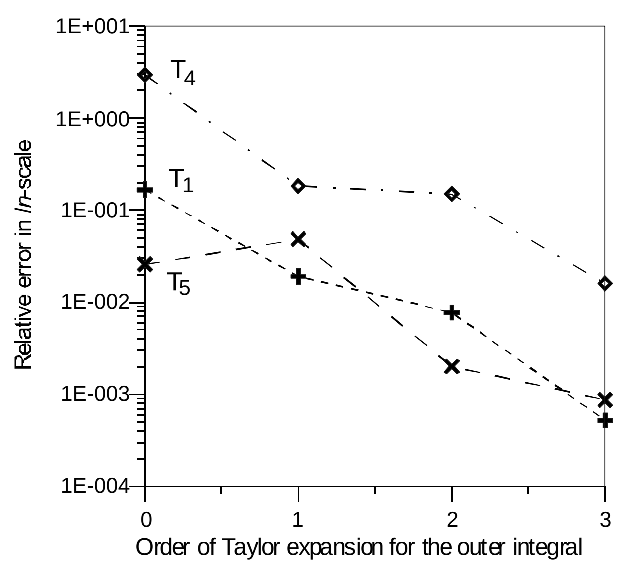


Figure 8 : Second example: convergence of the *PCMa* method for different triangles (η=0.2).

$T_1, d_1 = 0.5651$				
	Relative error		$CPU - time(sec)$	
	PCM	*PCMa*	*PCM*	*PCMa*
$n = 1$	$1.579E-1$	$1.667E-1$	0.16	–
$n = 2$	$2.796E-2$	$1.910E-2$	0.17	–
$n = 3$	$1.072E-2$	$7.735E-3$	19.45	0.06
$n = 4$	$6.552E-4$	$5.217E-4$	62.18	0.11

$T_4, d_4 = 0.5894$				
	Relative error		$CPU - time(sec)$	
	PCM	*PCMa*	*PCM*	*PCMa*
$n = 1$	$3.253E-0$	$2.952E-0$	0.06	–
$n = 2$	$2.682E-1$	$1.830E-1$	0.11	–
$n = 3$	$2.010E-1$	$1.494E-1$	16.69	0.05
$n = 4$	$2.827E-2$	$1.603E-2$	106.89	0.17

$T_5, d_5 = 0.6346$				
	Relative error		$CPU - time(sec)$	
	PCM	*PCMa*	*PCM*	*PCMa*
$n = 1$	$4.578E-2$	$2.576E-2$	0.16	–
$n = 2$	$5.752E-2$	$4.820E-2$	0.11	–
$n = 3$	$1.678E-3$	$2.009E-3$	35.37	0.11
$n = 4$	$1.584E-3$	$8.747E-4$	73.93	0.21

Table 3 : Second example: comparison between the results of *PCM* and *PCMa* methods. Relative errors for different triangles in the η-admissible cluster (η=0.2) and different orders $n-1$ of expansion (T_1, T_4, T_5 lye on different faces - see fig. 7).

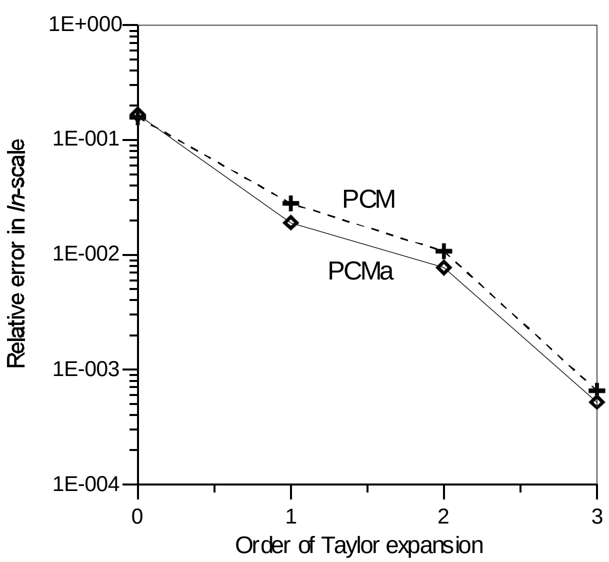


Figure 9 : Second example: speed of convergence of *PCMa* and *PCM* methods for the triangle T_1 (η=0.2).

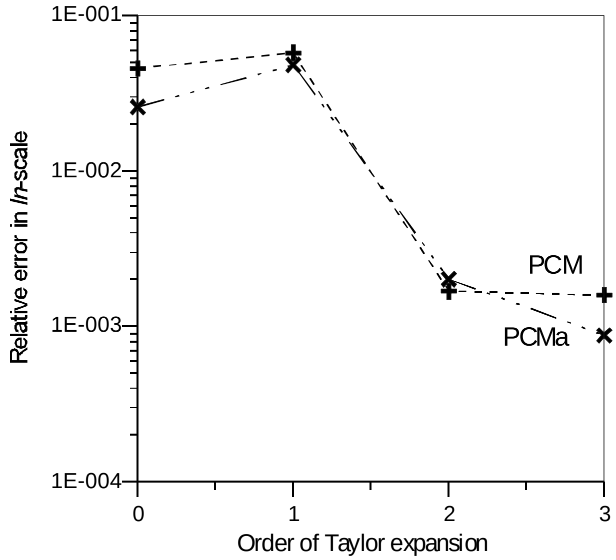


Figure 10 : Second example: speed of convergence of *PCMa* and *PCM* methods for the triangle T_5 (η=0.2).

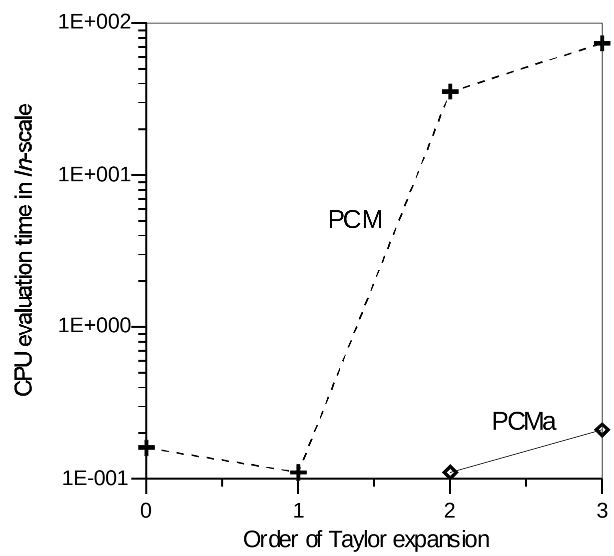
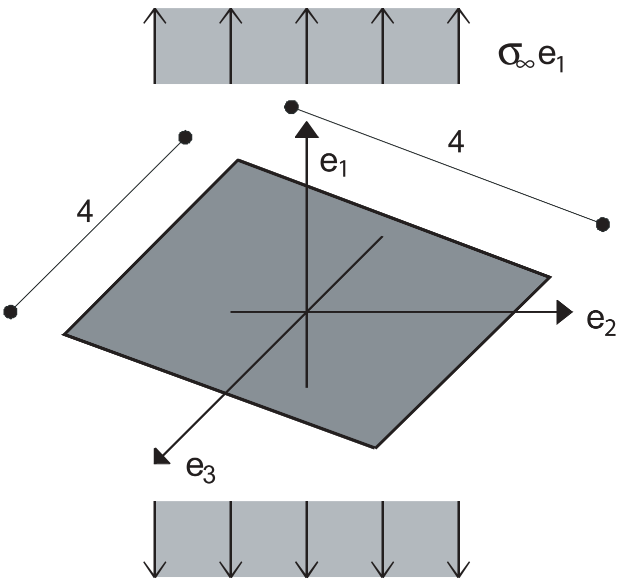
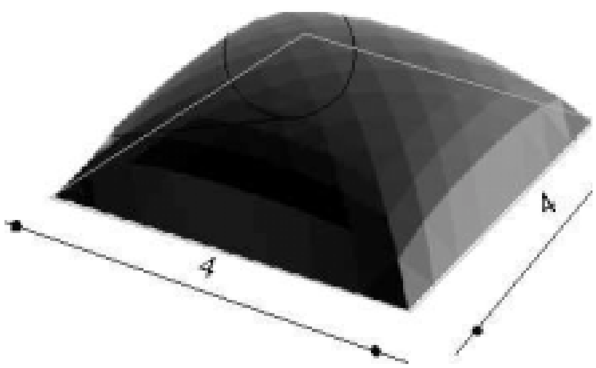


Figure 11 : Second example: comparison between CPU-evaluation times for *PCM* and *PCMa* methods calculated for the triangle T_5 (η=0.2).



(a)



(b)

Figure 12 : Scheme of the problem of a square plane crack in an infinite elastic medium (a). Illustration of the numerical solution with $\sigma_\infty = E = 10^6$ (stress unit), $\nu = 0.2$ (b).

where kernel $\mathbf{G}_{pp}$ yields

$$\mathbf{G}_{pp}(\mathbf{r};-\mathbf{e}_1;-\mathbf{e}_1) =$$

$$\frac{E\nu}{8\pi(1-\nu^2)}\frac{1}{r^3}\left\{2(\mathbf{e}_1\otimes\mathbf{e}_1)+3\frac{\mathbf{r}\otimes\mathbf{r}}{r^2}+\frac{(1-2\nu)}{\nu}\mathbf{I}\right\}$$

and $\mathbf{w}$ is the crack opening at $\mathbf{x}\in\Gamma$, $\mathbf{r}=\mathbf{x}-\mathbf{y}$, E Young's modulus for the material and ν Poisson's coefficient.

If one writes equation (39) by components ($\mathbf{x}\in\Gamma$)

$$\frac{E\nu}{8\pi(1-\nu^2)} \,=\!\!\!\!\!\!\int_\Gamma \frac{1}{r^3}\begin{bmatrix} \frac{w_1(\mathbf{y})}{\nu} \\ \left(\frac{1-2\nu}{\nu}+3\frac{r_2^2}{r^2}\right)w_2(\mathbf{y})+3\frac{r_2 r_3}{r^2}w_3(\mathbf{y}) \\ 3\frac{r_2 r_3}{r^2}w_2(\mathbf{y})+\left(\frac{1-2\nu}{\nu}+3\frac{r_3^2}{r^2}\right)w_3(\mathbf{y}) \end{bmatrix} d\Gamma_{\mathbf{y}}$$

$$= \begin{bmatrix} -\sigma_\infty \\ 0 \\ 0 \end{bmatrix} \tag{40}$$

one notes that a solution for (39) is

$$w_2(\mathbf{y}) = w_3(\mathbf{y}) = 0\,,$$

$$w_1(\mathbf{y}) \quad \text{s. t.} \quad =\!\!\!\!\!\!\int_\Gamma \frac{w_1(\mathbf{y})}{r^3}\,d\Gamma_{\mathbf{y}} = -8\pi\frac{1-\nu^2}{E}\sigma_\infty. \tag{41}$$

It is worth stressing here that even if $\frac{w_1(\mathbf{y})}{r^3} \geq 0$ when $\mathbf{x},\mathbf{y}\in\Gamma$ its finite part of Hadamard can be negative, as it is expected in equation (41-b). For a larger comprehension, see [Carini and Salvadori (2002)]. Further, in the context of a more general 3D non-planar crack analysis, we recall here the efficient method recently proposed in [Nikishkov, Park and Atluri (2001)].

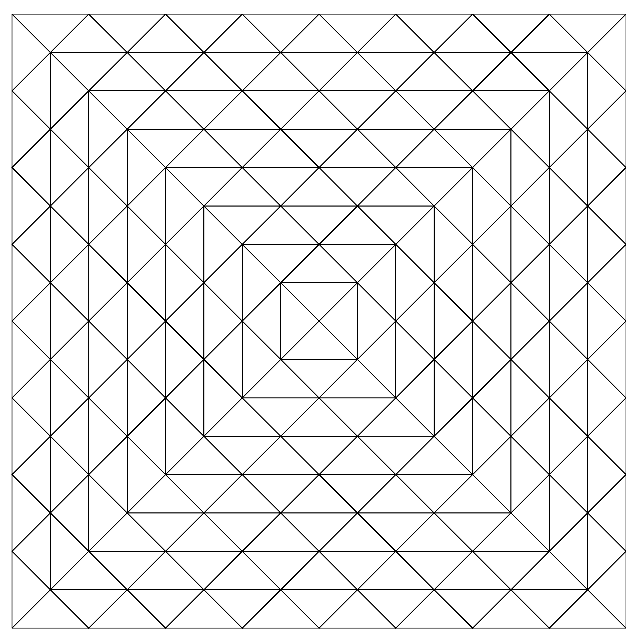


Figure 13 : decomposition of the square plane crack with 256 triangles (uniform triangulation).

	64 Triangles		256 Triangles		1024 Triangles	
η	El. time (sec)	%	El. time (sec)	%	El. time (sec)	%
0 up to 0.1	28	100%	602	100%	11059	100%
0.2	28	100%	352	68%	2247	20%
0.3	25	86%	209	35%	1202	11%
0.4	20	69%	140	23%	768	7%
0.5	15.5	54%				

Table 4 : Square plane crack: results of the implementation of the method in terms of elapsed time for three uniform grids.

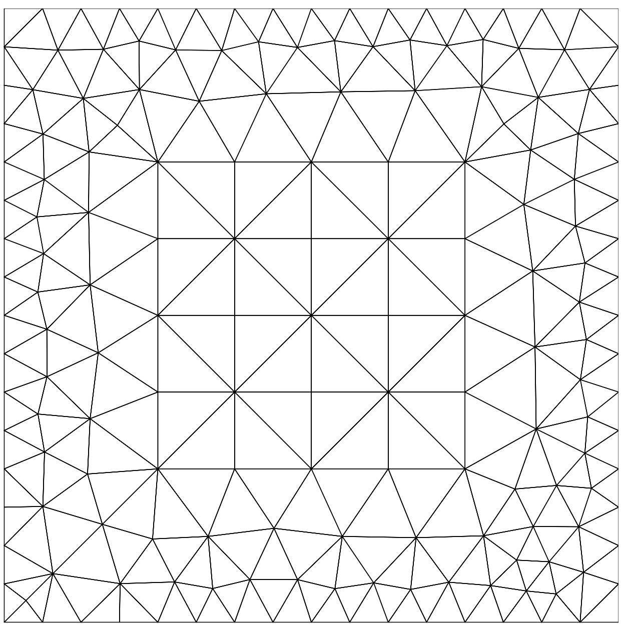


Figure 14 : decomposition of the square plane crack with a geometrical mesh (281 triangles).

In table 4 are presented the elapsed times for the evaluation, by using *PCMa* method, of the approximate solution of the problem using various uniform grids (fig. 13) and various values for η.

Table 5 illustrates the behavior of the method proposed in this note: for each triangulation, all the values and the the elapsed times are calculated with respect to the reference value calculated with $\eta = 0$ (that means no PCM); in the last column we show results obtained with the geometrical mesh presented in fig. 14. This type of decomposition has been used for the particular nature of the problem and can be used for a comparison with the uniform mesh with 256 triangles. Table 5 shows that, for each grid, if we increase η the error grows but, correspondingly, the elapsed time becomes smaller and smaller.

Appendix

In this appendix we want to study the error estimate in the general case in which T_i and T_j are not on the same plane (see Theorem 2). Let us consider a fixed triangle T_j with vertices P_j, P_{j+1}, O where we have performed the inner analytical integration. For the outer integral now we have to consider the sum of the functions (21) and (22), that in this appendix we simply indicate with g_2 and g_3, and we define $g := g_2 + g_3$.

The function g_2

We observe that the function g_2 is constituted by sum of functions of the same kind, that is

$$g_2(\mathbf{x}) = \sum_{i=1}^{3} c_i h_i(\mathbf{x}) \tag{42}$$

where, for instance

$$h_1(\mathbf{x}) = \ln \frac{[(\mathbf{x} - P_{j+1}) \cdot (P_j - P_{j+1}) + \|P_j - P_{j+1}\| \|\mathbf{x} - P_{j+1}\|]}{[(\mathbf{x} - P_j) \cdot (P_j - P_{j+1}) + \|P_j - P_{j+1}\| \|\mathbf{x} - P_j\|]}. \tag{43}$$

If we consider a Taylor expansion of order $n-1$ of h_1, the remainder $R_n(\mathbf{x}_\tau, \mathbf{x}_\tau - \mathbf{x})$ has the expression

$$R_n(\mathbf{x}_\tau, \mathbf{x}_\tau - \mathbf{x}) = \frac{1}{n!} \left(\frac{d}{dt} \right)^n \Phi_1(t)$$

where $\Phi_1(t) := h_1(\mathbf{x}_\tau - t(\mathbf{x}_\tau - \mathbf{x}))$, $t \in (0,1)$.

If we use the Cauchy integral formula, we have to consider $\Phi_1(z)$ (the analytical continuation of $\Phi_1(t)$ in the complex plane) and where it is holomorphic. We observe

	64 triangles		256 triangles		1024 triangles		geom. mesh	
η	rel. error	El. time	rel. error	El. time	rel. error	El. time	rel. error	El. time
0		**100%**		**100%**		**100%**		**100%**
0.2	0%	100%	0.31%	58%	1.48%	20%	0.45%	45%
0.3	0.8%	86%	1.95%	35%	6.01%	11%	2.75%	28%
0.4	0.68%	96%	5.45%	23%	14.03%	7%	8.50%	19%
0.5	2.66%	54%						

Table 5 : Square plane crack: relative errors and the Elapsed times calculated with respect to the reference value calculated with $\eta = 0$, that means no PCM.

that, having set

$$\xi := \mathbf{x}_\tau - P_{j+1}, \qquad \tilde{\xi} := \mathbf{x}_\tau - P_j,$$

$$\beta := P_j - P_{j+1}, \qquad \mu := \mathbf{x}_\tau - \mathbf{x}. \tag{44}$$

$$\Phi_1(t) = \ln \frac{[(\xi - t\mu)\cdot\beta + \|\beta\|\|\xi - t\mu\|]}{[(\tilde{\xi} - t\mu)\cdot\beta + \|\beta\|\|\tilde{\xi} - t\mu\|]} = \ln \frac{\left[|\xi - t\mu| + \frac{(\xi - t\mu)\cdot\beta}{\|\beta\|}\right]}{\left[|\tilde{\xi} - t\mu| + \frac{(\tilde{\xi} - t\mu)\cdot\beta}{\|\beta\|}\right]}. \tag{45}$$

The following fundamental result holds.

Theorem 6 *The function $\Phi_1(z)$, analytical continuation of* (45) *in $\mathbb{C}$, is holomorphic for*

$$|z| < \min\left\{\frac{\|\xi\|}{\|\mu\|}, \frac{\|\tilde{\xi}\|}{\|\mu\|}\right\}.$$

Proof. The analytical continuation in $\mathbb{C}$ of (45) is

$$\Phi_1(z) = \ln \frac{\left[\left(\sum_{i=1}^{3}(\xi_i - z\mu_i)^2\right)^{1/2} + \sum_{i=1}^{3}\frac{(\xi_i - z\mu_i)\beta_i}{\|\beta\|}\right]}{\left[\left(\sum_{i=1}^{3}(\tilde{\xi}_i - z\mu_i)^2\right)^{1/2} + \sum_{i=1}^{3}\frac{(\tilde{\xi}_i - z\mu_i)\beta_i}{\|\beta\|}\right]}. \tag{46}$$

We are looking for a circle with center in the origin of the complex plane in which the function Φ_1 is analytical.

Let us start considering $z = 0$. In this case the logarithm in (46) is well defined if

$$0 < \frac{\|\xi\|\|\beta\| + \xi\cdot\beta}{\|\tilde{\xi}\|\|\beta\| + \tilde{\xi}\cdot\beta} < \infty,$$

that is for $\xi \neq -\alpha_1\beta$ and $\tilde{\xi} \neq -\alpha_2\beta$, with $\alpha_1, \alpha_2 \in \mathbb{R}^+$ (note that, from the definitions (44) $\tilde{\xi} = \xi - \beta$). But even if this happens, we observe that ξ, $\tilde{\xi}$ and β have all the same direction in the complex plane and $\exists\gamma \in \mathbb{R}^+$, such that $\tilde{\xi} = \gamma\xi$. With a limit process we obtain

$$\Phi_1(0) = \ln(\gamma).$$

Since $\gamma > 0$ the logarithm is well defined also in this geometrical case.

Now our aim is to determine the radius of the circle centered in $z = 0$ where $\Phi_1(z)$ is holomorphic. Hence we must find the points where the numerator, the denominator, the arguments of the square roots in (46) vanish. Considering the numerator, we have to solve two equations

$$\left(\sum_{i=1}^{3}(\xi_i - z\mu_i)^2\right)^{1/2} = 0,$$
$$\left[\left(\sum_{i=1}^{3}(\xi_i - z\mu_i)^2\right)^{1/2} + \sum_{i=1}^{3}\frac{(\xi_i - z\mu_i)\beta_i}{\|\beta\|}\right] = 0. \tag{47}$$

The solutions of the first equation in (47) are $z_{1,2} = \frac{(\xi\cdot\mu)\pm i\|\xi\times\mu\|}{\|\mu\|^2}$, hence $|z_{1,2}| = \frac{\|\xi\|}{\|\mu\|}$. The solutions of the second equation in (47) are $z^*_{1,2} = \frac{(\mu\times\beta)\cdot(\xi\times\beta)\pm i\|\beta\|((\mu\times\xi)\cdot\beta)}{\|\mu\times\beta\|^2}$, hence $\left|z^*_{1,2}\right| = \frac{\|\xi\times\beta\|}{\|\mu\times\beta\|}$. Observe that $z^*_{1,2} \neq 0$ if and only if $\|\xi\times\beta\| \neq 0$, that implies $\|\mu\times\beta\| \neq 0$ (in fact if $\|\mu\times\beta\| = 0$, then from the definition (44), $\|\xi\times\beta\| = 0$ too).

Analogous results hold for the denominator of (46): $\tilde{z}_{1,2} = \frac{(\tilde{\xi}\cdot\mu)\pm i\|\tilde{\xi}\times\mu\|}{\|\mu\|^2}$, $|\tilde{z}_{1,2}| = \frac{\|\tilde{\xi}\|}{\|\mu\|}$; $\tilde{z}^*_{1,2} = \frac{(\mu\times\beta)\cdot(\tilde{\xi}\times\beta)\pm i\|\beta\|((\mu\times\tilde{\xi})\cdot\beta)}{\|\mu\times\beta\|^2}$ and $\left|\tilde{z}^*_{1,2}\right| = \frac{\|\tilde{\xi}\times\beta\|}{\|\mu\times\beta\|}$.

It is easy to prove that $z^*_{1,2} = \tilde{z}^*_{1,2}$, therefore the numerator and the denominator in (46) vanish for the same values of the variable z, hence we have an indeterminate form. Using L'Hôspital theorem, we calculate the following limit

$$\lim_{z\to z^*_{1,2}} \Phi_1(z) = \frac{\|\tilde{\xi}-z^*_{1,2}\mu\| \left[(\beta\cdot\mu)+\|\beta\|(\mu\cdot(\xi-z^*_{1,2}\mu))\right]}{\|\xi-z^*_{1,2}\mu\| \left[(\beta\cdot\mu)+\|\beta\|(\mu\cdot(\tilde{\xi}-z^*_{1,2}\mu))\right]}. \tag{48}$$

The result in (48) is finite and different from zero (and therefore $z^*_{1,2} = \tilde{z}^*_{1,2}$ give no problem), except in the following four cases:

$$\exists\alpha_1 \in \mathbb{R} \text{ s. t. } \beta = \alpha_1(\mu\times\xi), \tag{49}$$

$$\exists\alpha_2 \in \mathbb{R} \text{ s. t. } \beta = \alpha_2(\mu\times\tilde{\xi}), \tag{50}$$

$$\exists\gamma_1 \in \mathbb{R} \text{ s. t. } \mu = \gamma_1\xi, \tag{51}$$

$$\exists\gamma_2 \in \mathbb{R} \text{ s. t. } \mu = \gamma_2\tilde{\xi}. \tag{52}$$

In the hypotheses (49) or (50) we have that $z^*_1 = z^*_2 = \tilde{z}^*_1 = \tilde{z}^*_2 = z^* = \frac{(\mu\cdot\xi)}{\|\mu\|^2}$, and this means that the result in (48) is an indeterminate form again. Using L'Hôspital theorem iteratively, in the hypothesis (49), we obtain

$$\lim_{z\to z^*} \Phi_1(z) = \sqrt{\frac{\|\tilde{\xi}\|^2\|\mu\|^2-(\mu\cdot\tilde{\xi})^2}{\|\xi\|^2\|\mu\|^2-(\mu\cdot\xi)^2}}. \tag{53}$$

The result in (53) is finite and different from zero except when $\exists c_1 \in \mathbb{R}$ such that $\mu = c_1\xi$ or $\exists c_2 \in \mathbb{R}$ such that $\mu = c_2\tilde{\xi}$, that implies $|z^*| = \frac{\|\xi\|}{\|\mu\|}$ or $|z^*| = \frac{\|\tilde{\xi}\|}{\|\mu\|}$ respectively. The same result is obtained if we consider the hypothesis (50).

Now we must consider (51). In this case $z^*_{1,2} = \tilde{z}^*_{1,2} = 1/\gamma_1$. This means that the result in (48) is an indeterminate form again. As before, using L'Hôspital theorem, in the hypothesis (51), we obtain

$$\lim_{z\to 1/\gamma_1} \Phi_1(z) = 0. \tag{54}$$

In this situation we have that $\frac{1}{|\gamma_1|} = \frac{\|\xi\|}{\|\mu\|}$. Correspondingly, if we consider hypothesis (52), we obtain that $z^*_{1,2} = \tilde{z}^*_{1,2} = 1/\gamma_2$ and using the same procedure as before we obtain that the limit is infinite. In this case $\frac{1}{|\gamma_2|} = \frac{\|\tilde{\xi}\|}{\|\mu\|}$.

We can conclude that, considering all the possible situations, the function $\Phi_1(z)$ is holomorphic for $|z| < \min\left\{\frac{\|\xi\|}{\|\mu\|}, \frac{\|\tilde{\xi}\|}{\|\mu\|}\right\}$.

Analogous results hold starting from h_j, $j = 2,3$ (see (42)).

Theorem 7 *The function $\Phi_{g_2}(z)$, $z \in \mathbb{C}$, analytical continuation of $g_2(\mathbf{x}_\tau - t(\mathbf{x}_\tau - \mathbf{x}))$, $t \in (0,1)$, of (42), is holomorphic for*

$$|z| < R, \quad \textit{where } R := \frac{dist(\mathbf{x}_\tau, T_j)}{\|\mu\|}. \tag{55}$$

Proof: Analogous to that of Theorem 4.

The function g_3

Now we consider the function (22). Following the same procedure as in the previous subsection, in order to study the remainder of the Taylor expansion, we introduce the function

$$\Phi_{g_3}(t) := g_3(\mathbf{x}_\tau - t(\mathbf{x}_\tau - \mathbf{x})), \qquad t \in (0,1).$$

If we use the Cauchy integral formula

$$R_n(\mathbf{x}_\tau, \mathbf{x}_\tau - \mathbf{x}) = \frac{1}{n!}\left(\frac{d}{dt}\right)^n \Phi_{g_3}(t)$$

$$= \frac{1}{2\pi i}\oint_{|z|=r} \frac{\Phi_{g_3}(z)}{(z-t)^{n+1}} \mathrm{dz} \quad \text{ for } |t| < r < R$$

where $\Phi_{g_3}(z)$ is the analytical continuation of $\Phi_{g_3}(t)$ in the complex plane and $|z| < R$ is a region where $\Phi_{g_3}(z)$ is holomorphic. Using the definitions (44) the function $\Phi_{g_3}(z)$ has the expression

$$\Phi_{g_3}(z) = -c_4\Big\{$$

$$\mathrm{Arctan}\frac{(P_j\times(z\mu-\xi))\cdot(\beta\times(\tilde{\xi}-z\mu))\|P_j\|}{\|(P_j\times P_{j+1})\times(P_j\times(\mathbf{x}_\tau-z\mu))\|\|z\mu-\xi\|} -$$

$$\mathrm{Arctan}\frac{(P_j\times(\mathbf{x}_\tau-z\mu))\cdot((z\mu-\tilde{\xi})\times\beta)\|P_j\|}{\|(P_j\times P_{j+1})\times(P_j\times(\mathbf{x}_\tau-z\mu))\|\|z\mu-\tilde{\xi}\|} +$$

$$\mathrm{Arctan}\frac{(P_{j+1}\times(\mathbf{x}_\tau-z\mu))\cdot(P_j\times(\xi-z\mu))\|P_j\|}{\|(P_j\times P_{j+1})\times(P_j\times(\mathbf{x}_\tau-z\mu))\|\|z\mu-\xi\|} -$$

$$\mathrm{Arctan}\frac{(P_{j+1}\times(\mathbf{x}_\tau-z\mu))\cdot(P_j\times(\mathbf{x}_\tau-z\mu))\|P_j\|}{\|(P_j\times P_{j+1})\times(P_j\times(\mathbf{x}_\tau-z\mu))\|\|\mathbf{x}_\tau-z\mu\|}\Big\}. \tag{56}$$

	$z_{1,2}, \quad \lvert z_{1,2}\rvert$	$z_{3,4}, \quad \lvert z_{3,4}\rvert$
I-Arctan	$z_{1,2}^{(I)}=\frac{(\mu\times P_j)\cdot(\xi\times P_j)\mp i\lVert P_j\rVert\lvert P_j\cdot(\xi\times\mu)\rvert}{\lVert P_j\times\mu\rVert^2}$ $\lvert z_{1,2}^{(I)}\rvert=\frac{\lVert P_j\times\xi\rVert}{\lVert P_j\times\mu\rVert}$	$z_{3,4}^{(I)}=\frac{(\mu\times\beta)\cdot(\xi\times\beta)\mp i\lVert\beta\rVert\lvert\beta\cdot(\xi\times\mu)\rvert}{\lVert\beta\times\mu\rVert^2}$ $\lvert z_{3,4}^{(I)}\rvert=\frac{\lVert\beta\times\xi\rVert}{\lVert\beta\times\mu\rVert}$
II-Arctan	$z_{1,2}^{(II)}=\frac{(\mu\times P_j)\cdot(\tilde{\xi}\times P_j)\mp i\lVert P_j\rVert\lvert P_j\cdot(\tilde{\xi}\times\mu)\rvert}{\lVert P_j\times\mu\rVert^2}$ $\lvert z_{1,2}^{(II)}\rvert=\frac{\lVert P_j\times\tilde{\xi}\rVert}{\lVert P_j\times\mu\rVert}$	$z_{3,4}^{(II)}=z_{3,4}^{(I)}$ $\lvert z_{3,4}^{(II)}\rvert=\lvert z_{3,4}^{(I)}\rvert$
III-Arctan	$z_{1,2}^{(III)}=z_{1,2}^{(I)}$ $\lvert z_{1,2}^{(III)}\rvert=\lvert z_{1,2}^{(I)}\rvert$	$z_{3,4}^{(III)}=\frac{(\mu\times P_{j+1})\cdot(\xi\times P_{j+1})\mp i\lVert P_{j+1}\rVert\lvert P_{j+1}\cdot(\xi\times\mu)\rvert}{\lVert P_{j+1}\times\mu\rVert^2}$ $\lvert z_{3,4}^{(III)}\rvert=\frac{\lVert P_{j+1}\times\xi\rVert}{\lVert P_{j+1}\times\mu\rVert}$
IV-Arctan	$z_{1,2}^{(IV)}=z_{1,2}^{(II)}$ $\lvert z_{1,2}^{(IV)}\rvert=\lvert z_{1,2}^{(II)}\rvert$	$z_{3,4}^{(IV)}=z_{3,4}^{(III)}$ $\lvert z_{3,4}^{(IV)}\rvert=\lvert z_{3,4}^{(III)}\rvert$

Table 6 : Points in the complex plane where each Arctan in (56) is not defined.

In order to verify where the function $\Phi_{g_3}(z)$ is holomorphic, we can consider each of the four Arctan$(\cdot)$ in the sum (56) and study where they are not defined. In table 6 one can find the points in the complex plane in which each Arctan in (56) is not defined and the expression of their modulus.

In order to deeper analyze these critical points in the complex plane, we rewrite $\Phi_{g_3}(z)$ in the following form

$$\Phi_{g_3}(z)=-\frac{c_4}{2i}\ln\prod_{j=1}^{4}\frac{F_j(z)}{G_j(z)} \tag{57}$$

where

$$\begin{aligned}
F_1(z)&=\nu(z)\lVert z\mu-\xi\rVert+i\alpha_1(z)\lVert P_j\rVert\\
F_2(z)&=\nu(z)\lVert z\mu-\tilde{\xi}\rVert-i\alpha_2(z)\lVert P_j\rVert\\
F_3(z)&=\nu(z)\lVert z\mu-\xi\rVert+i\alpha_3(z)\lVert P_j\rVert\\
F_4(z)&=\nu(z)\lVert \mathbf{x}_\tau-z\mu\rVert-i\alpha_4(z)\lVert P_j\rVert\\
G_1(z)&=\nu(z)\lVert z\mu-\xi\rVert-i\alpha_1(z)\lVert P_j\rVert\\
G_2(z)&=\nu(z)\lVert z\mu-\tilde{\xi}\rVert+i\alpha_2(z)\lVert P_j\rVert\\
G_3(z)&=\nu(z)\lVert z\mu-\xi\rVert-i\alpha_3(z)\lVert P_j\rVert\\
G_4(z)&=\nu(z)\lVert \mathbf{x}_\tau-z\mu\rVert+i\alpha_4(z)\lVert P_j\rVert
\end{aligned} \tag{58}$$

and

$$\begin{aligned}
\alpha_1(z)&:=P_j\times(z\mu-\xi))\cdot(\beta\times(\tilde{\xi}-z\mu))\\
\alpha_2(z)&:=P_j\times(\mathbf{x}_\tau-z\mu))\cdot(z\mu-\tilde{\xi})\times\beta)\\
\alpha_3(z)&:=P_{j+1}\times(\mathbf{x}_\tau-z\mu))\cdot(P_j\times(\xi-z\mu))\\
\alpha_4(z)&:=(P_{j+1}\times(\mathbf{x}_\tau-z\mu))\cdot(P_j\times(\mathbf{x}_\tau-z\mu))\\
\nu(z)&:=\lVert(P_j\times P_{j+1})\times(P_j\times(\mathbf{x}_\tau-z\mu)\rVert
\end{aligned} \tag{59}$$

The argument of the logarithmic function in (57) is an indeterminate form for each of the eight distinct values included in table 6. In fact for each $z_j^{(J)}\in\left\{z_{1,2}^{(I)},z_{3,4}^{(I)},z_{1,2}^{(II)},z_{3,4}^{(III)}\right\}$ there exists a unique couple of indexes k,m such that $F_k(z_j^{(J)})=G_m(z_j^{(J)})=0$. For this reason we must apply L'Hôspital theorem, and we obtain that

$$\lim_{z\to z_j^{(J)}}\frac{F_k(z)}{G_m(z)}=\ell_{km}\in\mathbb{C} \tag{60}$$

It can be proved that $\ell_{km}\neq 0$ except in some simple geometric situations.

Considering all possible cases, we finally proved the following two theorems (the proofs are similar to those of theorems 4 and 5)

Theorem 8 *The function* $\Phi_{g_3}(z)$, $z \in \mathbb{C}$, *analytical continuation of* $g_3(\mathbf{x}_\tau - t(\mathbf{x}_\tau - \mathbf{x}))$, $t \in (0,1)$ *is holomorphic for*

$$|z| < R, \quad \text{where } R := \frac{dist(\mathbf{x}_\tau, T_j)}{\|\mu\|}. \tag{61}$$

Theorem 9 *The remainder of the Taylor expansion of order* $n-1$ *of the function* $g = g_2 + g_3$ *satisfies the inequality*

$$|R_n(\mathbf{x}_\tau, \mathbf{x}_\tau - \mathbf{x})| \leq M 2^{n+1} \eta^n \quad \text{where}$$

$$M := \max_{|z| \leq r} |\Phi_{g_2}(z) + \Phi_{g_3}(z)|, \quad r < R \tag{62}$$

where R is defined in (61).

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