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2.5D Green's Functions for Elastodynamic Problems in Layered Acoustic and Elastic Formations

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Abstract: This paper presents analytical solutions, together with explicit expressions, for the steady state response of homogeneous three-dimensional layered acoustic and elastic formations subjected to a spatially sinusoidal harmonic line load. These formulas are theoretically interesting in themselves and they are also useful as benchmark solutions for numerical applications. In particular, they are very important in formulating three-dimensional elastodynamic problems in layered fluid and solid formations using integral transform methods and/or boundary elements, avoiding the discretization of the solid-fluid interfaces. The proposed Green's functions will allow the solution to be obtained for high frequencies, for which the conventional boundary elements' solution would require an inordinate computational effort, ruling out its use. In order to validate the final expressions, the results were compared with those provided by the Boundary Element Method (BEM) solution, for which the interfaces between layers are discretized with boundary elements.

keyword: Green's functions, analytical solutions, spatially sinusoidal harmonic line load.

1 Introduction

The derivation and development of Green's functions has been object of research over the years because these expressions can be used as benchmark solutions and they can be incorporated in the development of numerical methods such as the Boundary Elements Method (BEM) [Kögl and Gaul (2000); Katsikadelis and Nerantzaki (2000); Zheng and Dravinski (2000); Polyzos, Dassios and Beskos (1994)]. This research has been extended to many areas of engineering, such as fracture mechanics, fluid dynamics, heat transfer, structural mechanics, elastodynamics [Guimarães and Telles (2000); Melnikov and Melnikov (2001)]. The present study extends the work performed by the authors in the derivation of analytical solutions for the steady state response of a homogeneous three-dimensional half-space subjected to a spatially sinusoidal, harmonic line load. In the present case, this work presents the Green's functions for calculating the wavefield in a formation formed by an elastic solid medium, bounded by one or two acoustic flat fluid media, as in Fig. 3 and Fig. 4, when subjected to either a spatially sinusoidal harmonic point load placed in the solid or a spatially sinusoidal harmonic pressure load submerged in the fluid. These functions, or fundamental solutions, relate the field variables (stresses or displacements) at some location in the solid or in the fluid domains caused by a dynamic source located at a different point in the solid-fluid formation.

The technique presented here requires the knowledge of solid displacement potentials and fluid pressure potentials. The solid displacement potentials employed to define the present Green's functions are those defined by the methodology used by the authors [Tadeu and Kausel (2000)] to evaluate the Green's functions for a harmonic (steady state) line load with a sinusoidally varying amplitude in the third dimension in an unbounded medium. For the fluid pressure potential, a similar technique is used. All these displacement and pressure potentials are written as a superposition of plane waves following the approach used first by Lamb (1904) for the two-dimensional case and then by Bouchon (1979) and Kim and Papgeorgiou (1993) to calculate the three-dimensional field by means of a discrete wave number representation. The Green's functions for the solid-fluid formation are then derived, assuming the continuity of normal displacements and stresses, and ascribing null tangential stresses at the interface between the solid and the fluid media. The final Green's functions are then written as the sum of the Green's function for a full-space with surface terms, using a technique similar to that described by Kawase (1988).

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The fundamental solutions presented here, namely the Green's functions for a spatially sinusoidal line harmonic (steady state) load, or pressure load in a layered solid fluid formation, often referred to in the literature as the 2.5D problem, are of great value in formulating 3D elastodynamic problems, such as those involving the solid-fluid interaction via boundary elements together with integral transforms.

This paper describes first how the Green's functions for a sinusoidal line load applied in an unbounded solid formation along the x, y and z directions, can be written as a continuous superposition of homogeneous plane waves. A similar procedure is applied to the Green's function for a sinusoidal line pressure load applied in an unbounded fluid medium. Next, the Green's functions for an elastic formation, bounded by one or two flat fluid media, are established, using the required boundary conditions at the solid-fluid interfaces. Finally, the full set of expressions is compared with those provided by the Boundary Element Method, for which a full discretization of the boundary interfaces is required.

2 Green's Functions in an Unbounded Medium

2.1 Solid Formation

An infinite homogeneous space is subjected, at the origin of coordinates, to a spatially varying line load of the form $p(x,y,z,t) = \delta(x)\,\delta(y)\,e^{i(\omega t - k_z z)}$ acting in one of the three coordinate directions. Here, $\delta(x)$ and $\delta(y)$ are Dirac-delta functions, ω is the frequency of the load and k_z is the wavenumber in z (see Fig. 1). The response to this

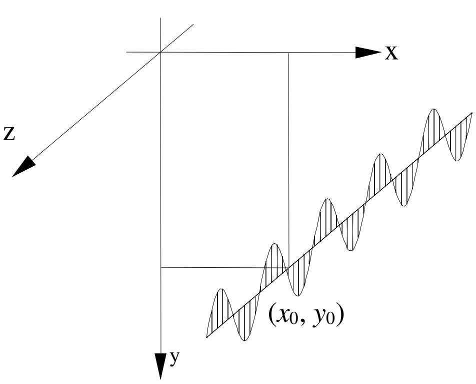

Figure 1 : Geometry of the problem: Full-space

load can be calculated by applying a spatial Fourier transform in the z direction to the Helmholtz equations for a point load (see e.g. Gradshteyn and Ryzhik (1980)). The z transformed equations are then

$$\left(\frac{\partial^2 \hat{A}_p}{\partial x^2} + \frac{\partial^2 \hat{A}_p}{\partial y^2} + k_\alpha^2 \hat{A}_p\right) = \frac{-iH_0^{(2)}(-ik_z r)}{4\rho\alpha^2}$$

$$\left(\frac{\partial^2 \hat{A}_s}{\partial x^2} + \frac{\partial^2 \hat{A}_s}{\partial y^2} + k_\beta^2 \hat{A}_s\right) = \frac{-iH_0^{(2)}(-ik_z r)}{4\rho\beta^2} \qquad (1)$$

where $k_\alpha = \sqrt{k_p^2 - k_z^2}$ with $(\mathrm{Im}(k_\alpha) \leq 0)$ and $k_p = \omega/\alpha$, $k_\beta = \sqrt{k_s^2 - k_z^2}$ with $\left(\mathrm{Im}(k_\beta) \leq 0\right)$ and $k_s = \omega/\beta$, $\alpha = \sqrt{(\lambda + 2\mu)/\rho}$ and $\beta = \sqrt{\mu/\rho}$ are the velocities for P (pressure) waves and S (shear) waves, respectively, λ and μ are the Lamé constants, ρ is the mass density, $\hat{A}_p(x,y,k_z,\omega)$ and $\hat{A}_s(x,y,k_z,\omega)$ are the Fourier transforms of the two potentials $A_p(x,y,z,\omega)$ and $A_s(x,y,z,\omega)$ for the irrotational and equivoluminal parts of the displacement vector, $H_n^{(2)}()$ are Hankel functions of the second kind and n^{th} order, $r = \sqrt{x^2+y^2}$ and $i = \sqrt{-1}$. From equilibrium conditions we find $\hat{A}_p$ and $\hat{A}_s$,

$$\hat{A}_p = \frac{i}{4\rho\omega^2}\left[H_0^{(2)}(k_\alpha r) - H_0^{(2)}(-ik_z r)\right]$$

$$\hat{A}_s = \frac{i}{4\rho\omega^2}\left[H_0^{(2)}(k_\beta r) - H_0^{(2)}(-ik_z r)\right] \qquad (2)$$

It is now possible to compute displacements G_{ij} in direction i due to a load applied in direction j from the relation

$$G_{ij} = \frac{\partial^2\left(\hat{A}_p - \hat{A}_s\right)}{\partial x_i \partial x_j} + \delta_{ij}\overline{\nabla}^2 \hat{A}_s \qquad (3)$$

in which δ_{ij} is the Kronecker delta, $x_j = x,\ y,\ z$ for $j = 1,\ 2,\ 3$, and $\frac{\partial}{\partial z} = -ik_z$. We may observe that

$$\hat{A}_p - \hat{A}_s = \frac{1}{4i\rho\omega^2}\left[H_0^{(2)}(k_\beta r) - H_0^{(2)}(k_\alpha r)\right]$$

$$\overline{\nabla}^2 \hat{A}_s = \frac{1}{4i\rho\beta^2} H_0^{(2)}(k_\beta r) \qquad (4)$$

A full set of Green's functions, expressions for the strains and stresses, are presented in Tadeu and Kausel (2000), which are in complete agreement with the solution for moving loads given earlier by Pedersen, Sánchez-Sesma and Campillo (1994) and Papageorgiou and Pei (1998).

These same equations can be expressed as a continuous superposition of homogeneous and inhomogeneous plane waves when the load acts in the direction x, y and z.

2.1.1 Load acting in the direction of the x-axis

The displacement potentials that result from a spatially sinusoidal harmonic line load along the z direction, applied at the point (x_0, y_0) in the x direction, are then given by the expressions,

$$\phi^x = \frac{1}{4\pi\rho\omega^2}\int_{-\infty}^{+\infty}\left(\frac{k}{\nu}e^{-i\nu|y-y_0|}\right)e^{-ik(x-x_0)}dk$$

$$\psi_x^x = 0$$

$$\psi_y^x = \frac{i(-ik_z)}{4\pi\rho\omega^2}\int_{-\infty}^{+\infty}\left(\frac{e^{-i\gamma|y-y_0|}}{\gamma}\right)e^{-ik(x-x_0)}dk$$

$$\psi_z^x = \frac{-\operatorname{sgn}(y-y_0)}{4\pi\rho\omega^2}\int_{-\infty}^{+\infty}\left(e^{-i\gamma|y-y_0|}\right)e^{-ik(x-x_0)}dk \tag{5}$$

where $\nu = \sqrt{k_p^2 - k_z^2 - k^2}$ with $(\operatorname{Im}(\nu) \le 0)$, $\gamma = \sqrt{k_s^2 - k_z^2 - k^2}$ with $(\operatorname{Im}(\gamma) \le 0)$, and integration is performed with respect to the horizontal wave number (k) along the x direction.

The transformation of these integrals into a summation can be achieved if an infinite number of such sources are distributed along the x direction, at equal intervals L_x. The above compressional and rotational potentials can then be written as

$$\phi^x = E_a\sum_{n=-\infty}^{n=+\infty}\left(\frac{k_n}{\nu_n}E_b\right)E_d$$

$$\psi_x^x = 0$$

$$\psi_y^x = E_a k_z\sum_{n=-\infty}^{n=+\infty}\left(\frac{E_c}{\gamma_n}\right)E_d$$

$$\psi_z^x = -\operatorname{sgn}(y-y_0)E_a\sum_{n=-\infty}^{n=+\infty}(E_c)E_d \tag{6}$$

where $E_a = \frac{1}{2\rho\omega^2 L_x}$, $E_b = e^{-i\nu_n|y-y_0|}$, $E_c = e^{-i\gamma_n|y-y_0|}$, $E_d = e^{-ik_n(x-x_0)}$, $\nu_n = \sqrt{k_p^2 - k_z^2 - k_n^2}$ with $(\operatorname{Im}(\nu_n) \le 0)$, $\gamma_n = \sqrt{k_s^2 - k_z^2 - k_n^2}$ with $(\operatorname{Im}(\gamma_n) \le 0)$, $k_n = \frac{2\pi}{L_x}n$, which can in turn be approximated by a finite sum of equations (N).

The Green's functions can be expressed in terms of the compressional and rotational potentials, ϕ^x, ψ_x^x, ψ_y^x and ψ_z^x, from which the following three components of displacement can be calculated,

$$G_{xx}^{full} = E_a\sum_{n=-N}^{n=+N}\left[\frac{-ik_n^2}{\nu_n}E_b + \left(-i\gamma_n - \frac{ik_z^2}{\gamma_n}\right)E_c\right]E_d$$

$$G_{yx}^{full} = E_a\sum_{n=-N}^{n=+N}[-i\operatorname{sgn}(y-y_0)k_nE_b + i\operatorname{sgn}(y-y_0)k_nE_c]E_d$$

$$G_{zx}^{full} = E_a\sum_{n=-N}^{n=+N}\left(\frac{-ik_zk_n}{\nu_n}E_b + \frac{ik_zk_n}{\gamma_n}E_c\right)E_d \tag{7}$$

The corresponding expressions for forces applied along the y and z directions can be calculated in the same way. The derivation of these solutions is then presented, but in condensed form.

2.1.2 Load acting in the direction of the y-axis

The displacement potentials resulting from a spatially sinusoidal harmonic line load along the z direction, applied at the point (x_0, y_0) in the y direction, are then given, in a discrete form, by the expressions,

$$\phi^y = E_a\sum_{n=-N}^{n=+N}[\operatorname{sgn}(y-y_0)E_b]E_d$$

$$\psi_x^y = E_a k_z\sum_{n=-N}^{n=+N}\left(\frac{-E_c}{\gamma_n}\right)E_d$$

$$\psi_y^y = 0$$

$$\psi_z^y = E_a\sum_{n=-N}^{n=+N}\left(\frac{k_n}{\gamma_n}E_c\right)E_d \tag{8}$$

The Green's functions for a two-and-a-half dimensional full space are thus,

$$G_{xy}^{full} = G_{yx}^{full} = E_a\sum_{n=-N}^{n=+N}[-i\operatorname{sgn}(y-y_0)k_nE_b + i\operatorname{sgn}(y-y_0)k_nE_c]E_d$$

$$G_{yy}^{full} = E_a\sum_{n=-N}^{n=+N}\left[-i\nu_nE_b + \left(\frac{i\nu_{zn}^2}{\gamma_n}\right)E_c\right]E_d$$

$$G_{zy}^{full} = E_a\sum_{n=-N}^{n=+N}[-i\operatorname{sgn}(y-y_0)k_zE_b + i\operatorname{sgn}(y-y_0)k_zE_c]E_d \tag{9}$$

2.1.3 Load acting in the direction of the z-axis

Similarly, the discrete form of the displacement potentials resulting from a spatially sinusoidal harmonic line load along the z direction, applied at the point (x_0, y_0) in the z direction, is given by the expressions,

$$\phi^z = E_a k_z \sum_{n=-N}^{n=+N} \left(\frac{E_b}{\nu_n} \right) E_d$$

$$\psi_x^z = E_a \sum_{n=-N}^{n=+N} [\operatorname{sgn}(y-y_0) E_c] E_d$$

$$\psi_y^z = E_a \sum_{n=-N}^{n=+N} \left(\frac{-k_n}{\gamma_n} E_c \right) E_d$$

$$\psi_z^z = 0 \qquad (10)$$

The Green's functions for a two-and-a-half dimensional full space are then,

$$G_{xz}^{full} = G_{zx}^{full} = E_a \sum_{n=-N}^{n=+N} \left(\frac{-ik_z k_n}{\nu_n} E_b + \frac{ik_z k_n}{\gamma_n} E_c \right) E_d$$

$$G_{yz}^{full} = G_{zy}^{full} = E_a \sum_{n=-N}^{n=+N} [-i \operatorname{sgn}(y-y_0) k_z E_b + \; i \operatorname{sgn}(y-y_0) k_z E_c] E_d$$

$$G_{zz}^{full} = E_a \sum_{n=-N}^{n=+N} \left[\frac{-ik_z^2}{\nu_n} E_b + \left(\frac{-ik_n^2}{\gamma_n} - i\gamma_n \right) E_c \right] E_d \qquad (11)$$

2.2 Fluid formation

The fluid dilatational potential for a sinusoidal pressure line load applied at the point (x_0, y_0) can be obtained using a similar process described above, leading to the expression

$$\phi_{fluid}(\omega, x, y, k_z) = \frac{-i}{2} \left(-\frac{\alpha_f^2}{\omega^2 \lambda_f} \right) H_0^{(2)} \left(k_{\alpha_f} \sqrt{(x-x_0)^2 + (y-y_0)^2} \right) \qquad (12)$$

in which

$$k_{\alpha_f} = \sqrt{\frac{\omega^2}{\alpha_f^2} - k_z^2}, \quad \operatorname{Im} k_{\alpha_f} < 0$$

λ_f is the fluid Lamé constant, $\alpha_f = \sqrt{\lambda_f / \rho_f}$ is the acoustic (dilatational) wave velocity of the medium and ρ_f is the mass density of the fluid.

The discrete form of the fluid dilatational potential, as described above, can be written in the form,

$$\phi_{fluid} = -\frac{i}{L_x} \sum_{n=-N}^{n=+N} \left[\left(\frac{-\alpha_f^2}{\omega^2 \lambda_f} \right) \frac{E_f}{\nu_n^f} \right] E_d \qquad (13)$$

where $E_f = e^{-i\nu_n^f |y-y_0|}$, $\nu_n^f = \sqrt{k_{p_f}^2 - k_z^2 - k_n^2}$ with $\left(\operatorname{Im}\left(\nu_n^f\right) \leq 0 \right)$ and $k_{p_f} = \omega / \alpha_f$.

The Green's function for a two-and-a-half dimensional full space can then be written as,

$$G_{fx}^{full} = -\frac{1}{L_x} \sum_{n=-N}^{n=+N} \left[\left(\frac{-\alpha_f^2}{\omega^2 \lambda_f} \right) \frac{k_n}{\nu_n^f} E_f \right] E_d$$

$$G_{fy}^{full} = -\frac{1}{L_x} \sum_{n=-N}^{n=+N} \left[\left(\frac{-\alpha_f^2}{\omega^2 \lambda_f} \right) \operatorname{sgn}(y-y_0) E_f \right] E_d$$

$$G_{fz}^{full} = -\frac{1}{L_x} \sum_{n=-N}^{n=+N} \left[\left(\frac{-\alpha_f^2}{\omega^2 \lambda_f} \right) \frac{k_z}{\nu_n^f} E_f \right] E_d \qquad (14)$$

3 Green's Functions in a Fluid-solid Formation

3.1 Load in the Solid Formation Acting in the Direction of the x-axis.

The Green's functions for a fluid-solid formation can be expressed as the sum of the source terms equal to those in the full-space and the surface terms needed to satisfy the fluid-solid interface conditions (continuity of normal displacements and stresses, and null tangential stresses). These surface terms can be expressed in a form similar to that of the source term, i.e.

Solid medium

$$\phi_0^x = E_a \sum_{n=-\infty}^{n=+\infty} \left(\frac{k_n}{\nu_n} E_{b0} A_n^x \right) E_d$$

$$\psi_{x0}^x = 0$$

$$\psi_{y0}^x = E_a k_z \sum_{n=-\infty}^{n=+\infty} \left(\frac{E_{c0}}{\gamma_n} B_n^x \right) E_d$$

$$\psi_{z0}^x = -E_a \sum_{n=-\infty}^{n=+\infty} (E_{c0} C_n^x) E_d \qquad (15)$$

Fluid medium

$$\phi_{fluid} = -\frac{i}{L_x} \sum_{n=-N}^{n=+N} \left[\left(\frac{-\alpha_f^2}{\omega^2 \lambda_f} \right) \frac{E_{f0}}{\nu_n^f} D_n^x \right] E_d \qquad (16)$$

(when $y < 0$)

where $E_{b0} = e^{-i\nu_n y}$, $E_{c0} = e^{-i\gamma_n y}$, $E_{f0} = e^{-i\nu_n^f y}$, A_n^x, B_n^x, C_n^x and D_n^x, are as yet unknown coefficients to be determined from the appropriate boundary conditions, so that the field produced simultaneously by the source and surface terms should produce $\sigma_{yx}^s = \sigma_{yx}^f = 0$, $\sigma_{yz}^s = \sigma_{yz}^f = 0$, $\sigma_{yy}^s = \sigma_{yy}^f$ and $u_y^s = u_y^f$ at $y = 0$.

Imposing the four stated boundary conditions for each value of n thus leads to a system of four equations in the four unknown constants. This procedure is quite straightforward, but the details are rather complex, and for this reason are not presented here. The final system of equations alone is,

$$\left[a_{ij}^x \;\; i=1,4;\; j=1,4\right] \left[c_i^x \;\; i=1,4\right] = \left[b_i^x \;\; i=1,4\right] \quad (17)$$

which is described in Appendix B.

Once the constants have been obtained, the motions and pressures associated with the surface terms may be calculated using the equations which relate potentials to displacements and pressures. Essentially, this needs to consider Eq. 15 and the application of partial derivatives over the potentials to obtain displacements and pressures. The Green's functions for a solid formation are then obtained from the sum of the source terms and these surface terms. When this procedure has been carried out, expressions for the displacements in the solid formation are obtained in the following form:

$$G_{xx}^{fs} = G_{xx}^{full} + E_a \sum_{n=-N}^{n=+N} \left[A_n^x \frac{-ik_n^2}{\nu_n} E_{b0} + \left(-i\gamma_n C_n^x - \frac{ik_z^2}{\gamma_n} B_n^x\right) E_{c0}\right] E_d$$

$$G_{yx}^{fs} = G_{yx}^{full} + E_a \sum_{n=-N}^{n=+N} \left(-ik_n A_n^x E_{b0} + ik_n C_n^x E_{c0}\right) E_d$$

$$G_{zx}^{fs} = G_{zx}^{full} + E_a \sum_{n=-N}^{n=+N} \left(\frac{-ik_z k_n}{\nu_n} A_n^x E_{b0} + \frac{ik_z k_n}{\gamma_n} B_n^x E_{c0}\right) E_d \quad (18)$$

The expressions for the Green's function for two-and-a-half dimensional full space G_{xx}^{full}, G_{yx}^{full} and G_{zx}^{full} can be defined in explicit form, as listed in the Appendix A: [Tadeu and Kausel (2000)]. The well-known equations relating strains and displacements can be used to calculate expressions for stress in the solid formation. The displacements and the pressures in the fluid medium are only given by the surface fluid terms (Eq. 16). The final expression for the pressure field in the fluid medium is then given by

$$\sigma_{fx}^{fs} = -\frac{i}{L_x} \sum_{n=-N}^{n=+N} \left(\frac{E_{f0}}{\nu_n^f} D_n^x k_n\right) E_d \qquad (\text{when } y < 0) \quad (19)$$

The corresponding expressions for forces applied along the y and z directions can be calculated in a similar way. The derivation of these solutions is then presented, but in condensed form.

3.2 *Load in the Solid Formation Acting in the Direction of the y-axis.*

The potential surface terms generated by the solid formation can be expressed in a form similar to that of the source term (Eq. 8),

$$\phi_0^y = E_a \sum_{n=-N}^{n=+N} \left(E_{b0} A_n^y\right) E_d$$

$$\psi_{x0}^y = E_a k_z \sum_{n=-N}^{n=+N} \left(\frac{-E_{c0}}{\gamma_n} C_n^y\right) E_d$$

$$\psi_{y0}^y = 0$$

$$\psi_{z0}^y = E_a \sum_{n=-N}^{n=+N} \left(\frac{k_n}{\gamma_n} E_{c0} B_n^y\right) E_d \quad (20)$$

while the fluid pressure potential is given by the same expression defined above,

$$\phi_{fluid} = -\frac{i}{L_x} \sum_{n=-N}^{n=+N} \left[\left(\frac{-\alpha_f^2}{\omega^2 \lambda_f}\right) \frac{E_{f0}}{\nu_n^f} D_n^y\right] E_d \quad (\text{when } y < 0) \quad (21)$$

The imposition of the four stated boundary conditions ($\sigma_{yx}^s = \sigma_{yx}^f = 0$, $\sigma_{yz}^s = \sigma_{yz}^f = 0$, $\sigma_{yy}^s = \sigma_{yy}^f$ and $u_y^s = u_y^f$ at $y = 0$) for each value of n leads to a system of four equations,

$$\left[a_{ij}^y \;\; i=1,4;\; j=1,4\right] \left[c_i^y \;\; i=1,4\right] = \left[b_i^y \;\; i=1,4\right] \quad (22)$$

which is listed in Appendix C.

Once the amplitude of each potential has been calculated, the Green's functions for the solid formation are then given by the sum of the source terms and these surface terms. This gives the expressions:

$$G_{xy}^{fs} = G_{xy}^{full} + E_a \sum_{n=-N}^{n=+N} \left[-iA_n^y k_n E_{b0} + iB_n^y k_n E_{c0}\right] E_d$$

$$G_{yy}^{fs} = G_{yy}^{full} + E_a \sum_{n=-N}^{n=+N} \left[-i\nu_n A_n^y E_{b0} + \left(\frac{-ik_n^2}{\gamma_n} B_n^y + \frac{-ik_z^2}{\gamma_n} C_n^y \right) E_{c0} \right] E_d$$

$$G_{zy}^{fs} = G_{zy}^{full} + E_a \sum_{n=-N}^{n=+N} \left(-iA_n^y k_z E_{b0} + iC_n^y k_z E_{c0} \right) E_d \quad (23)$$

The expressions G_{xy}^{full}, G_{yy}^{full} and G_{zy}^{full}, can be used in explicit form as shown in the Appendix A. Once again, it is possible to use the well-known equations relating strains and displacements to derive expressions for the stresses from G_{ij}.

The final expression for the pressure field in the fluid medium is then given by

$$\sigma_{fy}^{fs} = -\frac{i}{L_x} \sum_{n=-N}^{n=+N} \left(\frac{E_{f0}}{\nu_n^f} D_n^y \right) E_d \quad (24)$$

3.3 Load in the solid formation acting in the direction of the z-axis

The potential surface terms generated by the solid formation can now be written as (Eq. 10),

$$\phi_0^z = E_a k_z \sum_{n=-N}^{n=+N} \left(\frac{E_{b0}}{\nu_n} A_n^z \right) E_d$$

$$\psi_{x0}^z = E_a \sum_{n=-N}^{n=+N} \left(E_{c0} B_n^z \right) E_d$$

$$\psi_{y0}^z = E_a \sum_{n=-N}^{n=+N} \left(\frac{-k_n}{\gamma_n} E_{c0} C_n^z \right) E_d$$

$$\psi_{z0}^z = 0 \quad (25)$$

while the fluid pressure potential is given by the expression,

$$\phi_{fluid} = -\frac{i}{L_x} \sum_{n=-N}^{n=+N} \left[\left(\frac{-\alpha_f^2}{\omega^2 \lambda_f} \right) \frac{E_{f0}}{\nu_n^f} D_n^z \right] E_d \quad \text{(when } y < 0\text{)} \quad (26)$$

The imposition of the boundary conditions ($\sigma_{yx}^s = \sigma_{yx}^f = 0$, $\sigma_{yz}^s = \sigma_{yz}^f = 0$, $\sigma_{yy}^s = \sigma_{yy}^f$ and $u_y^s = u_y^f$ at $y = 0$) for each value of n leads to the following system of four equations,

$$\left[a_{ij}^z \;\; i = 1,4;\; j = 1,4 \right] \; \left[c_i^z \;\; i = 1,4 \right] = \left[b_i^z \;\; i = 1,4 \right] \quad (27)$$

as listed in Appendix D. Once the amplitude of each potential has been obtained, the Green's functions for a solid formation are given by the sum of the source terms and these surface terms, giving the following expressions,

$$G_{xz}^{fs} = G_{xz}^{full} + E_a \sum_{n=-N}^{n=+N} \left(\frac{-ik_z k_n}{\nu_n} A_n^z E_{b0} + \frac{ik_z k_n}{\gamma_n} C_n^z E_{c0} \right) E_d$$

$$G_{yz}^{fs} = G_{yz}^{full} + E_a \sum_{n=-N}^{n=+N} \left(-ik_z A_n^z E_{b0} + iB_n^z k_z E_{c0} \right) E_d$$

$$G_{zz}^{fs} = G_{zz}^{full} + E_a \sum_{n=-N}^{n=+N} \left[\frac{-ik_z^2}{\nu_n} A_n^z E_{b0} + \left(\frac{-ik_n^2}{\gamma_n} C_n^z - i\gamma_n B_n^z \right) E_{c0} \right] E_d \quad (28)$$

The expressions G_{xy}^{full}, G_{yy}^{full} and G_{zy}^{full} can be used in explicit form as shown in the Appendix A. Once again, the well-known equations relating strains and displacements can be used to derive expressions for the stresses in the solid formation from G_{ij}.

The final expression for the pressure field in the fluid medium is given, as before, by

$$\sigma_{fz}^{fs} = -\frac{i}{L_x} \sum_{n=-N}^{n=+N} \left(\frac{E_{f0}}{\nu_n^f} D_n^z \right) E_d \quad \text{(when } y < 0\text{)} \quad (29)$$

3.4 Pressure Load Acting in the Fluid

The potential surface terms generated by the solid formation can be expressed in a form similar to that of one of the source terms used before. The source terms generated when the load is acting along the y direction are used (Eq. 8),

$$\phi_0^y = E_a \sum_{n=-N}^{n=+N} \left(E_{b0} A_n^f \right) E_d$$

$$\psi_{x0}^y = E_a k_z \sum_{n=-N}^{n=+N} \left(\frac{-E_{c0}}{\gamma_n} C_n^f \right) E_d$$

$$\psi_{y0}^y = 0$$

$$\psi_{z0}^y = E_a \sum_{n=-N}^{n=+N} \left(\frac{k_n}{\gamma_n} E_{c0} B_n^f \right) E_d \quad (30)$$

while the fluid pressure potential is given by the expression already defined above,

$$\phi_{fluid} = -\frac{i}{L_x} \sum_{n=-N}^{n=+N} \left[\left(\frac{-\alpha_f^2}{\omega^2 \lambda_f} \right) \frac{E_{f0}}{\nu_n^f} D_n^f \right] E_d \quad \text{(when } y < 0\text{)} \quad (31)$$

The imposition of the four stated boundary conditions ($\sigma^s_{yx} = \sigma^f_{yx} = 0$, $\sigma^s_{yz} = \sigma^f_{yz} = 0$, $\sigma^s_{yy} = \sigma^f_{yy}$ and $u^s_y = u^f_y$ at $y = 0$) for each value of n leads to a system of four equations,

$$\left[a^f_{ij}\ \ i = 1,4;\ j = 1,4\right]\ \left[c^f_i\ \ i = 1,4\right] = \left[b^f_i\ \ i = 1,4\right] \tag{32}$$

as listed in Appendix E.

Once the amplitude of each potential has been calculated, the Green's functions for the solid formation are then given by these surface terms,

$$G^{fs}_{xf} = E_a \sum_{n=-N}^{n=+N} \left(-iA^f_n k_n E_{b0} + iB^f_n k_n E_{c0}\right) E_d$$

$$G^{fs}_{yf} = E_a \sum_{n=-N}^{n=+N} \left[-i\nu_n A^f_n E_{b0} + \left(\tfrac{-ik_n^2}{\gamma_n} B^f_n + \tfrac{-ik_z^2}{\gamma_n} C^f_n\right) E_{c0}\right] E_d$$

$$G^{fs}_{zf} = E_a \sum_{n=-N}^{n=+N} \left(-iA^f_n k_z E_{b0} + iC^f_n k_z E_{c0}\right) E_d \tag{33}$$

Again, it is possible to use the well-known equations relating strains and displacements to derive expressions for the stresses from G_{if}.

The final expression for the pressure field in the fluid medium is then given by the sum of the source terms and the fluid surface terms, giving the following expressions,

$$\sigma^{fs} = \sigma^{full} - \tfrac{i}{L_x} \sum_{n=-N}^{n=+N} \left(\tfrac{E_{f0}}{\nu^f_n} D^f_n\right) E_d \quad \text{(when } y < 0\text{)} \tag{34}$$

The expression σ^{full} is listed in the Appendix A.

Notice that, if $k_z = 0$ is used, the above system of equations is reduced to three unknowns, leading to the two-dimensional Green's function for plane strain line-loads.

4 Green's Functions in a Solid Layer Formation Bounded by Two Fluid Media

4.1 Load in the solid formation acting in the direction of the x-axis.

The Green's functions for a solid layer formation, with thickness h, bounded by two fluid media, can be expressed as the sum of the source terms equal to those in the full-space and the surface terms needed to satisfy the boundary conditions at the two fluid-solid interfaces (continuity of normal displacements and stresses, and null tangential stresses). With this specific problem, the two interfaces (top and bottom) generate surface terms which can be expressed in a form similar to that of the source term,

Solid medium (top interface)

$$\phi_0^{x\_top} = E_a \sum_{n=-\infty}^{n=+\infty} \left(\frac{k_n}{\nu_n} E_{b0} A^x_n\right) E_d$$

$$\psi_{x0}^{x\_top} = 0$$

$$\psi_{y0}^{x\_top} = E_a k_z \sum_{n=-\infty}^{n=+\infty} \left(\frac{E_{c0}}{\gamma_n} B^x_n\right) E_d$$

$$\psi_{z0}^{x\_top} = -E_a \sum_{n=-\infty}^{n=+\infty} \left(E_{c0} C^x_n\right) E_d \tag{35}$$

Fluid medium (top interface)

$$\phi_{fluid}^{top} = -\tfrac{i}{L_x} \sum_{n=-N}^{n=+N} \left[\left(\tfrac{-\alpha_f^2}{\omega^2 \lambda_f}\right) \tfrac{E_{f0}}{\nu^f_n} D^x_n\right] E_d \quad \text{(when } y < 0\text{)} \tag{36}$$

Solid medium (bottom interface)

$$\phi_0^{x\_bottom} = E_a \sum_{n=-\infty}^{n=+\infty} \left(\frac{k_n}{\nu_n} E^b_{b0} E^x_n\right) E_d$$

$$\psi_{x0}^{x\_bottom} = 0$$

$$\psi_{y0}^{x\_bottom} = E_a k_z \sum_{n=-\infty}^{n=+\infty} \left(\frac{E^b_{c0}}{\gamma_n} F^x_n\right) E_d$$

$$\psi_{z0}^{x\_bottom} = -E_a \sum_{n=-\infty}^{n=+\infty} \left(E^b_{c0} G^x_n\right) E_d \tag{37}$$

Fluid medium (bottom interface)

$$\phi_{fluid}^{bottom} = -\tfrac{i}{L_x} \sum_{n=-N}^{n=+N} \left[\left(\tfrac{-\alpha_f^2}{\omega^2 \lambda_f}\right) \tfrac{E^b_{f0}}{\nu^f_n} H^x_n\right] E_d \quad \text{(when } y > h\text{)} \tag{38}$$

where, $E^b_{b0} = e^{-i\nu_n|y-h|}$, $E^b_{c0} = e^{-i\gamma_n|y-h|}$, $E^b_{f0} = e^{-i\nu^f_n|y-h|}$. A^x_n, B^x_n, C^x_n, D^x_n, E^x_n, F^x_n, G^x_n and H^x_n are as yet unknown coefficients to be determined from the appropriate boundary conditions, so that the field produced simultaneously by the source and surface terms should produce $\sigma^s_{yx} = \sigma^f_{yx} = 0$, $\sigma^s_{yz} = \sigma^f_{yz} = 0$, $\sigma^s_{yy} = \sigma^f_{yy}$ and $u^s_y = u^f_y$ at $y = 0$ and at $y = h$.

Imposing the eight stated boundary conditions for each value of n thus leads to a system of eight equations in the eight unknown constants. This procedure is quite straightforward, but the details are rather complex, and for this reason are not presented here. The final system of equations is of the form

$$\left[a^x_{ij} \;\; i=1,8;\; j=1,8\right] \; \left[c^x_i \;\; i=1,8\right] = \left[b^x_i \;\; i=1,8\right] \quad (39)$$

which is fully described in Appendix B.

Once the unknown coefficients have been calculated, the motions and pressures associated with the surface terms can be obtained using the equations relating the potentials to displacements and pressures. The Green's functions for a solid formation are then obtained from the sum of the source terms and the surface terms originated at the two interfaces. This procedure produces the following expressions for the displacements in the solid formation:

$$\begin{aligned}
G^{fsf}_{xx} &= G^{full}_{xx} + \\
&E_a \sum_{n=-N}^{n=+N} \left[A^x_n \frac{-ik_n^2}{\nu_n} E_{b0} + \left(-i\gamma_n C^x_n - \frac{ik_z^2}{\gamma_n} B^x_n \right) E_{c0} \right] E_d + \\
&E_a \sum_{n=-N}^{n=+N} \left[E^x_n \frac{-ik_n^2}{\nu_n} E^b_{b0} + \left(-i\gamma_n G^x_n - \frac{ik_z^2}{\gamma_n} F^x_n \right) E^b_{c0} \right] E_d \\
G^{fsf}_{yx} &= G^{full}_{yx} + E_a \sum_{n=-N}^{n=+N} \left(-ik_n A^x_n E_{b0} + ik_n C^x_n E_{c0} \right) E_d + \\
&E_a \sum_{n=-N}^{n=+N} \left(ik_n E^x_n E^b_{b0} - ik_n G^x_n E^b_{c0} \right) E_d \\
G^{fsf}_{zx} &= G^{full}_{zx} + \\
&E_a \sum_{n=-N}^{n=+N} \left(\frac{-ik_z k_n}{\nu_n} A^x_n E_{b0} + \frac{ik_z k_n}{\gamma_n} B^x_n E_{c0} \right) E_d + \\
&E_a \sum_{n=-N}^{n=+N} \left(\frac{-ik_z k_n}{\nu_n} E^x_n E^b_{b0} + \frac{ik_z k_n}{\gamma_n} F^x_n E^b_{c0} \right) E_d
\end{aligned} \quad (40)$$

The final expression for the pressure field in the two fluid media are then given by

$$\begin{aligned}
\sigma^{fsf_top}_{fx} &= -\frac{i}{L_x} \sum_{n=-N}^{n=+N} \left(\frac{k_n E_{f0}}{\nu^f_n} D^x_n \right) E_d \qquad (\text{when } y<0) \\
\sigma^{fsf_bottom}_{fx} &= -\frac{i}{L_x} \sum_{n=-N}^{n=+N} \left(\frac{k_n E^b_{f0}}{\nu^f_n} H^x_n \right) E_d \\
&(\text{when } y>h)
\end{aligned} \quad (41)$$

The expressions for forces applied along the y and z directions can be derived in the same way. Thus, only the final system of equations is presented.

4.2 Load in the Solid Formation Acting in the Direction of the y-axis.

The surface terms generated at the two interfaces can be expressed through the following potentials,

Solid medium (top interface)

$$\begin{aligned}
\phi^{y_top}_0 &= E_a \sum_{n=-N}^{n=+N} \left(E_{b0} A^y_n \right) E_d \\
\psi^{y_top}_{x0} &= E_a k_z \sum_{n=-N}^{n=+N} \left(\frac{-E_{c0}}{\gamma_n} C^y_n \right) E_d \\
\psi^{y_top}_{y0} &= 0 \\
\psi^{y_top}_{z0} &= E_a \sum_{n=-N}^{n=+N} \left(\frac{k_n}{\gamma_n} E_{c0} B^y_n \right) E_d
\end{aligned} \quad (42)$$

Fluid medium (top interface)

$$\begin{aligned}
\phi^{top}_{fluid} &= -\frac{i}{L_x} \sum_{n=-N}^{n=+N} \left[\left(\frac{-\alpha_f^2}{\omega^2 \lambda_f} \right) \frac{E_{f0}}{\nu^f_n} D^y_n \right] E_d \\
&(\text{when } y<0)
\end{aligned} \quad (43)$$

Solid medium (bottom interface)

$$\begin{aligned}
\phi^{y_bottom}_0 &= E_a \sum_{n=-N}^{n=+N} \left(E^b_{b0} E^y_n \right) E_d \\
\psi^{y_bottom}_{x0} &= E_a k_z \sum_{n=-N}^{n=+N} \left(\frac{-E^b_{c0}}{\gamma_n} G^y_n \right) E_d \\
\psi^{y_bottom}_{y0} &= 0 \\
\psi^{y_bottom}_{z0} &= E_a \sum_{n=-N}^{n=+N} \left(\frac{k_n}{\gamma_n} E^b_{c0} F^y_n \right) E_d
\end{aligned} \quad (44)$$

Fluid medium (bottom interface)

$$\begin{aligned}
\phi^{bottom}_{fluid} &= -\frac{i}{L_x} \sum_{n=-N}^{n=+N} \left[\left(\frac{-\alpha_f^2}{\omega^2 \lambda_f} \right) \frac{E^b_{f0}}{\nu^f_n} H^y_n \right] E_d \\
&(\text{when } y>h)
\end{aligned} \quad (45)$$

The imposition of the eight stated boundary conditions for each value of n leads to a system of eight equations in the eight unknown constants,

$$\left[a^y_{ij} \;\; i=1,8;\; j=1,8\right] \; \left[c^y_i \;\; i=1,8\right] = \left[b^y_i \;\; i=1,8\right] \quad (46)$$

which is fully described in Appendix C.

Once the amplitude of each potential has been calculated, the Green's functions for the displacements in the solid formation are then given by the sum of the source terms and the surface terms originated at the two solid-fluid interfaces,

$$G_{xy}^{fsf} = G_{xy}^{full} + E_a \sum_{n=-N}^{n=+N} \left(-iA_n^y k_n E_{b0} + iB_n^y k_n E_{c0}\right) E_d + E_a \sum_{n=-N}^{n=+N} \left(-iE_n^y k_n E_{b0}^b + iF_n^y k_n E_{c0}^b\right) E_d$$

$$G_{yy}^{fsf} = G_{yy}^{full} + E_a \sum_{n=-N}^{n=+N} \left[-i\nu_n A_n^y E_{b0} + \left(\tfrac{-ik_n^2}{\gamma_n} B_n^y + \tfrac{-ik_z^2}{\gamma_n} C_n^y\right) E_{c0}\right] E_d + E_a \sum_{n=-N}^{n=+N} \left[-i\nu_n E_n^y E_{b0}^b + \left(\tfrac{-ik_n^2}{\gamma_n} F_n^y + \tfrac{-ik_z^2}{\gamma_n} G_n^y\right) E_{c0}^b\right] E_d$$

$$G_{zy}^{fsf} = G_{zy}^{full} + E_a \sum_{n=-N}^{n=+N} \left(-iA_n^y k_z E_{b0} + iC_n^y k_z E_{c0}\right) E_d + E_a \sum_{n=-N}^{n=+N} \left(-iE_n^y k_z E_{b0}^b + iG_n^y k_z E_{c0}^b\right) E_d \tag{47}$$

The final expressions for the pressure field in the two fluid media are then given by

$$\sigma_{fy}^{fsf\_top} = -\frac{i}{L_x} \sum_{n=-N}^{n=+N} \left(\frac{E_{f0}}{\nu_n^f} D_n^y\right) E_d \quad (\text{when } y < 0)$$

$$\sigma_{fy}^{fsf\_bottom} = -\tfrac{i}{L_x} \sum_{n=-N}^{n=+N} \left(\tfrac{E_{f0}^b}{\nu_n^f} H_n^y\right) E_d \quad (\text{when } y > h) \tag{48}$$

4.3 *Load in the solid formation acting in the direction of the z-axis*

The surface terms generated at the two interfaces can be expressed using the following potentials, which have been derived using the technique described above,

Solid medium (top interface)

$$\phi_0^{z\_top} = E_a k_z \sum_{n=-N}^{n=+N} \left(\frac{E_{b0}}{\nu_n} A_n^z\right) E_d$$

$$\psi_{x0}^{z\_top} = E_a \sum_{n=-N}^{n=+N} \left(E_{c0} B_n^z\right) E_d$$

$$\psi_{y0}^{z\_top} = E_a \sum_{n=-N}^{n=+N} \left(\frac{-k_n}{\gamma_n} E_{c0} C_n^z\right) E_d$$

$$\psi_{z0}^{z\_top} = 0 \tag{49}$$

Fluid medium (top interface)

$$\phi_{fluid}^{top} = -\tfrac{i}{L_x} \sum_{n=-N}^{n=+N} \left[\left(\tfrac{-\alpha_f^2}{\omega^2 \lambda_f}\right) \tfrac{E_{f0}}{\nu_n^f} D_n^z\right] E_d \quad (\text{when } y < 0) \tag{50}$$

Solid medium (bottom interface)

$$\phi_0^{z\_bottom} = E_a k_z \sum_{n=-N}^{n=+N} \left(\frac{E_{b0}^b}{\nu_n} E_n^z\right) E_d$$

$$\psi_{x0}^{z\_bottom} = E_a \sum_{n=-N}^{n=+N} \left(E_{c0}^b F_n^z\right) E_d$$

$$\psi_{y0}^{z\_bottom} = E_a \sum_{n=-N}^{n=+N} \left(\frac{-k_n}{\gamma_n} E_{c0}^b G_n^z\right) E_d$$

$$\psi_{z0}^{z\_bottom} = 0 \tag{51}$$

Fluid medium (bottom interface)

$$\phi_{fluid}^{bottom} = -\tfrac{i}{L_x} \sum_{n=-N}^{n=+N} \left[\left(\tfrac{-\alpha_f^2}{\omega^2 \lambda_f}\right) \tfrac{E_{f0}^b}{\nu_n^f} H_n^z\right] E_d \quad (\text{when } y > h) \tag{52}$$

The imposition of the eight stated boundary conditions for each value of n leads to a system of eight equations in the eight unknown constants,

$$\left[a_{ij}^z \quad i = 1,8;\ j = 1,8\right] \ [c_i^z \quad i = 1,8] = [b_i^z \quad i = 1,8] \tag{53}$$

which is fully described in Appendix D.

Once the unknown amplitude of each potential has been calculated, the Green's functions for the solid formation are given by the sum of the source terms and the surface terms originated at the two fluid-solid interfaces, leading to the following expressions,

$$G_{xz}^{fsf} = G_{xz}^{full} + E_a \sum_{n=-N}^{n=+N} \left(\tfrac{-ik_z k_n}{\nu_n} A_n^z E_{b0} + \tfrac{ik_z k_n}{\gamma_n} C_n^z E_{c0}\right) E_d + E_a \sum_{n=-N}^{n=+N} \left(\tfrac{-ik_z k_n}{\nu_n} E_n^z E_{b0}^b + \tfrac{ik_z k_n}{\gamma_n} G_n^z E_{c0}^b\right) E_d$$

$$G_{yz}^{fsf} = G_{yz}^{full} + E_a \sum_{n=-N}^{n=+N} \left(-ik_z A_n^z E_{b0} + iB_n^z k_z E_{c0}\right) E_d + E_a \sum_{n=-N}^{n=+N} \left(-ik_z E_n^z E_{bo}^b + iF_n^z k_z E_{c0}^b\right) E_d$$

$$G_{zz}^{fsf} = G_{zz}^{full} + E_a \sum_{n=-N}^{n=+N} \left[\tfrac{-ik_z^2}{\nu_n} A_n^z E_{b0} + \left(\tfrac{-ik_n^2}{\gamma_n} C_n^z - i\gamma_n B_n^z\right) E_{c0}\right] E_d + E_a \sum_{n=-N}^{n=+N} \left[\tfrac{-ik_z^2}{\nu_n} E_n^z E_{b0}^b + \left(\tfrac{-ik_n^2}{\gamma_n} G_n^z - i\gamma_n F_n^z\right) E_{c0}^b\right] E_d$$

(54)

The final expression for the pressure field in the two fluid media is given by

$$\sigma_{fz}^{fsf\_top} = -\frac{i}{L_x}\sum_{n=-N}^{n=+N}\left(\frac{E_{f0}}{\nu_n^f}D_n^z\right)E_d \quad (\text{when } y<0)$$

$$\sigma_{fz}^{fsf\_bottom} = -\frac{i}{L_x}\sum_{n=-N}^{n=+N}\left(\frac{E_{f0}^b}{\nu_n^f}H_n^z\right)E_d \quad (\text{when } y>h) \tag{55}$$

4.4 Pressure Load acting in the top layer of fluid

The surface terms produced at the two interfaces (top and bottom) can be expressed using the following potentials,

Solid medium (top interface)

$$\phi_0^{y\_top} = E_a\sum_{n=-N}^{n=+N}\left(E_{b0}A_n^f\right)E_d$$

$$\psi_{x0}^{y\_top} = E_a k_z\sum_{n=-N}^{n=+N}\left(\frac{-E_{c0}}{\gamma_n}C_n^f\right)E_d$$

$$\psi_{y0}^{y\_top} = 0$$

$$\psi_{z0}^{y\_top} = E_a\sum_{n=-N}^{n=+N}\left(\frac{k_n}{\gamma_n}E_{c0}B_n^f\right)E_d \tag{56}$$

Fluid medium (top interface)

$$\phi_{fluid}^{top} = -\frac{i}{L_x}\sum_{n=-N}^{n=+N}\left[\left(\frac{-\alpha_f^2}{\omega^2\lambda_f}\right)\frac{E_{f0}}{\nu_n^f}D_n^f\right]E_d \quad (\text{when } y<0) \tag{57}$$

Solid medium (bottom interface)

$$\phi_0^{y\_bottom} = E_a\sum_{n=-N}^{n=+N}\left(E_{b0}^bE_n^f\right)E_d$$

$$\psi_{x0}^{y\_bottom} = E_a k_z\sum_{n=-N}^{n=+N}\left(\frac{-E_{c0}^b}{\gamma_n}G_n^f\right)E_d$$

$$\psi_{y0}^{y\_bottom} = 0$$

$$\psi_{z0}^{y\_bottom} = E_a\sum_{n=-N}^{n=+N}\left(\frac{k_n}{\gamma_n}E_{c0}^bF_n^f\right)E_d \tag{58}$$

Fluid medium (bottom interface)

$$\phi_{fluid}^{bottom} = -\frac{i}{L_x}\sum_{n=-N}^{n=+N}\left[\left(\frac{-\alpha_f^2}{\omega^2\lambda_f}\right)\frac{E_{f0}^b}{\nu_n^f}H_n^f\right]E_d \quad (\text{when } y>h) \tag{59}$$

After the eight stated boundary conditions, for each value of n, have been imposed, a system of eight equations in the eight unknown constants is built up,

$$\left[a_{ij}^f \quad i=1,8;\ j=1,8\right]\left[c_i^f \quad i=1,8\right] = \left[b_i^f \quad i=1,8\right] \tag{60}$$

which is fully listed in Appendix E.

After the system of equations has been solved, the Green's functions for the solid formation are given by the sum of the source terms and the surface terms originated at the two fluid-solid interfaces, generating the following expressions,

$$G_{xf}^{fsf} = E_a\sum_{n=-N}^{n=+N}\left(-iA_n^fk_nE_{b0}+iB_n^fk_nE_{c0}\right)E_d + E_a\sum_{n=-N}^{n=+N}\left(-iE_n^fk_nE_{b0}^b+iF_n^fk_nE_{c0}^b\right)E_d$$

$$G_{yf}^{fsf} = E_a\sum_{n=-N}^{n=+N}\left[-i\nu_nA_n^fE_{b0}+\left(\frac{-ik_n^2}{\gamma_n}B_n^f+\frac{-ik_z^2}{\gamma_n}C_n^f\right)E_{c0}\right]E_d + E_a\sum_{n=-N}^{n=+N}\left[-i\nu_nE_n^fE_{b0}^b+\left(\frac{-ik_n^2}{\gamma_n}F_n^f+\frac{-ik_z^2}{\gamma_n}G_n^f\right)E_{c0}^b\right]E_d$$

$$G_{zf}^{fsf} = E_a\sum_{n=-N}^{n=+N}\left(-iA_n^fk_zE_{b0}+iC_n^fk_zE_{c0}\right)E_d + E_a\sum_{n=-N}^{n=+N}\left(-iE_n^fk_zE_{b0}^b+iG_n^fk_zE_{c0}^b\right)E_d \tag{61}$$

The final expressions for the pressure field in the two fluid media are then given by

$$\sigma^{fsf\_top} = \sigma^{full} - \frac{i}{L_x}\sum_{n=-N}^{n=+N}\left(\frac{E_{f0}}{\nu_n^f}D_n^f\right)E_d \quad (\text{when y} < 0)$$

$$\sigma^{fsf\_bottom} = -\frac{i}{L_x}\sum_{n=-N}^{n=+N}\left(\frac{E_{f0}^b}{\nu_n^f}H_n^f\right)E_d \quad (\text{when } y>h) \tag{62}$$

Note that, if $k_z = 0$ is used, the system of equations derived above is reduced to six unknowns, leading to the two-dimensional Green's function for plane strain line-loads.

5 Summary of Green's Functions

The Tables 1 - 3 summarizes the Green's functions presented along the paper.

Table 1 : Green's Functions in an Unbounded Medium

Location of output	Location of source		Direction of load	Equations
Solid	x_0, y_0	in solid	x axis	(7)
Solid	x_0, y_0	in solid	y axis	(9)
Solid	x_0, y_0	in solid	z axis	(11)
Fluid	x_0, y_0	in fluid	–	(14)

Table 2 : Green's Functions in a Fluid-solid Formation

Location of output	Location of source		Direction of load	Equations
Solid	x_0, y_0	in solid	x axis	(18)
Fluid	x_0, y_0	in solid	x axis	(19)
Solid	x_0, y_0	in solid	y axis	(23)
Fluid	x_0, y_0	in solid	y axis	(24)
Solid	x_0, y_0	in solid	z axis	(28)
Fluid	x_0, y_0	in solid	z axis	(29)
Solid	x_0, y_0	in fluid	–	(33)
Fluid	x_0, y_0	in fluid	–	(34)

Table 3 : Green's Functions in a Solid Layer Formation Bounded by Two Fluid Media

Location of output	Location of source		Direction of load	Equations
Solid	x_0, y_0	in solid	x axis	(40)
Fluid	x_0, y_0	in solid	x axis	(41)
Solid	x_0, y_0	in solid	y axis	(47)
Fluid	x_0, y_0	in solid	y axis	(48)
Solid	x_0, y_0	in solid	z axis	(54)
Fluid	x_0, y_0	in solid	z axis	(55)
Solid	x_0, y_0	in fluid	–	(61)
Fluid	x_0, y_0	in fluid	–	(62)

6 Validation of the Solution

The expressions described in the previous sections were applied to two cases: a solid formation bounded by one flat fluid medium (see Fig. 3) and a solid layer bounded by two parallel fluid media (see Fig. 4). The calculations are performed for the three displacement fields in the solid medium and the pressure field within the fluid. The spatially harmonic varying line load is assumed to be buried either in the solid formation, or in the fluid medium. The results provided were then validated with those arrived at by applying the BEM model, which requires the discretization of the solid-fluid interfaces using the Green's functions for a full space. The BEM code has been previously validated for the case of a circular inclusion, for which analytical solutions exist. It should be pointed out that the author's solutions do not require discretization of the material interfaces. This affords an enormous computational advantage and allows the computation of problems which cannot be tackled by other numerical methods, particularly for high frequencies.

The examples presented do not require an infinite number of boundary elements along the material interfaces because complex frequencies are used, with a small imaginary part of the form $\omega_c = \omega - i\eta$ (with $\eta = 0.7\frac{2\pi}{T}$) [Bouchon and Aki (1977); Phinney (1965)]. Boundary elements make a significant contribution to the response for a certain value of damping, but are otherwise unnecessary. These elements are distributed along the surface up to a spatial distance (L_{dist}) from the center, given by $L_{dist} = \alpha T$. This gives a discretized surface with a length $2L_{dist}$. Many simulations were conducted in order to study the effect of varying the size of boundary elements on the accuracy of the response. Improved performance was obtained by placing smaller elements close to where the response is required. Boundary elements of varying sizes were therefore used, with the shorter elements being placed nearer to the center of the discretized surface.

The scheme adopted here for determining the placement and size of the boundary elements uses the following geometrical construction: an auxiliary circular arc is divided into equal segments according to a previously defined ratio between the wavelength of the dilatational waves and the length of boundary elements. The boundary elements are then defined on the topographic surface by the vertical projection of these segments. The radius of the required circular arc (R) is greater than $(2L_{dist})/2$ and is placed at a tangent to the topographic interfaces at its boundary discretization end, thus avoiding unduly small boundary elements. In this work R is assumed to be $\left[(2L_{dist})/2\right]/\cos 10^o$ (see Fig. 2).

Next, the results are obtained for the two scenarios studied here. First, the solid formation is assumed to be bounded by one flat fluid medium (see Fig. 3). Then, a solid layer, 10.0 m thick, is bounded by two paral-

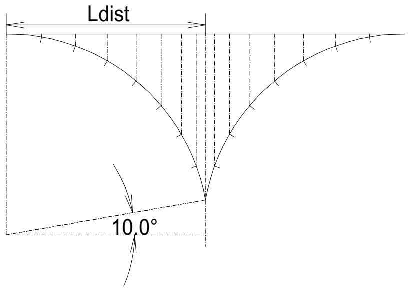

Figure 2 : Definition of the boundary elements

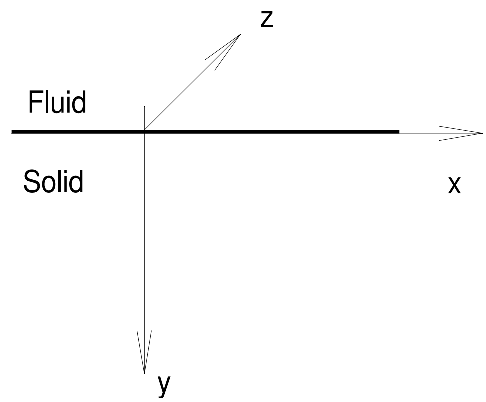

Figure 3 : Solid formation bounded by a flat fluid medium

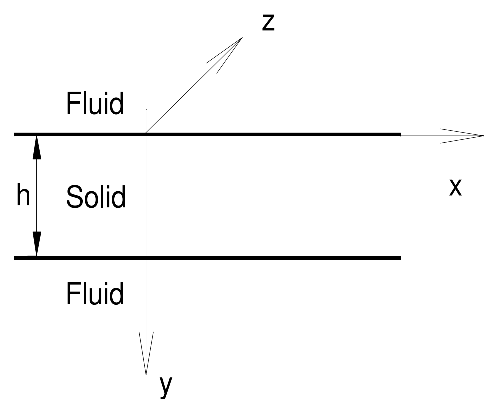

Figure 4 : Solid formation bounded by two flat fluid parallel media

lel fluid media (see Fig. 4). For the two scenarios the solid medium takes $\alpha = 4208\,\mathrm{m/s}$, $\beta = 2656\,\mathrm{m/s}$ with $\rho = 2140\,\mathrm{Kg/m^3}$, while the fluid medium allows $\alpha = 1500\,\mathrm{m/s}$ with $\rho = 1000\,\mathrm{Kg/m^3}$. The solid-fluid structures are illuminated either by a harmonic point source applied to the solid medium at the source point ($x = 1.0\,\mathrm{m}$, $y = 2.0\,\mathrm{m}$), acting along the directions x, y and z, or by a harmonic point pressure source applied to the fluid medium at ($x = 1.0\,\mathrm{m}$, $y = -2.0\,\mathrm{m}$). Calculations are performed in the frequency range [2.50, 320.0 Hz] with a frequency increment of 2.5 Hz. The imaginary part of the frequency has been set to $\eta = 0.7\frac{2\pi}{T}$ with $T = 0.0466$ s. To validate the results, the response is computed at a single value of k_z ($k_z = 0.4\,\mathrm{rad/m}$). The real and imaginary parts of the responses are shown in Fig. 5 - Fig. 6. The analytical responses are represented by the solid lines, while the marked points correspond to the BEM solution. The square and the round marks indicate the real and imaginary part of the responses, respectively. The BEM solution was computed for a very large number of boundary elements defined by the ratio between the wavelength of the incident waves and the length of the boundary elements, which was kept to a minimum of 40 elements at each interface.

6.1 Case 1: Solid formation bounded by a single flat fluid medium

To demonstrate the correctness of the equations described during the course of this work, the results are only illustrated when the load is applied at the solid formation along the y direction. The surface terms of the displacement fields, G_{xy}^{surf}, G_{yy}^{surf} and G_{zy}^{surf} are calculated at receivers placed at $x = 3.0\,\mathrm{m}$ and $y = 5.0\,\mathrm{m}$, within the solid, while the pressure response σ_{fy}^{fs} is computed at receivers $x = 3.0\,\mathrm{m}$ and $y = -1.0\,\mathrm{m}$ (see Fig. 5).

6.2 Case 2: Solid layer bounded by two flat fluid media

The results presented refer to the case where a harmonic point pressure source is applied to the top fluid medium. The surface terms of the displacement fields, G_{xf}^{fsf}, G_{yf}^{fsf} and G_{zf}^{fsf} are calculated at receivers placed at $x = 3.0\,\mathrm{m}$ and $y = 5.0\,\mathrm{m}$, within the solid, while the pressure response $\sigma^{fsf\_top\_surf}(\sigma^{fsf\_top} - \sigma^{full})$ and σ^{fsf_bottom} are computed at receivers $x = 3.0\,\mathrm{m}$ and $y = -1.0\,\mathrm{m}$, and $x = 3.0\,\mathrm{m}$ and $y = 15.0\,\mathrm{m}$, respectively (see Fig. 6).

These calculations are restricted to low frequencies because, for higher frequencies, the conventional BEM solution would require the use of a very large number of boundary elements, which would make its solution

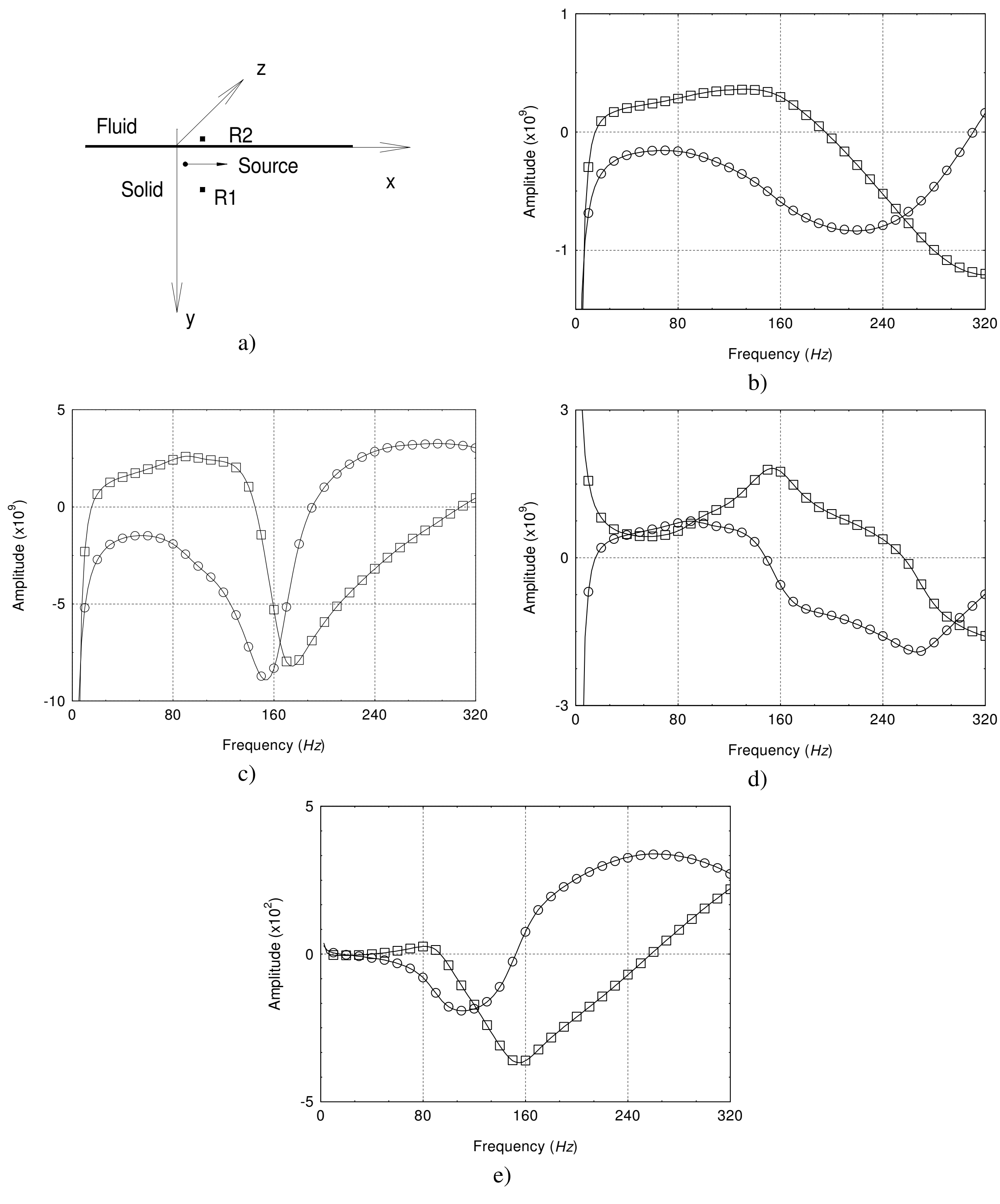

Figure 5 : Spatially sinusoidal harmonic line load along the z direction in a solid formation bounded by a flat fluid medium, applied in the y direction: a) Geometry of the problem; b) G_{xy}^{surf} solutions; c) G_{yy}^{surf} solutions; d) G_{zy}^{surf} solutions; e) σ_{fy}^{fs} solutions

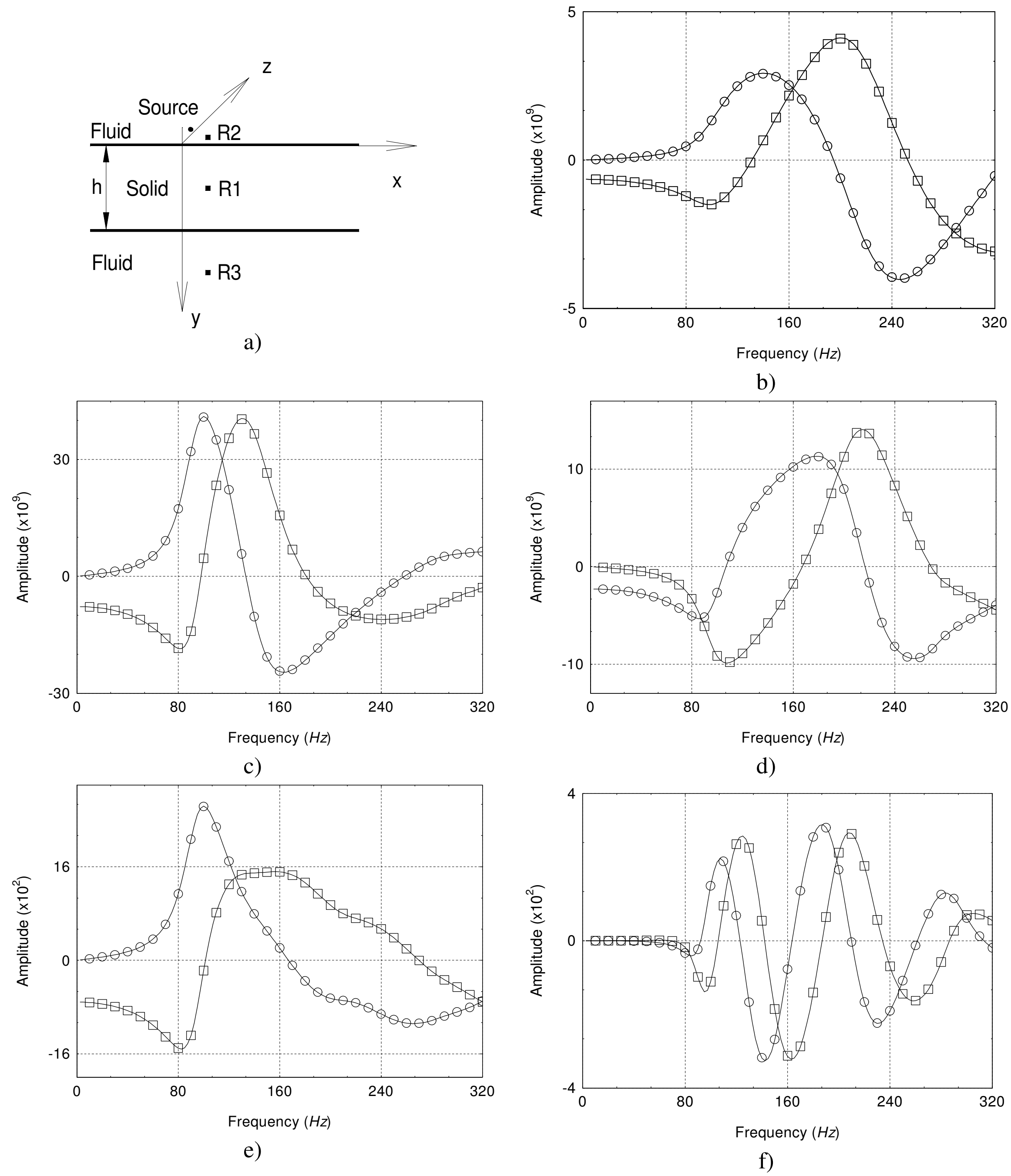

Figure 6 : Spatially sinusoidal harmonic line pressure load applied to the top fluid medium of a solid formation bounded by two fluid parallel media: a) Geometry of the problem; b) G_{xf}^{fsf} solutions; c) G_{yf}^{fsf} solutions; d) G_{zf}^{fsf} solutions; e) $\sigma^{fsf\_top\_surf}$ solutions; f) σ^{fsf_bottom} solutions

impossible, owing to computational cost. It may further be mentioned that the BEM solutions provided in the present examples were obtained by limiting the discretization of the solid-fluid interfaces, through the use of complex frequencies, in such a way as to diminish the contribution of the waves generated at sources placed at the end of the discretization.

As can be seen, these two solutions are in very close agreement, and equally good results were obtained from tests in which loads and receivers were situated at different points.

The proposed Green's functions are most useful in the context of boundary elements, such as in the solution of an inclusion buried within a flat solid layer. The use of these functions will avoid the discretization of the solid-fluid interfaces, and the discretization would be restricted to the boundary of the inclusion.

7 Conclusions

Having successfully obtained a completely analytical solution for the steady state response of a spatially sinusoidal, harmonic line load in a homogeneous solid formation, bounded by one or two flat fluid media, we compared the final expressions with numerical results calculated by using the Boundary Element Method in order to validate them. The solutions were found to be in very close agreement when the solid-fluid interface was discretized with a large number of boundary elements.

The analytical solutions presented in this paper are interesting in themselves. They provide the displacement, strain or stress in a formation formed by an elastic solid medium, bounded by one or two acoustic flat fluid media illuminated by a spatially sinusoidal harmonic load buried in solid formation or in the fluid medium. The solutions applied in conjunction with numerical methods, such as the BEM, make the discretization of the solid-fluid interfaces unnecessary, and may prove to be very useful in many engineering applications, such as calculating the acoustic insulation provided by solid walls, and in the context of interpreting seismic responses.

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Appendix A: The Green's function for a two-and-a-half dimensional full-space

$$G_{xx}^{full} = \frac{i}{4\rho\omega^2}\left[k_s^2 H_{0\beta} - \frac{1}{r}B_1 + \left(\frac{x-x_0}{r}\right)^2 B_2\right] \tag{A.1}$$

$$G_{yy}^{full} = \frac{i}{4\rho\omega^2}\left[k_s^2 H_{0\beta} - \frac{1}{r}B_1 + \left(\frac{y-y_0}{r}\right)^2 B_2\right] \tag{A.2}$$

$$G_{zz}^{full} = \frac{i}{4\rho\omega^2}\left(k_s^2 H_{0\beta} - k_z^2 B_0\right) \tag{A.3}$$

$$G_{xy}^{full} = G_{yx}^{full} = \frac{i}{4\rho\omega^2}\left(\frac{x-x_0}{r}\right)\left(\frac{y-y_0}{r}\right)B_2 \tag{A.4}$$

$$G_{xz}^{full} = G_{zx}^{full} = \frac{-k_z}{4\rho\omega^2}\left(\frac{x-x_0}{r}\right)B_1 \tag{A.5}$$

$$G_{yz}^{full} = G_{zy}^{full} = \frac{-k_z}{4\rho\omega^2}\left(\frac{y-y_0}{r}\right)B_1 \tag{A.6}$$

with $r = \sqrt{(x-x_0)^2 + (y-y_0)^2}$, $H_{n\alpha} = H_n^{(2)}(k_\alpha r)$, $H_{n\beta} = H_n^{(2)}(k_\beta r)$ and $B_n = k_\beta^n H_{n\beta} - k_\alpha^n H_{n\alpha}$.

Appendix B: Definition of $\left(a_{ij}^x \;\; i=1,8;\; j=1,8\right)$, $(c_i^x \;\; i=1,8)$ and $(b_i^x \;\; i=1,8)$

$a_{ij}^x \;\; i=1,8;\; j=1,8$

$a_{11}^x = -2k_n^2$	$a_{12}^x = -k_z^2$	$a_{13}^x = k_n^2 - \gamma_n^2$
$a_{14}^x = 0$	$a_{15}^x = -a_{11}^x E_{bh}$	$a_{16}^x = -a_{12}^x E_{ch}$
$a_{17}^x = -a_{13}^x E_{ch}$	$a_{18}^x = 0$	$a_{21}^x = -2$
$a_{22}^x = 1$	$a_{23}^x = 1$	$a_{24}^x = 0$
$a_{25}^x = -a_{21}^x E_{bh}$	$a_{26}^x = -a_{22}^x E_{ch}$	$a_{27}^x = -a_{23}^x E_{ch}$
$a_{28}^x = 0$	$a_{31}^x = \frac{-k_s^2}{\nu_n} - \frac{2\nu_{zn}^2}{\nu_n}$	$a_{32}^x = 0$
$a_{33}^x = 2\gamma_n$	$a_{34}^x = \frac{-i\,2\rho\omega^2}{\nu_n^f \mu}$	$a_{35}^x = a_{31}^x E_{bh}$
$a_{36}^x = 0$	$a_{37}^x = a_{33}^x E_{ch}$	$a_{38}^x = 0$
$a_{41}^x = -i$	$a_{42}^x = 0$	$a_{43}^x = i$
$a_{44}^x = \frac{-2\rho\omega^2}{k_{p_f}^2 \lambda_f}$	$a_{45}^x = -a_{41}^x E_{bh}$	$a_{46}^x = 0$
$a_{47}^x = -a_{43}^x E_{ch}$	$a_{48}^x = 0$	$a_{51}^x = a_{11}^x E_{bh}$
$a_{52}^x = a_{12}^x E_{ch}$	$a_{53}^x = a_{13}^x E_{ch}$	$a_{54}^x = 0$
$a_{55}^x = -a_{11}^x$	$a_{56}^x = -a_{12}^x$	$a_{57}^x = -a_{13}^x$
$a_{58}^x = 0$	$a_{61}^x = a_{21}^x E_{bh}$	$a_{62}^x = a_{22}^x E_{ch}$
$a_{63}^x = a_{23}^x E_{ch}$	$a_{64}^x = 0$	$a_{65}^x = -a_{21}^x$
$a_{66}^x = -a_{22}^x$	$a_{67}^x = -a_{23}^x$	$a_{68}^x = 0$
$a_{71}^x = a_{31}^x E_{bh}$	$a_{72}^x = 0$	$a_{73}^x = a_{33}^x E_{ch}$
$a_{74}^x = 0$	$a_{75}^x = a_{31}^x$	$a_{76}^x = 0$
$a_{77}^x = a_{33}^x$	$a_{78}^x = a_{34}^x$	$a_{81}^x = a_{41}^x E_{bh}$
$a_{82}^x = 0$	$a_{83}^x = a_{43}^x E_{ch}$	$a_{84}^x = 0$
$a_{85}^x = -a_{41}^x$	$a_{86}^x = 0$	$a_{87}^x = -a_{43}^x$

$a_{88}^x = -a_{44}^x$

$c_i^x \;\; i=1,8$

$c_1^x = A_n^x$	$c_2^x = B_n^x$	$c_3^x = C_n^x$
$c_4^x = D_n^x / k_n$	$c_5^x = E_n^x$	$c_6^x = F_n^x$
$c_7^x = G_n^x$	$c_8^x = H_n^x / k_n$	

$b_i^x \;\; i=1,8$

$b_1^x = -2k_n^2 E_{b1} + \left(-k_s^2 + 2k_n^2\right) E_{c1}$	$b_2^x = -2E_{b1} + 2E_{c1}$
$b_3^x = \left(\frac{k_s^2}{\nu_n} + \frac{2\nu_{zn}^2}{\nu_n}\right) E_{b1} - 2\gamma_n E_{c1}$	$b_4^x = -iE_{b1} + iE_{c1}$
$b_5^x = 2k_n^2 E_{bh1} - \left(-k_s^2 + 2k_n^2\right) E_{ch1}$	$b_6^x = 2E_{bh1} - 2E_{ch1}$
$b_7^x = \left(\frac{k_s^2}{\nu_n} + \frac{2\nu_{zn}^2}{\nu_n}\right) E_{bh1} - 2\gamma_n E_{ch1}$	$b_8^x = iE_{bh1} - iE_{ch1}$

with $E_{b1} = e^{-i\nu_n y_0}$, $E_{c1} = e^{-i\gamma_n y_0}$, $\nu_{zn} = \sqrt{-k_z^2 - k_n^2}$, $E_{bh} = e^{-i\nu_n h}$, $E_{ch} = e^{-i\gamma_n h}$, $E_{bh1} = e^{-i\nu_n |h-y_0|}$ and $E_{ch1} = e^{-i\gamma_n |h-y_0|}$.

Appendix C: Definition of $\left(a_{ij}^y \;\; i=1,8;\; j=1,8\right)$, $\left(c_i^y \;\; i=1,8\right)$ and $\left(b_i^y \;\; i=1,8\right)$

$a_{ij}^y \;\; i=1,8;\; j=1,8$

$a_{11}^y = -2\nu_n$	$a_{12}^y = \frac{-k_n^2}{\gamma_n} + \gamma_n$	$a_{13}^y = \frac{-k_z^2}{\gamma_n}$
$a_{14}^y = 0$	$a_{15}^y = a_{11}^y E_{bh}$	$a_{16}^y = a_{12}^y E_{ch}$
$a_{17}^y = a_{13}^y E_{ch}$	$a_{18}^y = 0$	$a_{21}^y = -2\nu_n$
$a_{22}^y = \frac{-k_n^2}{\gamma_n}$	$a_{23}^y = \frac{-k_z^2}{\gamma_n} + \gamma_n$	$a_{24}^y = 0$
$a_{25}^y = a_{21}^y E_{bh}$	$a_{26}^y = a_{22}^y E_{ch}$	$a_{27}^y = a_{23}^y E_{ch}$
$a_{28}^y = 0$	$a_{31}^y = -k_s^2 - 2\nu_{zn}^2$	$a_{32}^y = -2k_n^2$
$a_{33}^y = -2k_z^2$	$a_{34}^y = \frac{-i\,2\rho\omega^2}{\nu_n^f \mu}$	$a_{35}^y = -a_{31}^y E_{bh}$
$a_{36}^y = -a_{32}^y E_{ch}$	$a_{37}^y = -a_{33}^y E_{ch}$	$a_{38}^y = 0$
$a_{41}^y = -i\nu_n$	$a_{42}^y = \frac{-ik_n^2}{\gamma_n}$	$a_{43}^y = \frac{-ik_z^2}{\gamma_n}$
$a_{44}^y = \frac{-2\rho\omega^2}{k_{p_f}^2 \lambda_f}$	$a_{45}^y = a_{41}^y E_{bh}$	$a_{46}^y = a_{42}^y E_{ch}$
$a_{47}^y = a_{43}^y E_{ch}$	$a_{48}^x = 0$	$a_{51}^y = a_{11}^y E_{bh}$
$a_{52}^y = a_{12}^y E_{ch}$	$a_{53}^y = a_{13}^y E_{ch}$	$a_{54}^y = 0$
$a_{55}^y = a_{11}^y$	$a_{56}^y = a_{12}^y$	$a_{57}^y = a_{13}^y$
$a_{58}^y = 0$	$a_{61}^y = a_{21}^y E_{bh}$	$a_{62}^y = a_{22}^y E_{ch}$

$a^y_{63} = a^y_{23}E_{ch}$ $\quad a^y_{64} = 0$ $\quad a^y_{65} = a^y_{21}$
$a^y_{66} = a^y_{22}$ $\quad a^y_{67} = a^y_{23}$ $\quad a^y_{68} = 0$
$a^y_{71} = a^y_{31}E_{bh}$ $\quad a^y_{72} = a^y_{32}E_{ch}$ $\quad a^y_{73} = a^y_{33}E_{ch}$
$a^x_{74} = 0$ $\quad a^y_{75} = -a^y_{31}$ $\quad a^y_{76} = -a^y_{32}$
$a^y_{77} = -a^y_{33}$ $\quad a^y_{78} = a^y_{34}$ $\quad a^y_{81} = a^y_{41}E_{bh}$
$a^y_{82} = a^y_{42}E_{ch}$ $\quad a^y_{83} = a^y_{43}E_{ch}$ $\quad a^x_{84} = 0$
$a^y_{85} = a^y_{41}$ $\quad a^y_{86} = a^y_{42}$ $\quad a^y_{87} = a^y_{43}$
$a^y_{88} = -a^y_{44}$

$c^y_i \;\; i = 1,8$

$c^y_1 = A^y_n$ $\quad c^y_2 = B^y_n$ $\quad c^y_3 = C^y_n$
$c^y_4 = D^y_n$ $\quad c^y_5 = E^y_n$ $\quad c^y_6 = F^y_n$
$c^y_7 = G^y_n$ $\quad c^y_8 = H^y_n$

$b^y_i \;\; i = 1,8$

$$b^y_1 = 2\nu_n E_{b1} - \left(\frac{\nu^2_{zn}}{\gamma_n} + \gamma_n\right) E_{c1}$$
$$b^y_2 = 2\nu_n E_{b1} - \left(\frac{\nu^2_{zn}}{\gamma_n} + \gamma_n\right) E_{c1}$$
$$b^y_3 = \left(-k^2_s - 2\nu^2_{zn}\right) E_{b1} + 2\nu^2_{zn} E_{c1}$$
$$b^y_4 = i\nu_n E_{b1} + \left(\frac{ik^2_n}{\gamma_n} + \frac{ik^2_z}{\gamma_n}\right) E_{c1}$$
$$b^y_5 = 2\nu_n E_{bh1} - \left(\frac{\nu^2_{zn}}{\gamma_n} + \gamma_n\right) E_{ch1}$$
$$b^y_6 = 2\nu_n E_{bh1} - \left(\frac{\nu^2_{zn}}{\gamma_n} + \gamma_n\right) E_{ch1}$$
$$b^y_7 = \left(k^2_s + 2\nu^2_{zn}\right) E_{bh1} - 2\nu^2_{zn} E_{ch1}$$
$$b^y_8 = i\nu_n E_{bh1} + \left(\frac{ik^2_n}{\gamma_n} + \frac{ik^2_z}{\gamma_n}\right) E_{ch1}$$

with $E_{b1} = e^{-i\nu_n y_0}$, $E_{c1} = e^{-i\gamma_n y_0}$, $\nu_{zn} = \sqrt{-k^2_z - k^2_n}$, $E_{bh} = e^{-i\nu_n h}$, $E_{ch} = e^{-i\gamma_n h}$, $E_{bh1} = e^{-i\nu_n |h-y_0|}$ and $E_{ch1} = e^{-i\gamma_n |h-y_0|}$.

Appendix D: Definition of $\left(a^z_{ij} \;\; i = 1,8;\; j = 1,8\right)$, $(c^z_i \;\; i = 1,8)$ and $(b^z_i \;\; i = 1,8)$

$a^z_{ij} \;\; i = 1,8;\; j = 1,8$

$a^z_{11} = -2$ $\quad a^z_{12} = 1$ $\quad a^z_{13} = 1$
$a^z_{14} = 0$ $\quad a^z_{15} = -a^z_{11}E_{bh}$ $\quad a^z_{16} = -a^z_{12}E_{ch}$
$a^z_{17} = -a^z_{13}E_{ch}$ $\quad a^z_{18} = 0$ $\quad a^z_{21} = -2k^2_z$
$a^z_{22} = k^2_z - \gamma^2_n$ $\quad a^z_{23} = -k^2_n$ $\quad a^z_{24} = 0$
$a^z_{25} = -a^z_{21}E_{bh}$ $\quad a^z_{26} = -a^z_{22}E_{ch}$ $\quad a^z_{27} = -a^z_{23}E_{ch}$
$a^z_{28} = 0$ $\quad a^z_{31} = -\frac{k^2_s}{\nu_n} - \frac{2\nu^2_{zn}}{\nu_n}$ $\quad a^z_{32} = 2\gamma_n$
$a^z_{33} = 0$ $\quad a^z_{34} = \frac{-i\,2\rho\omega^2}{\nu^f_n k_z \mu}$ $\quad a^z_{35} = a^z_{31}E_{bh}$
$a^z_{36} = a^z_{32}E_{ch}$ $\quad a^z_{37} = 0$ $\quad a^x_{38} = 0$
$a^z_{41} = -ik_z$ $\quad a^z_{42} = ik_z$ $\quad a^z_{43} = 0$
$a^z_{44} = \frac{-2\rho\omega^2}{k^2_{p_f}\lambda_f}$ $\quad a^z_{45} = -a^z_{41}E_{bh}$ $\quad a^z_{46} = -a^z_{42}E_{ch}$
$a^z_{47} = 0$ $\quad a^z_{48} = 0$ $\quad a^z_{51} = a^z_{11}E_{bh}$
$a^z_{52} = a^z_{12}E_{ch}$ $\quad a^z_{53} = a^z_{13}E_{ch}$ $\quad a^z_{54} = 0$
$a^z_{55} = -a^z_{11}$ $\quad a^z_{56} = -a^z_{12}$ $\quad a^z_{57} = -a^z_{13}$
$a^z_{58} = 0$ $\quad a^z_{61} = a^z_{21}E_{bh}$ $\quad a^z_{62} = a^z_{22}E_{ch}$
$a^z_{63} = a^z_{23}E_{ch}$ $\quad a^z_{64} = 0$ $\quad a^z_{65} = -a^z_{21}$
$a^z_{66} = -a^z_{22}$ $\quad a^z_{67} = -a^z_{23}$ $\quad a^z_{68} = 0$
$a^z_{71} = a^z_{31}E_{bh}$ $\quad a^z_{72} = a^z_{32}E_{ch}$ $\quad a^z_{73} = 0$
$a^z_{74} = 0$ $\quad a^z_{75} = a^z_{31}$ $\quad a^z_{76} = a^z_{32}$
$a^z_{77} = 0$ $\quad a^z_{78} = a^z_{34}$ $\quad a^z_{81} = a^z_{41}E_{bh}$
$a^z_{82} = a^z_{42}E_{ch}$ $\quad a^z_{83} = 0$ $\quad a^z_{84} = 0$
$a^z_{85} = -a^z_{41}$ $\quad a^z_{86} = -a^z_{42}$ $\quad a^z_{87} = 0$
$a^z_{88} = -a^z_{44}$

$c^z_i \;\; i = 1,8$

$c^z_1 = A^z_n$ $\quad c^z_2 = B^z_n$ $\quad c^z_3 = C^z_n$
$c^z_4 = D^z_n$ $\quad c^z_5 = E^z_n$ $\quad c^z_6 = F^z_n$
$c^z_7 = G^z_n$ $\quad c^z_8 = H^z_n$

$b^z_i \;\; i = 1,8$

$$b^z_1 = 2\,\left(-E_{b1} + E_{c1}\right)$$
$$b^z_2 = -2k^2_z E_{b1} + \left(k^2_z - \gamma^2_n - k^2_n\right) E_{c1}$$
$$b^z_3 = \left(\frac{k^2_s}{\nu_n} + \frac{2\nu^2_{zn}}{\nu_n}\right) E_{b1} - 2\gamma_n E_{c1}$$
$$b^z_4 = -ik_z E_{b1} + ik_z E_{c1}$$
$$b^z_5 = -2\,\left(-E_{bh1} + E_{ch1}\right)$$
$$b^z_6 = 2k^2_z E_{bh1} - \left(k^2_z - \gamma^2_n - k^2_n\right) E_{ch1}$$
$$b^z_7 = \left(\frac{k^2_s}{\nu_n} + \frac{2\nu^2_{zn}}{\nu_n}\right) E_{bh1} - 2\gamma_n E_{ch1}$$
$$b^z_8 = ik_z E_{bh1} - ik_z E_{ch1}$$

with $E_{b1} = e^{-i\nu_n y_0}$, $E_{c1} = e^{-i\gamma_n y_0}$, $\nu_{zn} = \sqrt{-k^2_z - k^2_n}$,

$E_{bh} = e^{-i\nu_n h}$, $E_{ch} = e^{-i\gamma_n h}$, $E_{bh1} = e^{-i\nu_n|h-y_0|}$ and $E_{ch1} = e^{-i\gamma_n|h-y_0|}$.

Appendix E: Definition of $\left(a_{ij}^f \;\; i=1,8;\; j=1,8\right)$, $\left(c_i^f \;\; i=1,8\right)$ and $\left(b_i^f \;\; i=1,8\right)$

$a_{ij}^f \;\; i=1,8;\; j=1,8$

$a_{11}^f = -2\nu_n$ $\quad a_{12}^f = \frac{-k_n^2}{\gamma_n} + \gamma_n$ $\quad a_{13}^f = \frac{-k_z^2}{\gamma_n}$

$a_{14}^f = 0$ $\quad a_{15}^f = a_{11}^f E_{bh}$ $\quad a_{16}^f = a_{12}^f E_{ch}$

$a_{17}^f = a_{13}^f E_{ch}$ $\quad a_{18}^f = 0$ $\quad a_{21}^f = -2\nu_n$

$a_{22}^f = \frac{-k_n^2}{\gamma_n}$ $\quad a_{23}^f = \frac{-k_z^2}{\gamma_n} + \gamma_n$ $\quad a_{24}^f = 0$

$a_{25}^f = a_{21}^f E_{bh}$ $\quad a_{26}^f = a_{22}^f E_{ch}$ $\quad a_{27}^f = a_{23}^f E_{ch}$

$a_{28}^f = 0$ $\quad a_{31}^f = -k_s^2 - 2\nu_{zn}^2$ $\quad a_{32}^f = -2k_n^2$

$a_{33}^f = -2k_z^2$ $\quad a_{34}^f = \frac{-i\,2\rho\omega^2}{\nu_n^f \mu}$ $\quad a_{35}^f = -a_{31}^y E_{bh}$

$a_{36}^f = -a_{32}^f E_{ch}$ $\quad a_{37}^f = -a_{33}^f E_{ch}$ $\quad a_{38}^f = 0$

$a_{41}^f = -i\nu_n$ $\quad a_{42}^f = \frac{-ik_n^2}{\gamma_n}$ $\quad a_{43}^f = \frac{-ik_z^2}{\gamma_n}$

$a_{44}^f = \frac{-2\rho\omega^2}{k_{p_f}^2 \lambda_f}$ $\quad a_{45}^f = a_{41}^f E_{bh}$ $\quad a_{46}^f = a_{42}^f E_{ch}$

$a_{47}^f = a_{43}^f E_{ch}$ $\quad a_{48}^f = 0$ $\quad a_{51}^f = a_{11}^f E_{bh}$

$a_{52}^f = a_{12}^f E_{ch}$ $\quad a_{53}^f = a_{13}^f E_{ch}$ $\quad a_{54}^f = 0$

$a_{55}^f = a_{11}^f$ $\quad a_{56}^f = a_{12}^f$ $\quad a_{57}^f = a_{13}^f$

$a_{58}^f = 0$ $\quad a_{61}^f = a_{21}^f E_{bh}$ $\quad a_{62}^f = a_{22}^f E_{ch}$

$a_{63}^f = a_{23}^f E_{ch}$ $\quad a_{64}^f = 0$ $\quad a_{65} = a_{21}^f$

$a_{66}^f = a_{22}^f$ $\quad a_{67}^f = a_{23}^f$ $\quad a_{68}^f = 0$

$a_{71}^f = a_{31}^f E_{bh}$ $\quad a_{72}^f = a_{32}^f E_{ch}$ $\quad a_{73}^f = a_{33}^f E_{ch}$

$a_{74}^f = 0$ $\quad a_{75}^f = -a_{31}^f$ $\quad a_{76}^f = -a_{32}^f$

$a_{77}^f = -a_{33}^f$ $\quad a_{78}^f = a_{34}^f$ $\quad a_{81}^f = a_{41}^f E_{bh}$

$a_{82}^f = a_{42}^f E_{ch}$ $\quad a_{83}^f = a_{43}^f E_{ch}$ $\quad a_{84}^f = 0$

$a_{85}^f = a_{41}^f$ $\quad a_{86}^f = a_{42}^f$ $\quad a_{87}^f = a_{43}^f$

$a_{88}^f = -a_{44}^f$

$c_i^f \;\; i=1,8$

$c_1^f = A_n^f$ $\quad c_2^f = B_n^f$ $\quad c_3^f = C_n^f$

$c_4^f = D_n^f$ $\quad c_5^f = E_n^f$ $\quad c_6^f = F_n^f$

$c_7^f = G_n^f$ $\quad c_8^f = H_n^f$

$b_i^f \;\; i=1,8$

$b_1^f = 0$ $\quad b_2^f = 0$

$b_3^f = \frac{-i2\rho\omega^2}{\nu_n^f \mu} E_{f1}$ $\quad b_4^f = \frac{2\rho\omega^2}{k_{p_f}^2 \lambda_f} E_{f1}$

$b_5^f = 0$ $\quad b_6^f = 0$

$b_7^f = 0$ $\quad b_8^f = 0$

with $E_{b1} = e^{-i\nu_n y_0}$, $E_{c1} = e^{-i\gamma_n y_0}$, $\nu_{zn} = \sqrt{-k_z^2 - k_n^2}$, $E_{f1} = e^{-i\nu_n^f y_0}$, $E_{bh} = e^{-i\nu_n h}$, $E_{ch} = e^{-i\gamma_n h}$, $E_{bh1} = e^{-i\nu_n|h-y_0|}$ and $E_{ch1} = e^{-i\gamma_n|h-y_0|}$.