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On the Equivalence Between Least-Squares and Kernel Approximations in Meshless Methods

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Abstract: Meshless methods using least-squares approximations and kernel approximations are based on non-shifted and shifted polynomial basis, respectively. We show that, mathematically, the shifted and non-shifted polynomial basis give rise to identical interpolation functions when the nodal volumes are set to unity in kernel approximations. This result indicates that mathematically the least-squares and kernel approximations are equivalent. However, for large point distributions or for higher-order polynomial basis the numerical errors with a non-shifted approach grow quickly compared to a shifted approach, resulting in violation of consistency conditions. Hence, a shifted polynomial basis is better suited from a numerical implementation point of view. Finally, we introduce an improved finite cloud method which uses a shifted polynomial basis and a fixed-kernel approximation for construction of interpolation functions and a collocation technique for discretization of the governing equations. Numerical results indicate that the improved finite cloud method exhibits superior convergence characteristics compared to our original implementation [Aluru and Li (2001)] of the finite cloud method.

1 Introduction

A class of meshless methods (see e.g. [Belytschko, et al. (1996)] for an overview on meshless methods) use a least-squares approach to construct interpolation functions and a number of other approaches use kernel approximations to construct interpolation functions. For example, the element-free Galerkin method [Belytschko, Lu, and Gu (1994)], hp-clouds [Duarte and Oden (1996)], local boundary integral equation (LBIE) [Zhu, Zhang, Atluri (1998)], meshless local Petrov-Galerkin (MLPG) [Atluri and Zhu (1998); Kim and Atluri (2000); Lin and Atluri (2000)], finite point method [Onate, et al. (1996); Wordelman et al. (2000)], boundary node method [Mukherjee and Mukherjee (1997)] and a number of other approaches use least-square approaches to construct interpolation functions. Reproducing kernel particle method [Liu, et al. (1995); Chen, et. al. (1996)], point collocation based on reproducing kernels [Aluru (2000)], and a number of other approaches use kernel approximations to construct interpolation functions.

A difference between least-squares and kernel based approaches is the use of non-shifted basis functions in least-squares type approaches and a shifted basis in kernel type approximations. In this paper, we show that both shifted and non-shifted polynomial basis produce mathematically equivalent interpolation functions. However, we also show that a non-shifted form of the base interpolating polynomial starts violating consistency conditions for large point distributions and higher order polynomial basis.

Collocation based meshless methods typically employ higher-order polynomial basis because of the need to compute higher-order derivatives when solving partial-differential equations. A consequence of our observation that the non-shifted form of the polynomial basis starts violating consistency conditions means that collocation meshless methods (or Galerkin based meshless methods when employing higher-order polynomial basis) employing non-shifted polynomial basis can produce inaccurate results. A shifted polynomial basis performs better compared to a non-shifted basis, however, the errors with a shifted basis can also grow with increasing point distributions because of the poor conditioning of the moment matrix. An alternative to polynomial basis is to use

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Chebychev or other basis functions.

In a recent paper [Aluru and Li (2001)], we have introduced a finite cloud method, which uses a fixed-kernel approximation to compute interpolation functions and a collocation technique to discretize the governing equations. The finite cloud method uses a non-shifted polynomial basis and has been shown to produce interpolation functions that are equivalent (under certain conditions) to those obtained from a fixed least-squares approach [Onate, et al. (1996)]. In this paper, we introduce an improved finite cloud method, which uses a shifted polynomial basis in the kernel approximation. Our results indicate that improved finite cloud method exhibits far superior convergence characteristics compared to the original implementation of the finite-cloud method which uses a non-shifted basis.

The rest of the paper is organized as follows: In Section 2, we introduce the construction of meshless methods using shifted and non-shifted polynomial basis and establish the equivalence between the two approaches. In section 3, we highlight the numerical issues with a non-shifted approach, in Section 4 we introduce the improved finite cloud method, in Section 5, we show results comparing convergence characteristics of both shifted and non-shifted methods and conclusions are given in Section 6.

2 Meshless Methods Based on Shifted and Non-Shifted Base Interpolating Polynomials

We introduce both shifted and non-shifted approaches by using kernel approximations and make remarks at the end of each section on the connection between the least-squares and kernel approaches.

2.1 Non-Shifted Approach

Consider the following form of the kernel approximation

$$u^a_{NS}(x,y) = \int_\Omega P^T(s,t) C_{NS}(x,y) \varphi(x-s, y-t) u(s,t)\, ds dt \tag{1}$$

where u^a_{NS} denotes a non-shifted approximation to u, φ is the kernel, C_{NS} denotes the unknown correction function vector in a non-shifted approach and $P(s,t)$ is the $m \times 1$ vector of non-shifted basis functions. A linear basis in two-dimensions is given by

$$P^T(s,t) = [p_1, p_2, \dots, p_m] = [1, s, t], \quad m = 3 \tag{2}$$

and a quadratic basis in two-dimensions is given by

$$P^T(s,t) = [p_1, p_2, \dots, p_m] = \left[1, s, t, s^2, st, t^2\right], \quad m = 6 \tag{3}$$

The unknown correction function coefficients are computed by satisfying the consistency conditions i.e.

$$\int_\Omega P^T(s,t) C_{NS}(x,y) \varphi(x-s, y-t) p_i(s,t)\, ds dt = p_i(x,y)$$
$$i = 1, 2, \dots, m \tag{4}$$

The above consistency conditions can be rewritten in a matrix form as

$$[M(x,y)]_{NS} C_{NS}(x,y) = P(x,y) \tag{5}$$

where $[M(x,y)]_{NS}$ denotes the moment matrix in a non-shifted approach and the ij-th entry in the moment matrix is given by

$$[M_{ij}]_{NS} = \int_\Omega p_j(s,t) \varphi(x-s, y-t) p_i(s,t)\, ds dt \tag{6}$$

In discrete form, the $\mathrm{n^{th}} \times \mathrm{m^{th}}$ element of the square moment matrix is given by (setting the nodal volumes, ΔV_I, to unity)

$$[M]_{NS} = \left[\sum_{I=1}^{NP} \varphi_I p_n(x_I, y_I) p_m(x_I, y_I)\right]_{n,m} \tag{7}$$

where $\varphi_I = \varphi(x - x_I, y - y_I)$. Note that the moment matrix, $[M]_{NS}$, in a non-shifted approach is symmetric. Substituting the definition for correction function coefficients (from equation (5)) in equation (1), a discrete form of the non-shifted kernel approach can be written as

$$u^a_{NS}(x,y) = \sum_{I=1}^{NP} N_I^{NS}(x,y) \hat{u}_I \tag{8}$$

where $\hat{u}_I$ is a nodal unknown and N_I^{NS} is the interpolation function in a non-shifted approach which is given by

$$N_I^{NS}(x,y) = P^T(x,y)[M^{-T}]_{NS} P(x_I, y_I) \varphi(x - x_I, y - y_I) \Delta V_I \tag{9}$$

where ΔV_I is referred to as a nodal volume.

Remarks:

1. The non-shifted kernel approach described above was introduced in [Aluru and Li (2001)] and referred to as a moving reproducing kernel or a moving kernel technique.

2. In [Aluru and Li (2001), it was shown that the moving repoducing kernel technique is equivalent to a moving least-squares approach when the nodal volumes are set to unity ($\Delta V_I = 1$) i.e. the interpolation functions computed by the moving reproducing kernel and the moving least-squares approaches are identical.

2.2 Shifted Approach

A shifted-form of the kernel approximation can be written as

$$u_S^a(x,y) = \int_\Omega P^T(x-s,y-t) C_S(x,y)\varphi(x-s,y-t)u(s,t)dsdt \qquad (10)$$

where u_S^a denotes a shifted approximation to unknown u, C_S is the unknown correction function vector in the shifted approach, and $P^T(x-s,y-t)$ is the shifted polynomial basis vector. A linear basis in two-dimensions is given by

$$P^T(x-s,y-t)=[p_1,p_2,\ldots,p_m]=[1,x-s,y-t], \quad m=3 \qquad (11)$$

and a quadratic basis is given by

$$P^T(x-s,y-t) = [p_1,p_2,\ldots,p_m] = \left[1,x-s,y-t,(x-s)^2,(x-s)(y-t),(y-t)^2\right], \; m=6 \quad (12)$$

The unknown correction function coefficients are computed by satisfying the consistency conditions i.e.

$$\int_\Omega P^T(x-s,y-t)C_S(x,y)\varphi(x-s,y-t)p_i(s,t)dsdt = p_i(x,y) \quad i=1,2,\ldots,m \qquad (13)$$

The above consistency conditions can be rewritten in a matrix form as

$$[M(x,y)]_S C_S(x,y) = P(x,y) \qquad (14)$$

where $[M(x,y)]_S$ denotes the moment matrix in a shifted approach and the ij-th entry in the moment matrix is given by

$$[M_{ij}]_S = \int_\Omega p_j(x-s,y-t)\varphi(x-s,y-t)p_i(s,t)dsdt \quad (15)$$

In discrete form, the $\mathrm{n^{th}} \times \mathrm{m^{th}}$ element of the square moment matrix is given by (again by setting the nodal volumes to unity)

$$[M]_S = \left[\sum_{I=1}^{NP} \varphi_I p_n(x_I,y_I) p_m(\tilde{x}_I,\tilde{y}_I)\right]_{n,m} \qquad (16)$$

where $\varphi_I = \varphi(x-x_I, y-y_I)$, $\tilde{x}_I = x - x_I$ and $\tilde{y}_I = y - y_I$. Note that the moment matrix, $[M]_S$, in the shifted approach is non-symmetric. Substituting the definition for correction function coefficients (from equation (14)) in equation (10), a discrete form of the shifted kernel approach can be written as

$$u_S^a(x,y) = \sum_{I=1}^{NP} N_I^S(x,y)\hat{u}_I \qquad (17)$$

where $\hat{u}_I$ is a nodal unknown and N_I^S is the interpolation function in a shifted approach which is given by

$$N_I^S(x,y)=P^T(x,y)[M^{-T}]_S P(x-x_I,y-y_I)\varphi(x-x_I,y-y_I)\Delta V_I \qquad (18)$$

Remarks:

1. The shifted approach described above was introduced in [Liu, et al. (1995)] and referred to as a reproducing kernel technique.

2.3 Equivalence between Shifted and Non-Shifted Approaches

Mathematically, the shifted and non-shifted approaches can be shown to be identical. The equivalence between moving least-squared and kernel approximations has been addressed in [Belytschko et al. (1996)]. For simplicity, we consider a one-dimensional setting, but the results can be easily extended to multiple dimensions. Assuming an m-th order polynomial basis, the non-shifted polynomial vector is given by

$$P^T(x_I) = [p_1,p_2,\ldots,p_m] = \left[1,x_I,x_I^2,\ldots,x_I^{(m-1)}\right] \qquad (19)$$

and the shifted polynomial vector is given by

$$P^T(x-x_I) = [p_1,p_2,\ldots,p_m] = \left[1,x-x_I,(x-x_I)^2,\ldots,(x-x_I)^{(m-1)}\right] \qquad (20)$$

Defining $[S]$ to be

$$[S] = \begin{pmatrix} 1 & 0 & 0 & \dots 0 \\ x & -1 & 0 & \dots 0 \\ x^2 & -2x & (-1)^2 & \dots 0 \\ \vdots & \vdots & \vdots & \ddots \vdots \\ x^{m-1} & \binom{m-1}{1}(-1)x^{m-2} & \binom{m-1}{2}(-1)^2x^{m-3} & \dots (-1)^{m-1} \end{pmatrix} \tag{21}$$

the shifted and non-shifted polynomial vectors are related by

$$[S]\,P(x_I) = P(x - x_I) \tag{22}$$

In equation (21), $\binom{m-1}{1}$ and $\binom{m-1}{2}$ are binomial coefficients. A binomial coefficient $\binom{n}{k}$ is defined as

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \tag{23}$$

The shifted and non-shifted moment matrices for a one-dimensional case are related by

$$[S]\,[M^T]_{NS} = [M^T]_S \tag{24}$$

From equations (22) and (24), it follows that

$$[M^T]^{-1}_{NS}P(x_I) = [M^T]^{-1}_S[S]P(x_I) = [M^T]^{-1}_S P(x - x_I) \tag{25}$$

Therefore

$$N^{NS}_I(x) = N^S_I(x) \tag{26}$$

where $N^{NS}_I(x)$ and $N^S_I(x)$ are the one-dimensional interpolation functions obtained with a non-shifted and shifted approach, respectively.

Remarks:

1. The result in equation (26) indicates that, mathematically, the interpolation functions computed by shifted and non-shifted approaches are identical.

2. As will be discussed in the next section, numerical errors in a non-shifted approach can grow quickly and start violating consistency conditions.

3 Numerical Issues with a Non-Shifted Approach

Numerical implementation of shifted and non-shifted approaches can produce different results because of the different numerical steps involved in computing moment matrices and correction function coefficients. Specifically, by implementing both approaches, we have tried to check if the consistency conditions are being satisfied. These results are summarized in Table 1 and Table 2. In Table 1, we look at two consistency conditions using a quadratic basis ($m = 3$) and compare the results obtained with shifted and non-shifted polynomial basis. The results indicate that for increasing number of points, the non-shifted approach starts violating consistency conditions. Similarly, in Table 2, we look at three consistency conditions using a cubic basis ($m = 4$) and compare the results obtained with shifted and non-shifted basis. The results again indicate that the non-shifted approach violates consistency conditions. For a cubic basis, the non-shifted approach starts violating consistency conditions for a fewer number of points compared to the results obtained with a quadratic basis.

We try to explain the behavior of the non-shifted approach by considering a one-dimensional setting and a quadratic basis. Shown in Figure 1 is a five-point cloud and the weights for the five-points at which the kernel does not vanish. For this example, the moment matrix

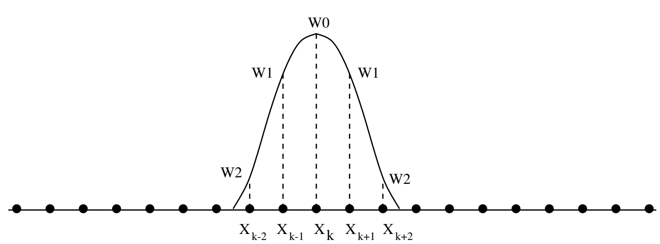


Figure 1 : A one-dimensional example discretized into points. Also shown is a five-point cloud where the kernel or weighting function does not vanish. w_0, w_1 and w_2 are the values of the weighting function for the points within the cloud

with a shifted polynomial basis is given by

$$M_S = \begin{pmatrix} s_1 & 0 & 2s_2h^2 \\ s_1x_k & -2s_2h^2 & 2s_2h^2x_k \\ s_1x_k^2 + 2s_2h^2 & -4s_2h^2x_k & 2s_2h^2x_k^2 + (2w_1 + 32w_2)h^4 \end{pmatrix} \tag{27}$$

and the moment matrix with a non-shifted polynomial

Table 1 : Comparison of the satisfaction of the consistency conditions for a quadratic basis with shifted and non-shifted approaches

numbers of nodes	$\sum N_{I,x} x_I = 1.0$		$\sum N_{I,xx} x_I^2 = 2.0$	
	non-shifted basis	shifted basis	non-shifted basis	shifted basis
81	1	1	2	2
161	1	1	2.00001	2
321	1	1	2.00016	2
641	0.999997	1	1.99799	2
1281	1	1	1.99114	2
2561	0.999849	1	2.17519	2
5121	1.00003	1	133.114	2

Table 2 : Comparison of the satisfaction of the consistency conditions for a cubic basis with shifted and non-shifted approaches

numbers of nodes	$\sum N_{I,x} x_I = 1.0$		$\sum N_{I,xx} x_I^2 = 2.0$		$\sum N_{I,xxx} x_I^3 = 6.0$	
	non-shifted	shifted	non-shifted	shifted	non-shifted	shifted
41	1	1	2	2	6.00035	6
81	1	1	1.99999	2	6.00473	6
161	0.999974	1	1.9994	2	8.32153	6
321	1.00185	1	2.01481	2	258.451	6
641	1.00185	1	2.00257	2	492.394	6
1281	1.00185	1	2.06534	2	492.394	5.9999
2561	1.00185	1	0.387417	2	492.394	6.00161
5121	1.00185	1	6685.12	2	-25771	6.00752

basis is given by

$$M_{NS} = \tag{28}$$
$$\begin{pmatrix} s_1 & s_1x_k & s_1x_k^2+2s_2h^2 \\ s_1x_k & s_1x_k^2+2s_2h^2 & s_1x_k^3+6s_2x_kh^2 \\ s_1x_k^2+2s_2h^2 & s_1x_k^3+6s_2x_kh^2 & s_1x_k^4+12s_2x_k^2h^2+(2w_1+32w_2)h^4 \end{pmatrix}$$

where $s_1 = (w_0+2w_1+2w_2)$, $s_2 = (w_1+4w_2)$, w_0, w_1 and w_2 are the weights, x_k is the point at which the kernel is centered, and h is the spacing between the points.

From equation (29), it is clear that when $x_k >> h$, the columns of M_{NS} can become linearly dependent. This is, however, not the case for M_S, even though the condition number of M_S can be bad. This indicates that the moment matrix in a non-shifted approach is very close to becoming singular whenever $x_k >> h$, and this leads to the violation of consistency conditions.

Remarks:

1. Even though both the shifted and non-shifted approaches are mathematically identical, a shifted approach is better suited for numerical implementation. As the order of the polynomial basis increases, the errors with the non-shifted approach start growing quickly.

2. As shown in Table 2, the shifted approach also starts violating consistency conditions for larger number of nodes. It is well-known that the use of a polynomial basis can generate these results. A remedy to this situation would be to employ Chebychev or other types of basis functions.

4 Improved Finite Cloud Method

We have recently introduced a finite cloud meshless method which uses a fixed kernel technique for the construction of interpolation functions and a collocation technique for the discretization of governing equations [Aluru and Li (2001)]. In the finite cloud method an approximation to an unknown function is given by

$$u^a = \int_\Omega \mathcal{C}(x,y,s,t)\varphi(x_K-s,y_K-t)u(s,t)dsdt \tag{29}$$

where $\mathcal{C}(x,y,s,t)$ is the correction function and is given by

$$\mathcal{C}(x,y,s,t) = P^T(s,t)C(x,y) \tag{30}$$

$P^T(s,t) = \{p_1, p_2, \ldots, p_m\}$ is the $1\times m$ vector of basis functions and $C^T(x,y) = \{c_1, c_2, \ldots, c_m\}$ is the $1\times m$ vector of correction function coefficients. The kernel function $\varphi(x_K-s, y_K-t)$ is centered at the point (x_K, y_K). The approximation in equation (29) can be written as

$$u^a(x,y) = \sum_{I=1}^{NP} N_I(x,y)\hat{u}_I \tag{31}$$

where $N_I(x,y)$ is the fixed kernel interpolation function defined as (see [Aluru and Li (2001)] for details)

$$N_I(x,y) = P^T(x,y)M^{-1}P(x_I,y_I)\varphi(x_K-x_I,y_K-y_I)\Delta V_I \tag{32}$$

Note that this approach uses a non-shifted polynomial basis.

We propose an improved finite cloud method using the following construction

$$u^a = \int_\Omega \mathcal{C}(x,y,x_K-s,y_K-t)\varphi(x_K-s,y_K-t)u(s,t)dsdt \tag{33}$$

where $\mathcal{C}(x,y,x_K-s,y_K-t)$ is the modified correction function and is given by

$$\mathcal{C}(x,y,x_K-s,y_K-t) = P^T(x_K-s,y_K-t)C(x,y) \tag{34}$$

where $P^T(x_K-s,y_K-t)$ is the shifted polynomial basis and $C(x,y)$ is the unknown correction function coefficient vector. The approximation in equation (33) can be written as

$$u^a(x,y) = \sum_{I=1}^{NP} N_I^S(x,y)\hat{u}_I \tag{35}$$

where $N_I^S(x,y)$ is referred to as the fixed kernel interpolation function using a shifted polynomial basis and is defined as

$$N_I^S(x,y) = P^T(x,y)[M^{-T}]_S P(x_K-x_I,y_K-y_I)\varphi(x_K-x_I,y_K-y_I)\Delta V_I \tag{36}$$

where $[M]_S$ is the moment matrix obtained using a shifted polynomial basis.

Remarks:

1. In [Aluru and Li (2001)], it was shown that when the nodal volumes are set to unity (i.e. $\Delta V_I = 1$), the interpolation functions computed in a finite cloud

method (by using a fixed kernel approximation and a non-shifted polynomial basis) is identical to the interpolation function computed by a fixed least-squares approach [Onate, et al. (1996a); Onate, et al. (1996b); Onate and Idelsohn (1998)].

2. The improved finite cloud method introduced above uses a shifted polynomial basis, instead of a non-shifted polynomial basis employed in the finite cloud method. Even though both the shifted and non-shifted approaches are mathematically identical, the interpolation functions computed in a shifted approach are more accurate for large point distributions and for increasing polynomial order.

5 Results

Numerical results are shown for several one and two-dimensional problems. Specifically, we compare the improved finite cloud method, which uses a shifted polynomial basis, with the original implementation of the finite cloud method, which uses a non-shifted polynomial basis, by employing quadratic and cubic basis. The convergence of the methods is measured by using a global error measure

$$\varepsilon = \frac{1}{\left|u^{(e)}\right|_{max}} \sqrt{\frac{1}{NP}\sum_{I=1}^{NP}\left[u_I^{(e)} - u_I^{(c)}\right]^2} \tag{37}$$

where ε is the error in the solution and the superscripts (e) and (c) denote, respectively, the exact and the computed solutions.

5.1 1-D Examples

The first example is a Poisson equation with a forcing term that is a function of x. The governing equation and boundary conditions are

$$\frac{\partial^2 u}{\partial x^2} = \frac{105}{2}x^2 - \frac{15}{2} \qquad -1 < x < 1 \tag{38}$$

$$u(x=-1) = 1 \tag{39}$$

$$\frac{\partial u}{\partial x}(x=1) = 10 \tag{40}$$

The exact solution for this problem is given by

$$u = \frac{35}{8}x^4 - \frac{15}{4}x^2 + \frac{3}{8} \tag{41}$$

This problem is analyzed by employing a uniform distribution of 41, 81, 161, 321, 641, 1281, and 5121 points to study the convergence behavior. The convergence of shifted and non-shifted methods by using a quadratic basis is summarized in Figure 2. The convergence rate of u for shifted and non-shifted methods is 2 and 1.98, respectively. The convergence rate of u_x is identical to the convergence rate of u for both shifted and non-shifted methods. The convergence plot indicates that the error with the shifted and non-shifted basis is identical up to a certain number of points. Beyond this, the error with the non-shifted basis either decreases slowly or starts growing. The growing errors with the non-shifted basis are explained by the growing numerical errors introduced into the computation of the moment matrix and the correction function coefficients.

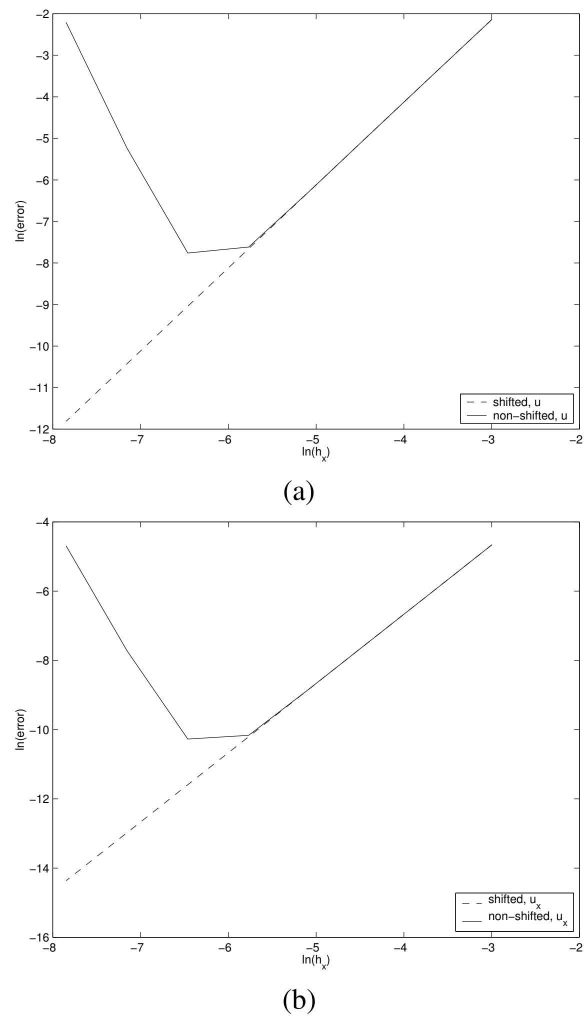
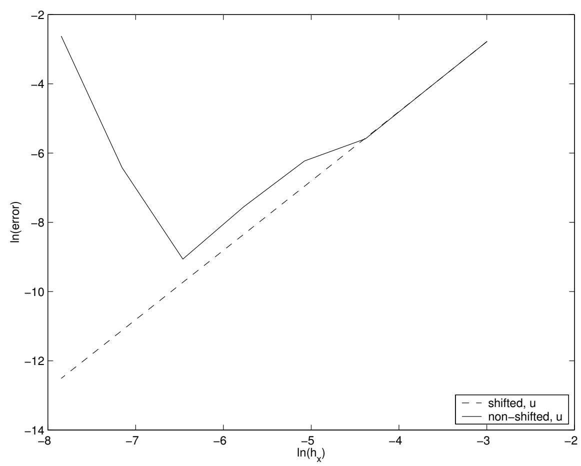
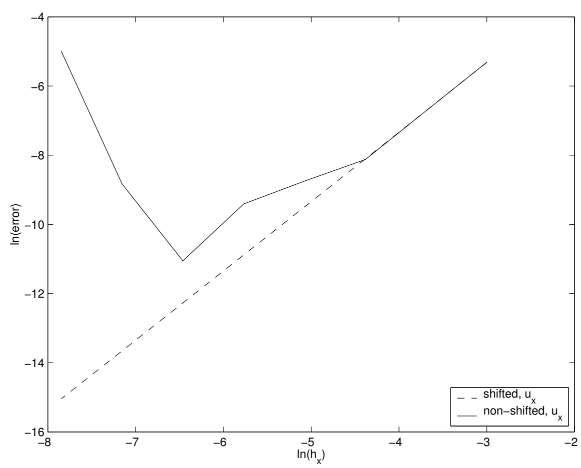


Figure 2 : Convergence of shifted and non-shifted methods for the Poisson equation using a quadratic basis (a) convergence in u (b) convergence in the derivative of u (u_x)

The convergence of shifted and non-shifted methods by using a cubic basis is shown in Figure 3. The convergence rate of u for shifted and non-shifted methods is 2.0 and 1.6, respectively. The convergence rate of u_x for shifted and non-shifted methods is 2.0 and 1.3, respectively. With a cubic basis, the deviation between shifted and non-shifted methods occurs sooner (for a fewer number of points) compared to the deviation observed with a quadratic basis. The deviation between shifted and non-shifted methods is again explained by the singularity of the moment matrix in a non-shifted approach. As the order of the polynomial increases, the moment matrix can become singular very quickly.



(a)



(b)

Figure 3 : Convergence of shifted and non-shifted methods for the Poisson equation using a cubic basis (a) convergence in u (b) convergence in the derivative of u (u_x)

The second 1-D example has a high gradient in a local region. The governing equation and boundary conditions for this example are

$$\frac{\partial^2 u}{\partial x^2} = -6x - \left[\frac{2}{\alpha^2} - 4\left(\frac{x-\beta}{\alpha^2}\right)^2\right] exp\left[-\left(\frac{x-\beta}{\alpha}\right)^2\right]$$

$$0 \le x \le 1 \tag{42}$$

$$u(x=0) = exp\left(-\frac{\beta^2}{\alpha^2}\right) \tag{43}$$

$$\frac{\partial u}{\partial x}(x=1) = -3 - 2\left(\frac{1-\beta}{\alpha^2}\right) exp\left[-\left(\frac{1-\beta}{\alpha}\right)^2\right] \tag{44}$$

The exact solution for this problem is given by

$$u = -x^3 + exp\left[-\left(\frac{x-\beta}{\alpha}\right)^2\right] \tag{45}$$

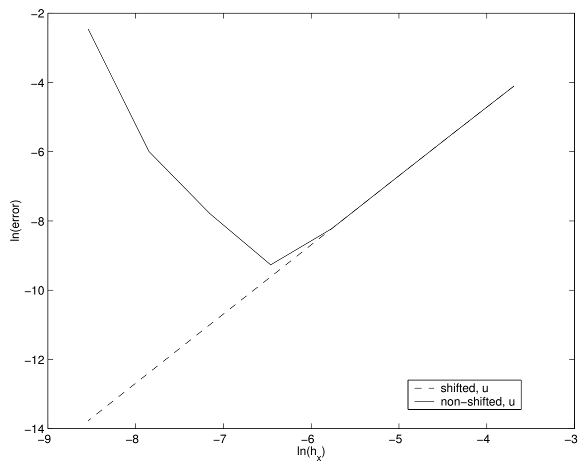
For the results shown in this paper, we use $\beta = 0.5$ and $\alpha = 0.05$. This problem is analyzed by employing a uniform distribution of 41, 81, 161, 321, 641, 1281, and 5121 points to study the convergence behavior. The convergence of shifted and non-shifted methods for quadratic and cubic basis is shown in Figure 4 and Figure 5, respectively. In the case of a quadratic basis, the convergence rate of u for shifted and non-shifted methods is 2.0 and 1.8, respectively. The convergence rate of u_x for shifted and non-shifted methods is 2.0 and 2.0, respectively. In the case of a cubic basis, the convergence rates of u for shifted and non-shifted methods is 2.0 and 1.6, respectively. The convergence rate of u_x for shifted and non-shifted methods is 1.98 and 1.36, respectively. Once again we observe that the shifted basis approach exhibits superior convergence compared to the non-shifted basis approach and with increasing polynomial order the performance of the non-shifted basis approach worsens.

5.2 2-D Examples

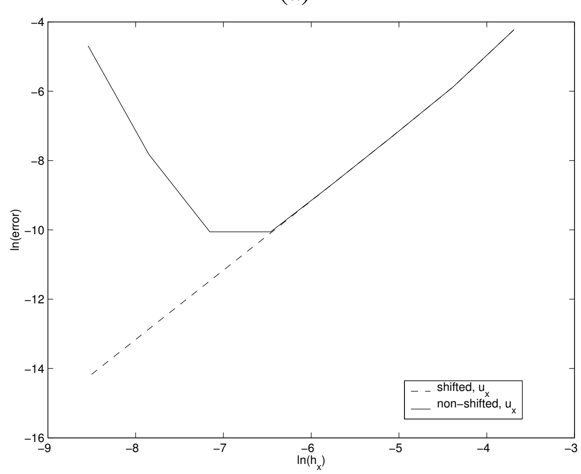
In this section we consider several two-dimensional examples. Numerical results again indicate that the shifted method exhibits superior convergence compared to the non-shifted method.

2-D Poisson Problem

We consider a two-dimensional extension of the 1-D Poisson example with a high local gradient. The governing equation along with the boundary conditions are

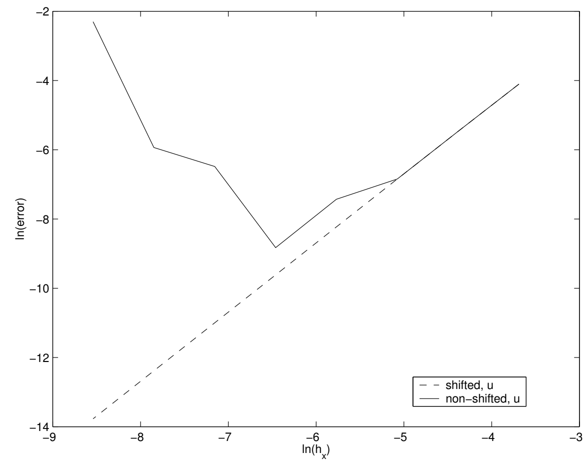


(a)

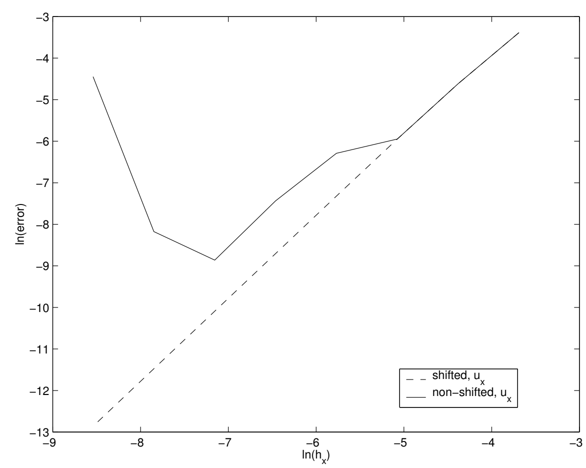


(b)

Figure 4 : Convergence of shifted and non-shifted methods for the Poisson equation with a high local gradient using a quadratic basis (a) convergence in *u* (b) convergence in the derivative of *u*



(a)



(b)

Figure 5 : Convergence of shifted and non-shifted methods for the Poisson equation with a high local gradient using a cubic basis (a) convergence in *u* (b) convergence in the derivative of *u*

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=-6x-6y \tag{46}$$
$$-\left[\frac{4}{\alpha^2}-4\left(\frac{x-\beta}{\alpha^2}\right)^2-4\left(\frac{y-\beta}{\alpha^2}\right)^2\right]\exp\left[-\left(\frac{x-\beta}{\alpha}\right)^2-\left(\frac{y-\beta}{\alpha}\right)^2\right]$$
$$0\le x\le .1 \qquad 0\le y\le .1$$

$$u(x=0)=-y^3+exp\left[-\left(\frac{\beta}{\alpha}\right)^2-\left(\frac{y-\beta}{\alpha}\right)^2\right] \tag{47}$$
$$u(x=0.1)=-0.001-y^3+exp\left[-\left(\frac{0.1-\beta}{\alpha}\right)^2-\left(\frac{y-\beta}{\alpha}\right)^2\right] \tag{48}$$
$$u_{,y}(y=0)=\frac{2\beta}{\alpha^2}exp\left[-\left(\frac{x-\beta}{\alpha}\right)^2-\left(\frac{\beta}{\alpha}\right)^2\right] \tag{49}$$
$$u_{,y}(y=0.1)=-0.03-2\left(\frac{0.1-\beta}{\alpha^2}\right)exp\left[-\left(\frac{x-\beta}{\alpha}\right)^2-\left(\frac{0.1-\beta}{\alpha}\right)^2\right] \tag{50}$$

The exact solution for this problem is given by

$$u=-x^3-y^3+exp\left[-(\frac{x-\beta}{\alpha})^2-(\frac{y-\beta}{\alpha})^2\right] \tag{51}$$

To perform convergence studies, we use a cubic basis and a uniform distribution of 9×9, 17×17, 33×33, and 65×65 points. The convergence of u with shifted and non-shifted methods is shown in Figure 6, and the convergence of the $x-$ and $y-$derivatives in u (denoted u_x and u_y) is shown in Figure 7. The convergence rate of u for shifted and non-shifted method is 1.96 and 1.9, respectively. The convergence rate of u_x for shifted and non-shifted method is 2.03 and 1.7, respectively. The convergence rate of u_y for shifted and non-shifted method is 2.15 and 1.2, respectively.

Heat Conduction

The steady-state heat conduction equation considered here is a rectangular plate ($0.5\times 1in^2$) with a heat source. The governing equation is given by

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=-2s^2sech^2[s(y-0.5)]\tanh[s(y-0.5)] \tag{52}$$

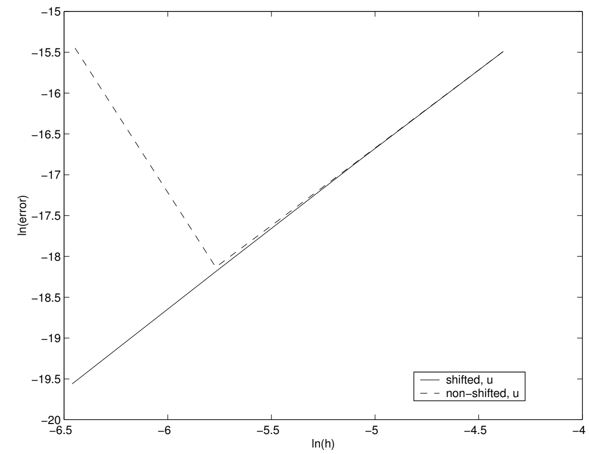
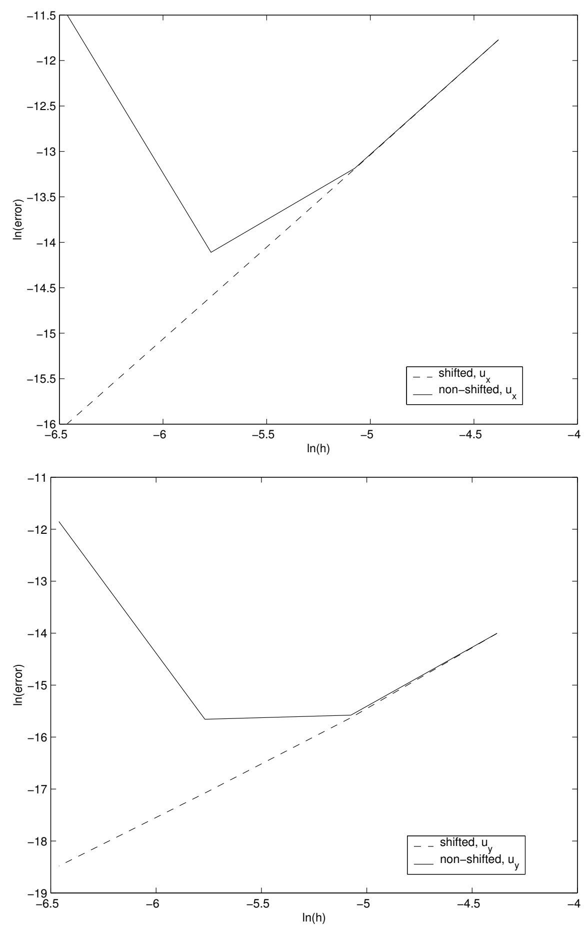


Figure 6 : Comparison of convergence in u with shifted and non-shifted methods using a cubic basis



(a) (b)

Figure 7 : Convergence of shifted and non-shifted methods for a 2-D Poisson problem using a cubic basis (a) convergence in u_x (b) convergence in u_y

The boundary conditions are given by

$$
\begin{aligned}
T(y=0) &= -\tanh(3s) \\
T(y=1) &= \tanh(3s) \\
T,x(x=0) &= 0 \\
T,x(x=0.5) &= 0
\end{aligned}
$$

The exact solution for this problem is given by

$$T = \tanh[s(y-0.5)] \tag{53}$$

We use a uniform distribution of 5×9, 5×17, 5×33, 5×65, 5×129, 5×257, 5×513, and 5×1025 points. The problem is analyzed with both quadratic and cubic basis. The convergence (of the solution, T) of shifted and non-shifted methods with quadratic and cubic basis is shown in Figure 8. With a quadratic basis, the convergence rate of T is 2.1 for the shifted method, and 1.9 for the non-shifted method. With a cubic basis, the convergence rate of T is 2.03 for the shifted method and 2.0 for the non-shifted method. The convergence (of the gradient of the solution, T_y) of shifted and non-shifted methods with quadratic and cubic basis is shown in Figure 9. With a quadratic basis, the convergence rate of T_y is 2.26 for the shifted method, and 1.2 for the non-shifted method. With a cubic basis, the convergence rate of T_y is 1.92 for the shifted method and 1.35 for the non-shifted method.

Convection-Diffussion Problem

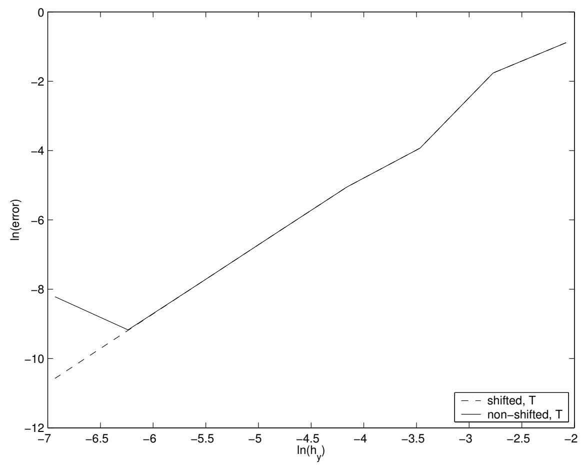
The convection-diffusion equation in two-dimensions is given by

$$
u_x \frac{\partial C}{\partial x} + u_y \frac{\partial C}{\partial y} - k_x \frac{\partial^2 C}{\partial x^2} - k_y \frac{\partial^2 C}{\partial y^2} = 0
$$

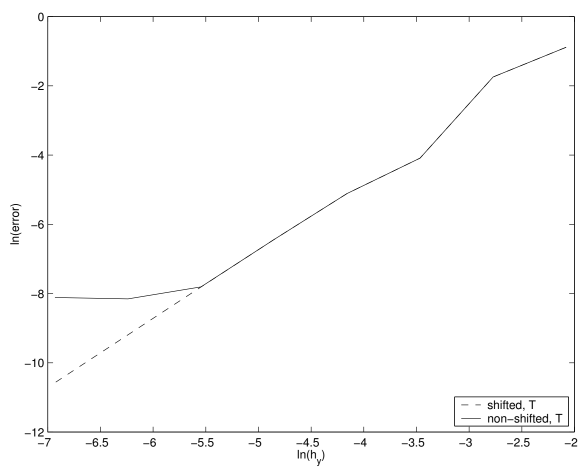
$$5 \le x \le 6 \qquad 5 \le y \le 6 \tag{54}$$

where u_x and u_y are velocities in the $x-$ and $y-$directions, respectively, and k_x and k_y are the diffusion coefficients. In this paper, we take $u_x = u_y = u$, $k_x = k_y = k$, the ratio of u to k is defined as the Peclet number (Pe), and we consider a Peclet number of 1. The following boundary conditions are considered for the convection-diffusion example

$$
\begin{aligned}
C(x=5) &= 0 \\
C(y=5) &= 0 \\
C(x=6) &= \frac{(1-e^{Pe})(1-e^{Pe(y-5)})}{(1-e^{Pe})^2} \\
C(y=6) &= \frac{(1-e^{Pe})(1-e^{Pe(x-5)})}{(1-e^{Pe})^2}
\end{aligned}
$$

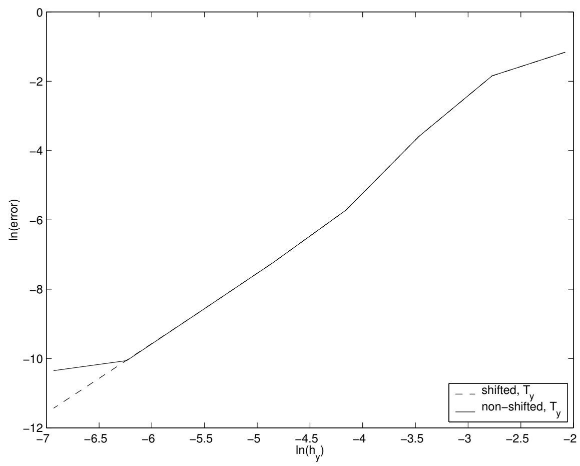


(a)

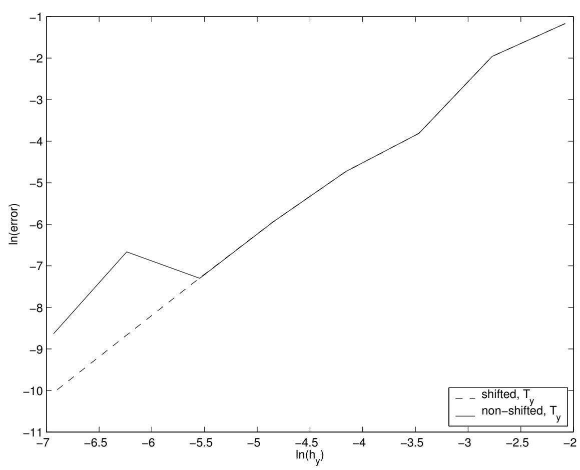


(b)

Figure 8 : Convergence of shifted and non-shifted methods for the heat conduction problem (a) convergence in T with quadratic basis (b) convergence in T with cubic basis



(a)



(b)

Figure 9 : Convergence of shifted and non-shifted methods for the heat conduction problem (a) convergence in T_y with quadratic basis (b) convergence in T_y with cubic basis

The exact solution of this problem is given by

$$C = \frac{(1 - exp(Pe(x-5)))\,(1 - exp(Pe(y-5)))}{(1 - e^{Pe})^2} \tag{55}$$

The convergence rate of C is studied by using a cubic basis and a uniform distribution of 5×5, 9×9, 17×17, and 33×33 points. The convergence of shifted and non-shifted methods is shown in Figure 10. The convergence rate of C for the shifted and non-shifted method is 1.81 and 1.75, repectively.

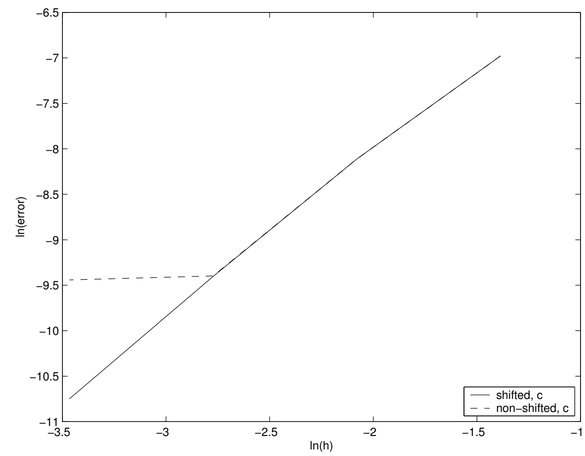


Figure 10 : Comparison of convergence of shifted and non-shifted methods for a convection-diffusion example using cubic basis

Elasticity Example

The governing equations for elasticity (in a plane stress situation) are

$$\frac{2}{1-\nu}\frac{\partial^2 u}{\partial x^2} + \frac{1+\nu}{1-\nu}\frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{56}$$

$$\frac{2}{1-\nu}\frac{\partial^2 v}{\partial y^2} + \frac{1+\nu}{1-\nu}\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial x^2} = 0 \tag{57}$$

where u and v are the $x-$ and $y-$components of the displacement and ν is the Poisson's ratio. We consider the solution of a beam subjected to a uniform load and a shear as shown in Figure 11. The beam is centered at $(a,b) = (3, 2.25)$, $l = 1$ unit, $c = 0.25$ and $t = 1$ unit. The modulus of elasticity is 3×10^7 and the Poisson's ratio is 0.25. The following boundary conditions are considered

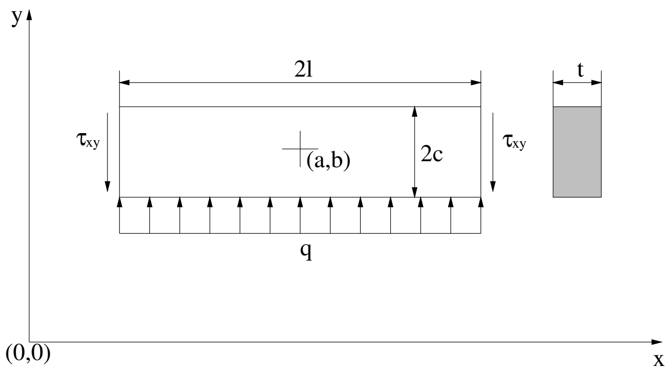


Figure 11 : A beam subjected to a uniform load and shear

$$
\begin{aligned}
u(x=a+l,y=b)=&\frac{\nu q l}{2E}\\
v(x=a\pm l,y=b)=&0\\
\tau_{xy}(y=b\pm c)=&0\\
\sigma_y(y=b-c)=&-q\\
\sigma_y(y=b+c)=&0\\
\sigma_x(x=a\pm l)=&\frac{q}{2I}\left[\frac{2}{3}(y-b)^3-\frac{2}{5}c^2(y-b)\right]\\
\tau_{xy}(x=a\pm l)=&-\frac{q}{2I}(x-a)\left[c^2-(y-b)^2\right]
\end{aligned}
$$

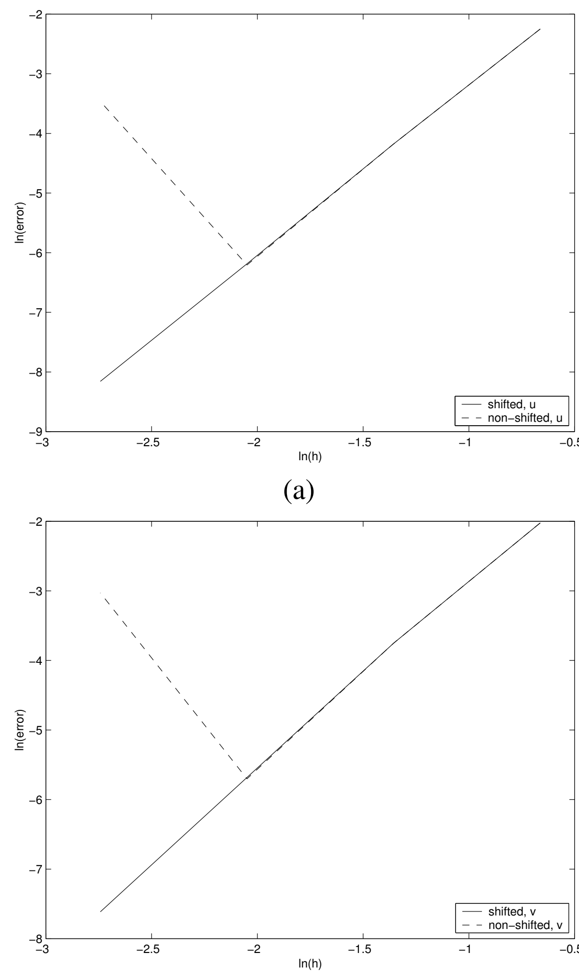
The exact solution for this problem is given by

$$
\begin{aligned}
u=&\frac{q}{2EI}\left\{\left[l^2(x-a)-\frac{(x-a)^3}{3}\right](y-b)\right.\\
&+(x-a)\left[\frac{2}{3}(y-b)^3-\frac{2}{5}c^2(y-b)\right]\\
&\left.+\nu(x-a)\left[\frac{(y-b)^3}{3}-c^2(y-b)+\frac{2}{3}c^3\right]\right\} \qquad (58)
\end{aligned}
$$

$$
\begin{aligned}
v=&-\frac{q}{2EI}\left\{\frac{(y-b)^4}{12}-\frac{c^2(y-b)^2}{2}+\frac{2c^3(y-b)}{3}\right.\\
&\left.+\nu\left[\left(l^2-(x-a)^2\right)\frac{(y-b)^2}{2}+\frac{(y-b)^4}{6}-\frac{c^2(y-b)^2}{5}\right]\right\}\\
&-\frac{q}{2EI}\left[\frac{l^2(x-a)^2}{2}-\frac{(x-a)^4}{12}-\frac{c^2(x-a)^2}{5}\right.\\
&\left.+\left(1+\frac{1}{2}\nu\right)c^2(x-a)^2\right]+\delta \qquad (59)
\end{aligned}
$$

$$
\delta=\frac{5}{24}\frac{ql^4}{EI}\left[1+\frac{12}{5}\frac{c^2}{l^2}\left(\frac{4}{5}+\frac{\nu}{2}\right)\right] \qquad (60)
$$

The convergence of shifted and non-shifted methods is studied by using a cubic basis and a uniform distribution of 5×5, 9×9, 17×17, and 33×33 points. The convergence of u and v for both methods is shown in Figure 12. The convergence of the x and y derivative of u is shown in Figure 13 and the convergence of the x and y derivative of v is shown in Figure 14. For the non-shifted method, the convergence rates of u and v are 2.85 and 2.65, respectively. For the shifted method, the convergence rates of u and v are 2.84 and 2.64, respectively.



(a)

(b)

Figure 12 : Convergence of shifted and non-shifted methods for the elasticity problem using a cubic basis (a) convergence in x displacement (u) (b) convergence in y displacement (v)

6 Conclusions

Many proposed meshless methods are distinguished either by the construction of the interpolation functions (e.g. least-squares, kernel approximations etc.) or by the

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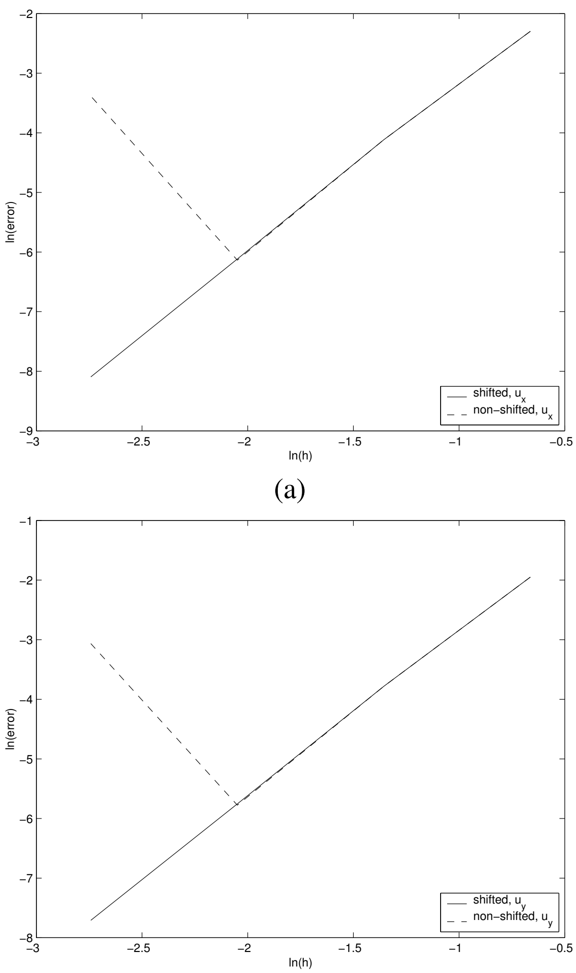
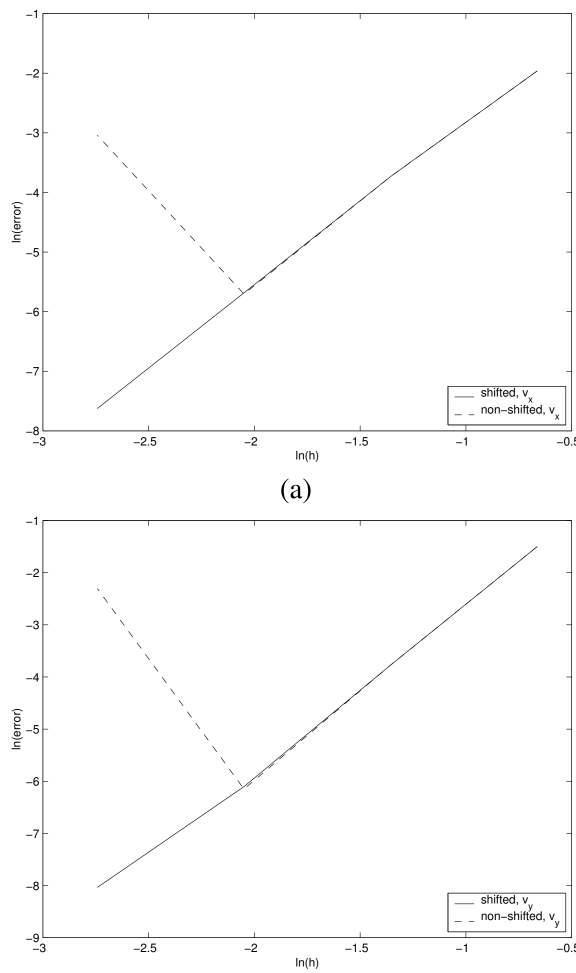


(a)

(b)

Figure 13 : Convergence of shifted and non-shifted methods for the elasticity problem using a cubic basis (a) convergence of u_x (x-derivative of u) (b) convergence of u_y (y-derivative of u)



(a)

(b)

Figure 14 : Convergence of shifted and non-shifted methods for the elasticity problem using a cubic basis (a) convergence of v_x (x-derivative of v) (b) convergence of v_y (y-derivative of v)

choice of the discretization technique (Galerkin and collocation). Least-squares based approaches, for example moving least-squares and fixed least-squares, use non-shifted polynomial basis. On the other hand, kernel approximations are based on shifted polynomial basis. In an earlier paper [Aluru and Li (2001)], we have shown that, when non-shifted polynomial basis is used in kernel approximations, least-squares and kernel approximations produce identical interpolation functions, if nodal volumes are set to unity in kernel approximations (see [Aluru and Li (2001)] for details). In this work, we have shown that the use of either shifted or non-shifted polynomial basis produces mathematically equivalent interpolation functions. However, when implemented numerically, the non-shifted approach can produce different results compared to the shifted approach. In particular, for large point distributions or for higher-order polynomial basis, the numerical errors with the non-shifted approach can grow quickly leading to the violation of consistency conditions. Hence, the use of shifted polynomial basis is recommended in meshless methods. Finally, we have introduced an improved finite cloud method which uses a shifted polynomial based fixed-kernel approximation for construction of interpolation functions and a collocation technique for discretization of the partial differential equations. The improved finite cloud method exhibits superior convergence behavior compared to our original implementation of the finite cloud method.

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References

Aluru N.R. (2000): A point collocation method based on reproducing kernel approximations, *Int. J. Numer. Methods in Engrg.*, Vol. 47, pp. 1083-1121.

Aluru N.R., Li G. (2001): Finite Cloud Method: A true meshless technique based on a fixed reproducing kernel approximation, *Int. J. Numer. Methods in Engrg.*, Vol. 50, No. 10, pp. 2373-2410.

Atluri S.N., Zhu T. (1998): A new meshless local Petrov-Galerkin (MLPG) approach in computational mechanics, *Computational Mechanics*, Vol. 22, pp. 117-127.

Belytschko T., Krongauz Y., Organ D., Fleming M., Krysl P. (1996): Meshless methods: An overview and recent developments, *Comput. Methods Appl. Mech. Engrg.*, Vol. 139, pp. 3-47.

Belytschko T., Lu Y.Y., Gu L. (1994): Element free Galerkin methods, *Int. J. Numer Methods in Engrg.*, Vol. 37, pp. 229-256.

Chen J.-S., Pan C., Wu C., Liu W.K. (1996): Reproducing kernel particle methods for large deformation analysis of non-linear structures, *Comput. Methods Appl. Mech. Engrg.*, Vol. 139, pp. 195-227.

Duarte C.A., Oden J.T. (1996): An h-p adaptive method using clouds,*Comput. Methods in Appl. Mech. Engrg.*, Vol. 139, pp. 237-262.

Kim H.G., Atluri S.N. (2000): Arbitrary Placement of Secondary Nodes and Error Control in the Meshless Local Petrov-Galerkin (MLPG) Method, *CMES: Computer Modeling in Engineering & Sciences*, Vol.1,No.3, pp. 11-32.

Lin H., Atluri S.N. (2000): Meshless Local Petrov-Galerkin (MLPG) Method for Convection - Diffusion Problems, *CMES: Computer Modeling in Engineerng & Sciences*, Vol.1,No.2, pp. 45-60.

Liu W.K., Jun S., Li S., Adee J., Belytschko T. (1995): Reproducing kernel particle methods for structural dynamics, *Int. J. Numer Methods in Engrg.*, Vol. 38, pp. 1655-1679.

Mukherjee Y.X., Mukherjee S. (1997): The boundary node method for potential problems, *Int. J. Numer. Methods Engrg.*, Vol. 40, pp. 797-815.

Oñate E., Idelsohn S., Zienkiewicz O.C., Taylor R.L. (1996): A finite point method in computational mechanics. Applications to convective transport and fluid flow, *Int. J. Numer. Methods in Engrg.*, Vol. 39, pp. 3839-3866.

Oñate E., Idelsohn S., Zienkiewicz O.C., Taylor R.L., Sacco C., (1996): A stabilized finite point method for analysis of fluid mechanics problems, *Comput. Methods Appl. Mech. Engrg.*, Vol. 139, pp. 315-346.

Oñate E., Idelsohn S. (1998): A mesh-free finite point method for advective-diffusive transport and fluid flow problems, *Computational Mechanics*, Vol. 21, pp. 283-292, 1998.

Wordelman C., Aluru N.R., Ravaioli U. (2000): A meshless method for the numerical solution of the 2- and 3-D semiconductor Poisson equation, *Computer Modeling in Engineering & Sciences*, Vol. 1, No. 1, pp. 123-128.

Zhu T., Zhang J.-D., Atluri S.N. (1998): A local boundary integral equation (LBIE) method in computational mechanics, and a meshless discretization approach, *Computational Mechanics*, Vol. 21, pp. 223-235.