

An Efficient Mesh-Free Method for Nonlinear Reaction-Diffusion Equations

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Abstract: The purpose of this paper is to develop a highly efficient mesh-free method for solving nonlinear diffusion-reaction equations in $\mathbb{R}^d, d = 2, 3$. Using various time difference schemes, a given time-dependent problem can be reduced to solving a series of inhomogeneous Helmholtz-type equations. The solution of these problems can then be further reduced to evaluating particular solutions and the solution of related homogeneous equations. Recently, radial basis functions have been successfully implemented to evaluate particular solutions for Poisson-type equations. A more general approach has been developed in extending this capability to obtain particular solutions for Helmholtz-type equations by using polyharmonic spline interpolants. The solution of the homogeneous equation may then be solved by a variety of boundary methods, such as the method of fundamental solutions. Preliminary work has shown that an increase in efficiency can be achieved compared to more traditional finite element, finite difference and boundary element methods without the need of either domain or surface meshing.

keyword: The method of fundamental solutions, radial basis functions, dual reciprocity method, polyharmonic splines, particular solution, reaction-diffusion equations, mesh-free method.

1 Introduction

In recent years there has been increasing interest in using mesh-free methods for solving partial differential equations. The most popular of these appear to be a variety of domain-based Galerkin methods which seek to eliminate the meshing restrictions of the classical finite element method. Although these methods have achieved some prominence in the past 5-6 years, mesh-free methods based on the ideas of Trefftz have been investigated since

the 1920's and some of these ideas have recently been incorporated into Babuska's and Melenk's generalized finite element method [Babuška and Melenk (1997)]. A particularly useful Trefftz method is the method of fundamental solutions (MFS) which has been intensively investigated by the authors and others [Golberg and Chen (1999); Fairweather and Karageorghis (1998)]. The MFS is a mesh-free boundary method. Traditionally, the MFS was restricted to solving homogeneous elliptic problems, but in recent years by combining the MFS with techniques from the Dual Reciprocity Method (DRM) [Partridge, Brebbia and Wrobel (1992)] we have shown how to extend this technique to solve inhomogeneous elliptic problems [Golberg and Chen (1999)]. Recently, due to the discovery of the analytic particular solutions for inhomogeneous Helmholtz-type equations with polyharmonic source terms, the above mesh-free approach was extended to solve linear diffusion equations [Chen, Rashed and Golberg (1998); Muleshkov, Golberg and Chen (1999)]. In Chen, Rashed and Golberg (1998) and in Zhu and Satravaha (1996), the Laplace transform was employed to remove the time dependence. However, this approach requires the solution of many elliptic boundary value problems and the numerical inversion of the Laplace transform is an ill-posed problem. To overcome these difficulties, a time-stepping algorithm [Polyzos and Beskos (1998)] was implemented making use of the particular solutions for polyharmonic splines. In this note, we show how to extend the approach for linear diffusion equations given in [Muleshkov, Golberg and Chen (1999)] to nonlinear reaction diffusion equations of the form $\Delta u - ku_t = f(u)$ ($\Delta =$ Laplacian). Such equations occur in a variety of heat transfer and biology problems [Britton (1986); Langdon (1999)] and present challenging problems to solve numerically. This work was motivated in part to improve the efficiency of the boundary integral equation method given by Langdon in his thesis [Langdon (1999)] and eventually to provide an extension of his approach to treat problems in \mathbb{R}^3 , since the method given in Langdon (1999) appears to be restricted

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to problems in \mathbb{R}^2 .

The paper is organized as follows. In Section 2 we use finite differencing in time to reduce initial-boundary value problems (IBVPs) for $\Delta u - ku_t = f(u)$ to solving a sequence of inhomogeneous modified Helmholtz equations. In Section 3 we review the approach given in Muleshkov, Golberg and Chen (1999) for solving such equation by approximating the source term by polyharmonic splines and then eliminating the inhomogeneity by using the analytic particular solutions derived in Muleshkov, Golberg and Chen (1999). In Section 4 we show how to solve the resulting homogeneous equations by the MFS. In Section 5 we present a number of numerical results. To validate our algorithm, we compare our approach to that given by Langdon in Langdon (1999) for Fisher's equation in \mathbb{R}^2 and then show how our approach can be easily extended to solve this equation in \mathbb{R}^3 . We then consider a well-known problem in combustion theory where $f(u) = \delta e^u$. In such problems it is often important to determine the steady state solution given by solving $\Delta u_\infty = f(u_\infty)$. Because f is nonlinear, such equations generally require iterative methods to solve, which can be quite time consuming. Here we show that if $\lim_{t \rightarrow \infty} u(t, P) = u_\infty(P)$, then the steady state solution u_∞ can be obtained efficiently by solving the time dependent problem for large values of t . For the case $f(u) = \delta e^u$, we show that this approach compares favorably with the monotone iteration scheme given in Chen (1995).

2 A Finite Difference Method

We consider the IBVP

$$\Delta u(P, t) - ku_t(P, t) = f(u(P, t)), \quad P \in D, \quad (1)$$

$$u(P, t) = u_0(P), \quad (2)$$

$$B(u(P, t)) = g(P), \quad P \in S, t > 0, \quad (3)$$

where D is a bounded open set in $\mathbb{R}^d, d = 2, 3$, with boundary S . B is a boundary operator which for this paper is given by $B(u) \equiv u$ so (3) is a Dirichlet boundary condition. (Note: this choice is merely for convenience, our algorithm does not generally depend on the choice of B .) We assume that (1)-(3) have a unique solution for all $t > 0$ [Britton (1986)]. In general, it is not possible to solve (1)-(3) analytically, so numerical techniques have to be used.

Following the approach in Muleshkov, Golberg and Chen (1999), we let $\tau > 0$ and define $U_n(P) = u(P, n\tau), n =$

$0, 1, 2, \dots$ and approximate

$$u_t(P, n\tau) \simeq \frac{u(P, n\tau) - u(P, (n-1)\tau)}{\tau}. \quad (4)$$

Using (4) in (1) we define an approximation $v_n(P)$ to $U_n(P)$ by solving

$$\Delta v_n(P) = \frac{k[v_n(P) - v_{n-1}(P)]}{\tau} + f(v_{n-1}(P)) \quad (5)$$

$$P \in D, n \geq 1,$$

$$v_0(P) = u_0(P), \quad (6)$$

$$v_n(P) = g(P), \quad P \in S, \quad (7)$$

Rearranging (5) gives v_n as the solution to

$$\Delta v_n(P) - \frac{kv_n(P)}{\tau} = -\frac{kv_{n-1}(P)}{\tau} + f(v_{n-1}(P)) \quad (8)$$

$$\equiv h_n(P), \quad n \geq 1.$$

Setting $\lambda^2 = k/\tau, k > 0$, we see that v_n satisfies the inhomogeneous modified Helmholtz equation

$$\Delta v_n(P) - \lambda^2 v_n(P) = h_n(P), \quad P \in D, \quad (9)$$

along with the initial condition (6) and boundary condition (7). Since we wish to avoid domain meshing for the solution of (6), (7), and (9), we propose to solve the problem by using a boundary-type approach. To do this we need to reduce the sequence of BVPs (5)-(7) to an equivalent sequence of homogeneous equations. For this we use the method introduced in Muleshkov, Golberg and Chen (1999) to solve linear diffusion equations.

3 Particular solutions

To reduce (9) to an equivalent homogeneous equation, let w_n be a particular solution to (9), which does not necessarily satisfy the boundary condition (7); i.e.,

$$\Delta w_n(P) - \lambda^2 w_n(P) = h_n(P), \quad P \in D. \quad (10)$$

Letting

$$z_n = v_n - w_n, \quad n \geq 1, \quad (11)$$

z_n satisfies the sequence of BVPs

$$\Delta z_n(P) - \lambda^2 z_n(P) = 0, \quad P \in D, n \geq 1, \quad (12)$$

$$z_n(P) = h_n(P) - w_n(P), \quad P \in S, \quad (13)$$

$$z_0 = u_0(P). \quad (14)$$

Since (12) is an homogeneous Helmholtz equation, it can be solved by standard boundary element methods. In Langdon (1999) used this approach to solve (12) in \mathbb{R}^2 by using an integral equation based on the double layer Helmholtz potential. However, because $k/\tau = \sqrt{\lambda}$ may be large since τ is small [Langdon (1999)], the kernel of the integral equation is highly peaked making numerical integration very difficult. In Langdon (1999) this problem was partly overcome by using a modified trapezoidal rule for $D \subseteq \mathbb{R}^2$ with a smooth boundary. However, the approach given there does not seem to be immediately generalizable to \mathbb{R}^3 . To avoid the numerical integration problem we will use the MFS to solve (12)-(14) which requires no numerical integration either in \mathbb{R}^2 or \mathbb{R}^3 . This will be discussed in Section 4.

Assuming that one can solve (12)-(14) efficiently, to complete our algorithm we need to be able to obtain a particular solution w_n . As is well known, one can often obtain w_n as the integral

$$w_n(P) = \int_D G(P,Q;\lambda) h_n(Q) dv(Q) \quad (15)$$

where $G(P,Q;\lambda)$ is the fundamental solution of $\Delta - \lambda^2$ [Golberg and Chen (1999)]. In general, the direct evaluation of (15) is difficult and it is preferable to find an alternative approach. If one replaces D by $\hat{D} \supseteq D$ where \hat{D} is a regular region, such as a circle in 2D or sphere in 3D, containing D , then the resulting integral can be evaluated by standard numerical methods without meshing D or \hat{D} [Golberg and Chen (1999)]. But this can still be a costly approach [Golberg and Chen (1999)]. In Langdon (1999) generalized the method of McKenney, Greengard and Mayo (1995) where D was embedded in a rectangular domain Ω and (9) solved by finite differences. In Langdon (1999) an efficient implementation was given using fast Fourier transform techniques. However, the generalization to 3D seems difficult because in 2D it requires the use of Cauchy's integral theorem to locate points in Ω/D [Langdon (1999)]. As a consequence, we have chosen to use the method developed in Muleshkov, Golberg and Chen (1999) and in Golberg and Chen (1999) for the linear-diffusion equation. We discuss this next.

3.1 Polyharmonic splines

To calculate particular solutions for

$$\Delta \psi - \lambda^2 \psi = F \quad (16)$$

we assume that F is approximated by a polyharmonic spline \hat{F} ; i.e.,

$$\hat{F}(P) = \sum_{j=1}^N a_j \phi_j^{[n]}(P) + p_n \quad (17)$$

where

$$p_n = \sum_{i=1}^{\ell_n} c_i b_i(P),$$

$$\phi_j^{[n]}(P) = \begin{cases} r_j^{2n} \log r_j, & n \geq 1 \text{ in } \mathbb{R}^2, \\ r_j^{2n-1}, & n \geq 1 \text{ in } \mathbb{R}^3, \end{cases} \quad (18)$$

$\{b_i\}_1^{\ell_n}$ is a basis for polynomials of degree $\leq n$, $r_j = \|P - P_j\|$ and $\{P_j\}_1^N \subseteq D$ is a unisolvent set of points for polynomial interpolation. In addition, if $\{a_j\}_1^N$ satisfy

$$\sum_{j=1}^N a_j b_i(P_j) = 0, \quad 1 \leq i \leq \ell_n, \quad (19)$$

then for a given \hat{F} , there is a unique polyharmonic spline interpolant to F of the form (17) on $\{P_j\}_1^N$. To be more specific, if the condition

$$F(P_k) = \hat{F}(P_k) \quad 1 \leq k \leq N,$$

is imposed, then the linear system

$$\begin{cases} \hat{F}(P_k) = \sum_{j=1}^N a_j \phi_j^{[n]}(P_k) + \sum_{i=1}^{\ell_n} c_i b_i(P_k), & 1 \leq k \leq N, \\ \sum_{j=1}^N a_j b_i(P_j) = 0, & 1 \leq i \leq \ell_n, \end{cases} \quad (20)$$

is uniquely solvable.

Hence, to obtain particular solutions to (16) we approximate F by \hat{F} and solve $\Delta \psi - \lambda^2 \psi = \hat{F}$. By linearity, it suffices to solve

$$\Delta \psi_j^{[n]} - \lambda^2 \psi_j^{[n]} = \phi_j^{[n]}, \quad (21)$$

$$\Delta \chi^{[n]} - \lambda^2 \chi^{[n]} = p_n. \quad (22)$$

In \mathbb{R}^2 it was shown [Muleshkov, Golberg and Chen (1999)] that $\psi_j^{[n]}(r)$ was given by

$$\psi_j^{[n]}(r) = AI_0(\lambda r) + BK_0(\lambda r) + \sum_{k=1}^{n+1} c_k r^{2k-2} \log r + \sum_{k=1}^n d_k r^{2k-2}, \quad (23)$$

where I_0 and K_0 are Bessel functions of order zero and the constants $((2n)!! \equiv 2 \cdot 4 \cdot \dots \cdot 2n = 2^n n!)$

$$\begin{cases} A = 0 \\ B = -\frac{[(2n)!!]^2}{\lambda^{2n+2}} \\ c_k = -\frac{[(2n)!!]^2}{[(2k-2)!!]^2} \lambda^{2k-2n-4}, \quad 1 \leq k \leq n+1, \\ d_k = c_k \sum_{j=k}^n \frac{1}{j}, \quad 1 \leq k \leq n. \end{cases}$$

Notice that the singularity of K_0 and $\log r$ in (23) are nicely cancelled out. The explicit forms of $\psi_j^{[n]}, 1 \leq n \leq 5$, are given in Table 1 in Muleshkov, Golberg and Chen (1999). In \mathbb{R}^3

$$\psi_j^{[n]}(r) = \frac{(2n)! \cosh(\lambda r)}{r \lambda^{2n+2}} - \sum_{k=0}^n \frac{(2n)! r^{2k-1}}{(2k)! \lambda^{2n-2k+2}}. \quad (24)$$

In Muleshkov, Golberg and Chen (1999) it was shown that

$$\chi^{[n]} = - \sum_{k=0}^{[n/2]} \frac{1}{\lambda^{2k+1}} \Delta^k p_n$$

where $[x]$ denotes the integer part of $[x]$. However, since p_n is usually expressed in terms of monomials, it is more convenient to calculate $\chi^{[n]}$ by using the formulas derived in Muleshkov, Chen, Golberg and Cheng (2000). There it was shown that a solution to

$$\Delta W - \lambda^2 W = x^m y^n$$

is given by

$$W(x, y) = \sum_{k=0}^{[m/2]} \sum_{\ell=0}^{[n/2]} \frac{(-1)^{k+\ell+1} (k+\ell)! m! n! x^{m-2k} y^{n-2\ell}}{\lambda^{2k+2\ell+2} k! \ell! (m-2k)! (n-2\ell)!}$$

with a similar formula in \mathbb{R}^3 [Muleshkov, Chen, Golberg and Cheng (2000)]. Note that W can be obtained easily by using symbolic computation such as MATHEMATICA or MAPLE.

Based on the above analysis, the approximate particular solution \hat{w}_n of w_n in (10) can be written as

$$\hat{w}_n = \sum_{j=1}^N a_j \psi_j^{[n]} + \chi^{[n]}. \quad (25)$$

4 The MFS

To solve the homogeneous boundary value problem, we approximate z_n by

$$\hat{z}_n(P) = \sum_{k=1}^m a_k G(P, Q_k; \lambda) \quad (26)$$

where

$$G(P, Q; \lambda) = \begin{cases} \frac{1}{2\pi} K_0(\lambda r), & \text{for 2D,} \\ \frac{1}{4\pi r} \exp(-r\lambda), & \text{for 3D,} \end{cases}$$

is the fundamental solution for $\Delta - \lambda^2$, K_0 is the modified Bessel function of the second kind of order 0, and $r = \|P - Q\|$ is the Euclidean distance between P and Q . Here $\{Q_k\}_1^m$ are source points on a fictitious surface \hat{S} containing D [Golberg and Chen (1999)]. The coefficients $\{a_k\}_1^m$ are determined by choosing m points $\{P_j\}_1^m$ on D and then setting

$$\sum_{k=1}^m a_k G(P_j, Q_k; \lambda) = h_n(P_j) - \hat{w}_n(P_j), \quad 1 \leq j \leq m \quad (27)$$

where \hat{w}_n is the approximation to w_n obtained in Section 3.1. An approximation \hat{v}_n to v_n is given by

$$\hat{v}_n(P) = \hat{z}_n(P) + \hat{w}_n(P). \quad (28)$$

In this paper \hat{S} is chosen as a circle of radius R in \mathbb{R}^2 and a sphere of radius R in \mathbb{R}^3 . Generally the accuracy of the MFS increases as R increases and as the number of source points increases [Golberg and Chen (1999)]. However, if R is chosen too large, then the equations (27) can become quite ill-conditioned. A simple remedy for this is to regularize the equations by adding a small number $< 10^{-6}$ to the diagonal [Golberg and Chen (1997)]. However, for the numerical examples considered in Section 5 we have found this tactic to be unnecessary. For further details on the implementation and convergence of the MFS we refer the reader to Golberg and Chen (1999) and Fairweather and Karageorghis (1998) and references therein.

5 Numerical results

To demonstrate the effectiveness of our algorithm, we consider Fisher's equation in 2D and 3D and a nonlinear thermal explosion problem considered in Chen (1995).

Since there is no exact solution available for these non-linear problems, we compare our results with those obtained in Langdon (1999) and in Chen (1995). In this section, we use the Quasi-Monte Carlo method [Press, Teukolsky, Vetterling and Flannery (1996)] to generate a sequence of quasi-random points which ensures that the interpolation points are uniformly distributed. There exist various kinds of general program subroutines that are ready to use. Here we choose the subroutine SOBSEQ in Press, Teukolsky, Vetterling and Flannery (1996) to generate these quasi-random points.

Our algorithm is extremely efficient. In the whole solution process, only two Gaussian eliminations are required; one to obtain the particular solution by polyharmonic spline interpolation as in (20), and the other for the MFS as in (27). Hence, at each time step, we need only to use backward substitution to find the coefficients in (20) and (27). All computation was performed using double precision.

Example 1. We first consider Fisher's equation [Britton (1986); Langdon (1999)] in 2D, which is given by

$$\frac{\partial u}{\partial t} = \Delta u + ku(1 - u), \quad \text{in } D \times (0, T], \quad (29)$$

with initial and boundary conditions

$$u(\mathbf{x}, 0) = J_0 \left(c \sqrt{x_1^2/9 + x_2^2} \right), \quad \text{in } D, \quad (30)$$

$$u(\mathbf{x}, t) = 0, \quad \text{on } \partial D \times (0, T], \quad (31)$$

where $k \geq 0$, J_0 is the first kind Bessel function of order zero, $c \simeq 2.4048$ is the first zero of J_0 and $\mathbf{x} = (x_1, x_2)$. The physical domain is given by $D \cup \partial D = \{(x_1, x_2) : x_1^2/9 + x_2^2 \leq 1\}$.

The existence and uniqueness of a solution to (29)-(31) is proved in Birtton (1986). It is also shown there that there exists a critical value $\delta^* > 0$ such that if $k < \delta^*$ then zero is a stable steady state, and if $k > \delta^*$ then zero is an unstable steady state. Furthermore, for $k > \delta^*$ there exists at least one non-trivial non-negative solution of (29)-(31). In his Ph.D. thesis, Langdon (1999) verified these results numerically by using a domain imbedding boundary integral method. As we shall see, our results are consistent with Langdon's Ph.D. thesis. Both approaches were carried out in a natural way and do not require spatial iteration; i.e., *a nonlinear problem can be solved the same way as a linear one.*

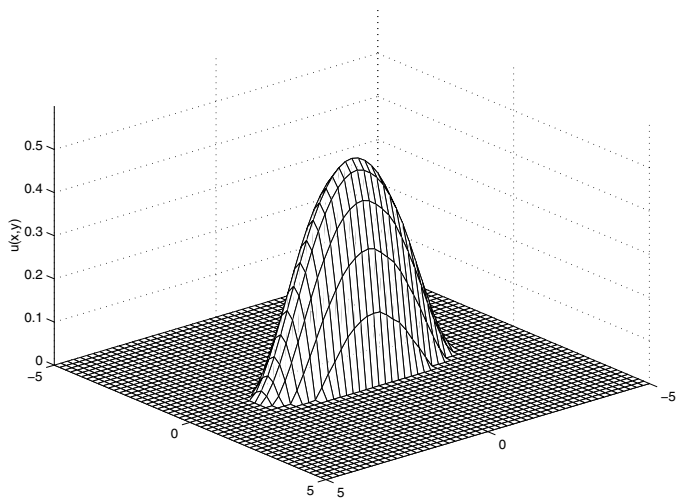


Figure 2 : Profile of steady state solution for $k=5$.

We used thin plate splines ($n = 1$ in (17)) as basis functions. 110 quasi-random points, including 32 uniformly distributed points on the boundary, in $D \cup \partial D$ were chosen for interpolating the forcing term h_n in (9). We used the MFS to evaluate the homogeneous solution. In doing so, the 32 evenly spaced boundary points mentioned above were chosen on the boundary ∂D . The same number of points were chosen on the fictitious boundary which is a circle with center at $(0, 0)$ and radius 15.

Fig. 1 shows the convergence to a trivial steady state for $k = 1, 3$ and to non-trivial steady states for $k = 5$ and 10 at the observation point $(0, 0)$. Here we used the backward time difference scheme with $\tau = 0.02$. From Fig. 1, we can deduce that the critical value δ^* satisfies $3 < \delta^* < 5$. The approximate value of δ^* can be obtained by using the bisection method [Chen (1995)].

In Table 1 we compare our results with Langdon's. They are in excellent agreement.

To evaluate the non-trivial steady state solution of (29)-(31), we take t sufficiently large such that $\|\hat{v}_{n+1} - \hat{v}_n\|_\infty < 10^{-6}$. The profiles of non-trivial steady state solutions for $k = 5$ and 10 are given in Figs. 2-3. Based on the numerical results that we have obtained on the 110 quasi-random points, we used the technique of surface reconstruction to reproduce these two graphs by using the thin plate splines shown in (17).

Example 2. To demonstrate that our proposed method can be easily extended to higher dimensional problems, we solve Fisher's equation in Example 1

Table 1. Comparison of $u(0,0)$ using different methods.

Time	$u(0,0)$ (Langdon)			$u(0,0)$ (mesh-free)		
	$k = 1$	$k = 5$	$k = 10$	$k = 1$	$k = 5$	$k = 10$
0.5	3.4684E-1	6.1473E-1	9.0441E-1	3.4547E-1	6.1570E-1	9.0389E-1
1.0	1.4151E-1	5.4890E-1	8.6755E-1	1.4044E-1	5.4944E-1	8.6933E-1
1.5	5.9663E-2	5.2275E-1	8.5080E-1	5.8980E-2	5.2303E-1	8.5462E-1
2.0	2.5426E-2	5.1070E-1	8.4215E-1	2.5038E-2	5.1084E-1	8.4733E-1
2.5	1.0878E-2	5.0484E-1	8.3723E-1	1.0672E-2	5.0489E-1	8.4332E-1
3.0	4.6619E-3	5.0192E-1	8.3424E-1	4.5564E-3	5.0192E-1	8.4097E-1
3.5	1.9990E-3	5.0045E-1	8.3237E-1	1.9466E-3	5.0042E-1	8.3958E-1
4.0	8.5740E-4	4.9971E-1	8.3116E-1	8.3183E-4	4.9967E-1	8.3873E-1
4.5	3.6779E-4	4.9934E-1	8.3037E-1	3.5551E-4	4.9928E-1	8.3822E-1
5.0	1.5777E-4	4.9915E-1	8.2985E-1	1.5194E-4	4.9909E-1	8.3792E-1

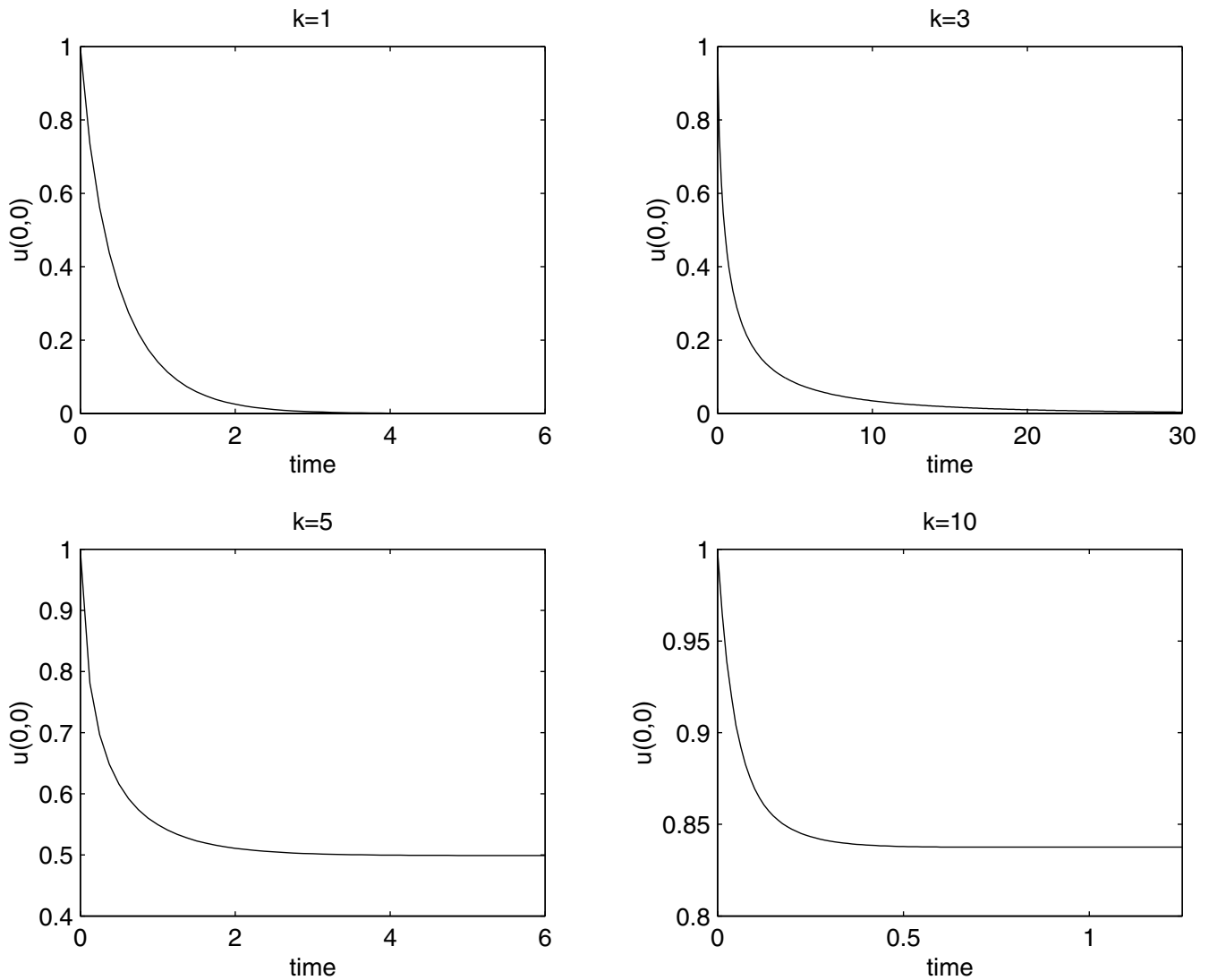


Figure 1 : Approximate value of $u(0,0)$ against time for $k = 1, 3, 5,$ and 10 .

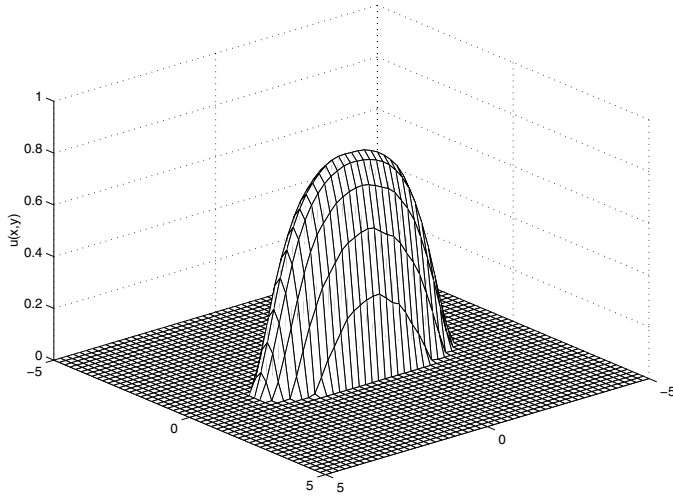


Figure 3 : Profile of steady state solution for $k=10$.

in 3D. The initial condition is given as $u(\mathbf{x},0) = J_0 \left(c\sqrt{x_1^2/9 + x_2^2 + x_3^2} \right)$ in D . The physical domain is defined as an ellipsoid; i.e. $D \cup \partial D = \{(x_1, x_2, x_3) : x_1^2/9 + x_2^2 + x_3^2 \leq 1\}$.

We chose 320 quasi-random points, including 120 points on the surface of the ellipsoid, in $D \cup \partial D$ to evaluate the particular solutions. We used thin plate splines $\varphi = r$ ($n = 1$ in (17)) as the basis functions. To be more specific, $\psi^{[1]}$ in (21) and χ in (22) is given by

$$\psi^{[1]} = \begin{cases} -\frac{r}{\lambda^2} - \frac{2}{\lambda^4 r} - \frac{2e^{-\lambda r}}{\lambda^4 r}, & r \neq 0, \\ -\frac{2}{\lambda^3}, & r = 0, \end{cases}$$

and

$$\chi^{[1]} = -\frac{1}{\lambda^2}(1 + x + y + z).$$

To obtain the homogeneous solution at each time step, we used the MFS. 120 quasi-random points were chosen on the surface of the ellipsoid as the collocation points. The same number of source points were chosen on the fictitious boundary which is a sphere with center at origin and radius 15.

The profiles of $u(0,0,0)$ for $k = 1, 5, 10$ and 15 are given in Fig. 4 which is similar to the 2D case. For $k = 1$ and 5 , the solutions converge to a trivial steady state solution. For $k = 10$ and 15 , there exist non-trivial solutions. It is clear that the critical value δ^* is between 5 and 10.

We note that the domain embedding method proposed in Langdon (1999) is not immediately applicable in the 3D case. Our proposed mesh-free method can be extended to even higher dimensions without any difficulty.

Example 3. In this example, we consider the thermal explosion problem [Chen (1995)]

$$u_t = \Delta u + \delta \exp(u), \text{ in } D, \quad (32)$$

$$u(\mathbf{x},0) = 0, \text{ in } D, \quad (33)$$

$$u(\mathbf{x},t) = 0, \text{ on } \partial D \times (0, T], \quad (34)$$

where δ is a parameter and $D \cup \partial D = \{\mathbf{x}=(x_1, x_2) : x_1^2/4 + x_2^2 \leq 1\}$. This problem has been studied both theoretically and numerically during the past three decades. It is known that u either converges to the steady state as t goes to infinity or blows up in finite time. Amann (1976) showed that there exists a critical value δ^* such that if $\delta < \delta^*$ there is a positive solution to the steady-state solution of (32)-(33), whereas if $\delta > \delta^*$ no solution exists. In this example, we do not intend to find the critical value δ^* . Instead, we approximate the steady-state solution by taking time sufficiently large as mentioned in Example 1. We then compare this with the result in Chen (1995) where Picard iteration was used to approximate the steady state solution. Let $U = \lim_{t \rightarrow \infty} u$, then the steady state solution U satisfies

$$\Delta U = \delta e^U \text{ in } D$$

$$U = 0 \text{ on } \partial D.$$

In order to do the comparison, we choose the same parameters for both approaches. We used thin plate splines ($n = 1$ in (17)) as basis functions. 80 quasi-random interpolation points, including 16 uniformly distributed points on the boundary, were chosen in $D \cup \partial D$. The same 16 points on the boundary were again used to approximate the homogeneous solution using the MFS. As usual, 16 uniformly distributed source points were chosen on the fictitious boundary which is a circle with center (0,0) and radius 15.

In Chen (1995), the critical value δ^* was found to be 1.235. In Table 2, we show the convergence of $u(0,0)$ for various values of δ . We consider $u(0,0)$ and $U(0,0)$ to converge when the difference between two consecutive iterations (space and time) is less than 10^{-6} . We denote T as the final time for convergence of the time-dependent problem. We choose the time step $\tau = 0.1$ to approximate u for time dependent problems. In order to compare the

Table 2. Approximation of $U(0,0)$ and $u(0,0)$ for various δ .

δ	Time Independent			Time Dependent		
	$U(0,0)$	# of Iterations	CPU (Sec.)	$u(0,0)$	Final Time (T)	CPU (Sec.)
1.0	0.6578	18	3.96	0.6581	8.9	4.39
1.1	0.8106	24	5.21	0.8082	11.1	5.39
1.2	1.0723	45	9.17	1.0573	18.5	8.35
1.23	1.2644	106	21.36	1.2135	31.7	13.62

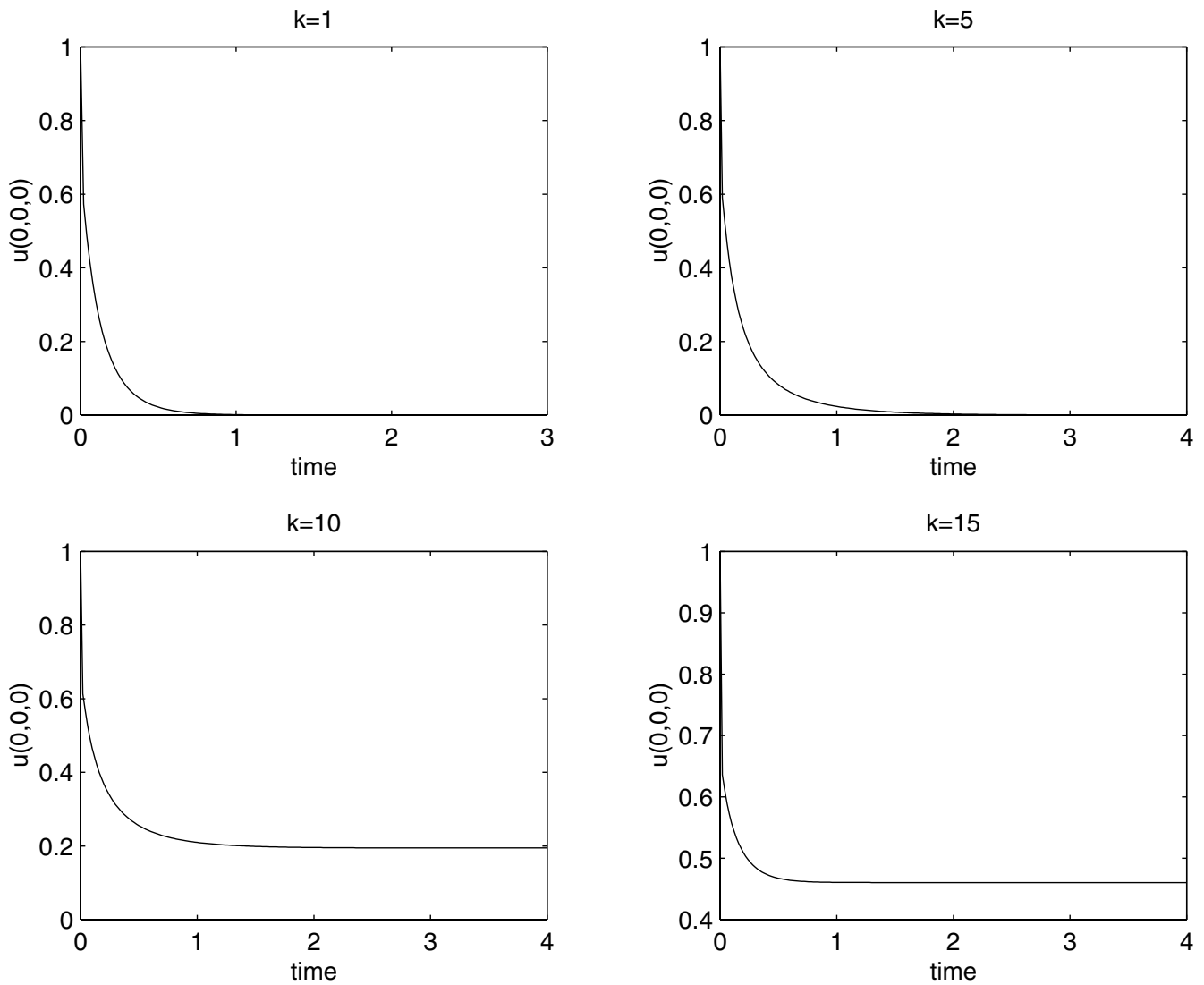


Figure 4 : Approximate value of $u(0,0,0)$ against time for $k = 1, 5, 10,$ and 15 .

efficiency of these two approaches, we deliberately tested both algorithms on a slower PC (Pentium 120 MH). For δ closer to the critical value, our proposed method is more efficient than algorithm in Chen (1995). For $\delta = 1.3$, thermal explosion occurs at $t \simeq 6.3$.

6 Conclusions

Using the radial basis functions and the MFS, we have developed a mesh-free method to solve nonlinear reaction-diffusion equations; i.e., no domain discretization and domain integration are required. Another attraction of our approach is that no spatial iteration is required and thus provides an efficient algorithm for solving such nonlinear problems, especially for 3D problems. It is expected that the accuracy can be further improved if a higher order time stepping algorithm and higher order polyharmonic splines are implemented. Further work in extending the current approach to other time-dependent problems is currently under investigation.

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