

Application of Multi-Region Trefftz Method to Elasticity

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Abstract: This paper presents an application of a direct Trefftz method with domain decomposition to the two-dimensional elasticity problem. Trefftz functions are substituted into Betti's reciprocity theorem to derive the boundary integral equations for each subdomain. The values of displacements and tractions on subdomain interfaces are tailored by continuity and equilibrium conditions, respectively. Since Trefftz functions are regular, much less requirements are put on numerical integration than in the traditional boundary integral method. Then, the method can be utilized to analyse also very narrow domains. Linear elements are used for modelling of the boundary geometry and approximation of boundary quantities. Numerical results for a rectangular plate with varying aspect ratio and cantilever beam are presented.

keyword: polynomial Trefftz functions, direct formulation, linear approximation, boundary integral equation

1 Introduction

The effort to predict more effectively the response of the complex problems of continuum mechanics motivates the authors to seek for formulations in which the governing equations inside the approximated domain are identically satisfied. Such analytically derived functions are called Trefftz functions [Trefftz (1926)] which can be found in the form of polynomials, Legendre, harmonic, Bessel, Hankel, singular Kupradze functions, etc. [Zielinski (1995)].

The main characteristic of the Trefftz method is the use of trial functions that satisfy the governing differential equations in a domain. The method can be classified into the indirect and the direct formulations [Jin, Cheung and Zienkiewicz (1990); Kita, Kamiya and Iao (1999)]. In the indirect formulation the trial function is given as a superposition of Trefftz functions. Then, the unknown parameters in the trial function are determined so that boundary conditions have to be satisfied. In the direct formulation the Trefftz functions are considered as a weighting (test) function in Betti's reciprocity theorem to derive boundary integral equations. Boundary values are approximated by standard polynomials within elements on discretized boundary. In this approach, unknown values have a physical interpretation in contrast to the indirect approach,

where unknowns are fictitious parameters. That is the main reason, why in this paper a direct formulation is used inspite of the more frequently occurred indirect formulation in the literature [Leitao (1998); Zielinski and Herrera (1987); Herrera (1995)]. If the number of Trefftz functions used as test functions is high the final system of algebraic equations can be ill-posed. To overcome this difficulty the whole domain is subdivided into smaller subregions. For the same purpose the special Trefftz finite element formulations are frequently used by many researchers [Jirousek and Leon (1977); Jirousek (1978); Jirousek (1987); Freitas and Ji (1996); Freitas (1998); Jirousek and Zielinski (1993); Kompis and Bury (1999)]. On interfaces of subregions displacement continuity and equilibrium of tractions have to be satisfied. Adding such restriction conditions at nodal points to the discretized integral equations, we create a complete set of algebraic equations for the unknown values at nodal points.

Boundary integral formulations combined with Trefftz functions as test functions can be used for the computation of boundary unknowns. An integral representation of displacements at internal points is not available in such a formulation. If such quantities are required, the integral representation with the fundamental (singular) solution has to be used [Balas, Sladek and Sladek (1989)].

Numerical tests were carried out on a rectangular plate with varying side aspect ratio and a cantilever beam.

2 Boundary integral equations

For linear isotropic problems of elasticity, the governing equations are force equilibrium equations expressed in displacements, the Lamé-Navier equations in domain

$$(\bar{\lambda} + \mu) u_{j,ji} + \mu u_{i,jj} = X_i, \quad (1)$$

where X_i is a body force vector. Recall that in two-dimensional problems under plane stress conditions special care should be paid to the association of the coefficients with the material parameters. In general,

$$\bar{\lambda} = \frac{2\mu\bar{\nu}}{1-2\nu}, \quad \bar{\nu} = \begin{cases} \frac{\nu}{1+\nu}, & \text{plane stress} \\ \nu, & \text{otherwise} \end{cases},$$

where ν is Poisson's ratio and μ is the shear modulus associated with the Young modulus as $E = 2\mu(1 + \nu)$.

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An equivalent to the Lamé-Navier equation is the force equilibrium in terms of stresses

$$\sigma_{ij,j} - X_i = 0. \quad (2)$$

Let Γ_u and Γ_t denote parts of the boundary Γ ($\Gamma = \Gamma_u \cup \Gamma_t$) with prescribed displacements and tractions, respectively.

Traction vector t_i is defined as a scalar product of stress tensor σ_{ij} and outward normal vector n_j . The weak formulation of eq. (2) can be expressed in weighted residual form

$$\int_{\Omega} W_{1i}(\sigma_{ij,j} - X_i) d\Omega + \int_{\Gamma_u} W_{2i}(u_i - \bar{u}_i) d\Gamma - \int_{\Gamma_t} W_{1i}(t_i - \bar{t}_i) d\Gamma = 0, \quad (3)$$

where prescribed quantities are denoted by overbar and u_i, t_i are trial functions. Weight functions are chosen in the following form

$$W_{1i} = u_i^*, \quad W_{2i} = t_i^*, \quad (4)$$

where u_i^* and t_i^* are the displacements and tractions corresponding to the weighting field.

Applying the Gauss-Green formula to the domain integral in eq. (3), we have

$$\int_{\Omega} u_i^*(\sigma_{ij,j} - X_i) d\Omega = \int_{\Gamma} u_i^* t_i d\Gamma - \int_{\Gamma} t_i^* u_i d\Gamma + \int_{\Gamma} \sigma_{ij,j}^* u_i d\Omega - \int_{\Omega} u_i^* X_i d\Omega. \quad (5)$$

Substituting eq. (5) into (3), with taking into account the prescribed boundary conditions, one obtains

$$\int_{\Gamma} u_i^* t_i d\Gamma - \int_{\Gamma} t_i^* u_i d\Gamma + \int_{\Omega} \sigma_{ij,j}^* u_i d\Omega = \int_{\Omega} X_i u_i^* d\Omega. \quad (6)$$

If the weighting field is selected as a Trefftz function the homogeneous governing equation is satisfied

$$\sigma_{ij,j}^* = 0 \quad (7)$$

in Ω' , where $\Omega \subset \Omega'$.

Finally, one obtains the boundary integral equation

$$\int_{\Gamma} u_i^* t_i d\Gamma - \int_{\Gamma} t_i^* u_i d\Gamma = \int_{\Omega} X_i u_i^* d\Omega. \quad (8)$$

The boundary integral equation (8) relates the boundary displacements and traction vectors. Then, the unprescribed quantities can be computed from that equation. In contrast to the conventional boundary integral equations the free term is missing here. The governing equation for the fundamental solution contains the Dirac delta function which causes a singular behaviour of the fundamental solution. Here, Trefftz functions are nonsingular and all the integrals exist in a regular sense.

It makes the numerical integrations much easier in the Trefftz boundary integral equation formulation than in the conventional formulation based on singular boundary integral equations. On the other hand, the numerical stability is more questionable in the former case because of a higher condition number of the discretized BIE.

The boundary integral equation (8) can be used only for the computation of unknown boundary quantities. If we are interested in displacement and stress values in the interior of the body, the Somigliana identity and integral representation for stresses, respectively, has to be utilized for the evaluation. Fundamental solutions corresponding to the Lamé-Navier governing equation (1), with the Dirac delta function distribution of body forces, are denoted here as $U_{ij}(x-y)$. Then, the governing equations can be written as

$$\left(\bar{\lambda} + \mu\right) U_{jm,ji}(x-y) + \mu U_{im,jj}(x-y) = \delta_{im} \delta(x-y), \quad (9)$$

with the fundamental solutions in two dimensions being given as

$$U_{ij}(r) = \frac{1}{8\pi\mu(1-\bar{\nu})} \{ (4\bar{\nu} - 3) \ln r \delta_{ij} + r_{,i} r_{,j} \}, \\ T_{ij}(x,y) = -\frac{1}{4\pi\mu(1-\bar{\nu})r} \left\{ [(1-2\bar{\nu})\delta_{ij} + 2r_{,i} r_{,j}] \frac{\partial r}{\partial n} - (1-2\bar{\nu})(r_{,j} n_i - r_{,i} n_j) \right\}. \quad (10)$$

The Somigliana identity has the form [Balas, Sladek and Sladek (1989)]

$$u_i(y) = \int_{\Gamma} t_j(x) U_{ij}(x-y) d\Gamma_x - \int_{\Gamma} u_j(x) T_{ij}(x,y) d\Gamma_x + \int_{\Omega} X_j(x) U_{ij}(x-y) d\Omega_x. \quad (11)$$

If an interior point y is lying very close to the boundary, the second integral on the r.h.s. of eq. (11) is nearly singular. The accurate evaluation of such integral requires special attention. It is well known that the integral representation (11) can be regularized (see e.g. [Balas, Sladek and Sladek (1989)]) as

$$u_i(y) = u_i(\zeta) + \int_{\Gamma} t_j(x) U_{ij}(x-y) d\Gamma_x - \int_{\Gamma} [u_j(x) - u_j(\zeta)] \times T_{ij}(x,y) d\Gamma_x + \int_{\Omega} X_j(x) U_{ij}(x-y) d\Omega_x, \quad (12)$$

where $\zeta \in \Gamma$ can be selected as the nearest boundary point to $y \in \Omega$.

Similarly, one can derive a regularized integral representation for stresses [Balas, Sladek and Sladek (1989)]. Note that in the integral representation (12) the strong singularity of the kernel T_{ij} (as y approaches $\zeta \in \Gamma$) is smoothed by the vanishing factor $u_j(x) - u_j(\zeta)$ in the numerical integration. A more advanced cancellation of divergent terms is discussed elsewhere [Sladek and Sladek (1998)].

The Trefftz formulation of the boundary integral equations supplemented with integral representations based on singular fundamental solutions can be utilized for the numerical analysis of the whole domain.

3 Trefftz functions for 2-d elasticity problems

The Trefftz function is a homogeneous solution of the partial differential equation (1). We try to find the Trefftz function in a polynomial form. Then, for displacements one can write

$$\begin{aligned} u_1^* &= \sum_{n=0}^s \sum_{m=0}^s a_{nm} x_1^n x_2^m \\ u_2^* &= \sum_{n=0}^s \sum_{m=0}^s b_{nm} x_1^n x_2^m \end{aligned} \quad (13)$$

where $2s$ is the order of polynomial. Substituting polynomials (13) into the governing equation (1) ($X_i = 0$), one obtains a system of algebraic equations for the computation of the unknown parameters a_{nm} and b_{nm} . For such purposes symbolic computations by MATHEMATICA software has been utilized. Explicit expressions of some of the Trefftz functions are given in Appendix.

4 Numerical implementation

In the direct BIE Trefftz formulation the boundary is discretized by conforming elements. Boundary quantities are approximated within the elements:

$$g_i(x) = \sum_{a=1}^n N^a(\xi) g_i^{aa}, \quad (14)$$

where g_i^{aa} is the nodal value of a physical quantity $g_i \in \{u_i, t_i\}$ on element q at the node with local number a . The value n denotes the number of nodes on a element. For a linear approximation, $n = 2$ and the shape functions $N^a(\xi)$ have the following form

$$\begin{aligned} N^1(\xi) &= 0.5(1 - \xi), \\ N^2(\xi) &= 0.5(1 + \xi) \quad \xi \in \langle -1, 1 \rangle. \end{aligned} \quad (15)$$

For the evaluation of the domain integral in eq. (8) we can use an analytical method if the domain and body force distribution are simple. Otherwise a numerical method has to be used. Standard isoparametric elements are very convenient [Balas, Sladek and Sladek (1989)]. Making use of the approximation formula (14) in the BIE (8), we obtain a system of algebraic equations for nodal displacements u_i^{aa} and tractions t_i^{aa} , respectively,

$$\begin{aligned} \sum_{q=1}^{N_q} \sum_{a=1}^2 \int_{\Gamma_q} \left[u_i^{*(b)}(\xi) t_i^{aa} - t_i^{*(b)}(\xi) u_i^{aa} \right] N^a(\xi) J^q d\xi = \\ \sum_{a=1}^{N_e} \int_{\Gamma_e} u_i^{*(b)}(\xi_1, \xi_2) N^a(\xi_1, \xi_2) X_i^{ae} J^e d\xi_1 d\xi_2 \end{aligned} \quad (16)$$

for $b = 1, 2, \dots$, where N_q and N_e is the number of boundary and domain elements, respectively. The symbol J^q denotes the Jacobian for transformation of Cartesian coordinates into isoparametric ones. A matrix form of eq. (16) is given by

$$\mathbf{T}\mathbf{u} - \mathbf{U}\mathbf{t} = \mathbf{F}, \quad (17)$$

where the definition of the matrices directly follows from eq. (16). The number of rows in eq. (17) is equal to the number of the Trefftz functions. Since the maximum number of the columns is twice the number of the boundary nodes, the number of the Trefftz functions must be equal to or larger than twice the number of the nodes. In this paper we have selected such a number of Trefftz functions that we obtained a square matrix.

Kita, Kamiya and Iio (1999) have analysed the conditioning of matrices in the Trefftz boundary integral equation method for potential problems. They considered a narrow cut of thick-walled cylinder with a high aspect ratio of the arc length over the difference of radii. From numerical results it follows that the condition number is dependent on the number of the functions rather than on the aspect ratio. Provided that the number of Trefftz functions is restricted, the accuracy is reasonable even if the objects with long and narrow profiles are analysed. Therefore, for solving the large-scaled problems accurately, the domain decomposition is required. For simplicity we will consider a domain divided into two subdomains. The subdomains are referred to as Ω_1 and Ω_2 , respectively, with boundaries Γ_1 Γ_2 and Γ_I (interface). Then, eq. (17) on the subdomains Ω_α can be written as

$$\mathbf{T}^\alpha \mathbf{u}^\alpha - \mathbf{U}^\alpha \mathbf{t}^\alpha = \mathbf{F}^\alpha \quad (18)$$

for $\alpha = 1, 2$, where the superscript is related to the subdomain.

On the interface boundary both the displacements and traction vector are unknown. The system of integral equations (18) is not sufficient to get a unique solution. Therefore, they have to be supplemented by additional equations which satisfy displacement continuity and force equilibrium on the interface:

$$\begin{aligned} u_i^1(x) &= u_i^2(x), \\ t_i^1(x) &= -t_i^2(x), \end{aligned} \quad (19)$$

for $x \in \Gamma_I$. Then, the integral equations (18), supplemented with the tailored conditions (19), give a complete system for a large-scaled boundary value problem.

5 Numerical examples

Two numerical examples will be presented here to show the accuracy of the Trefftz method in comparison with the conventional boundary integral equation method.

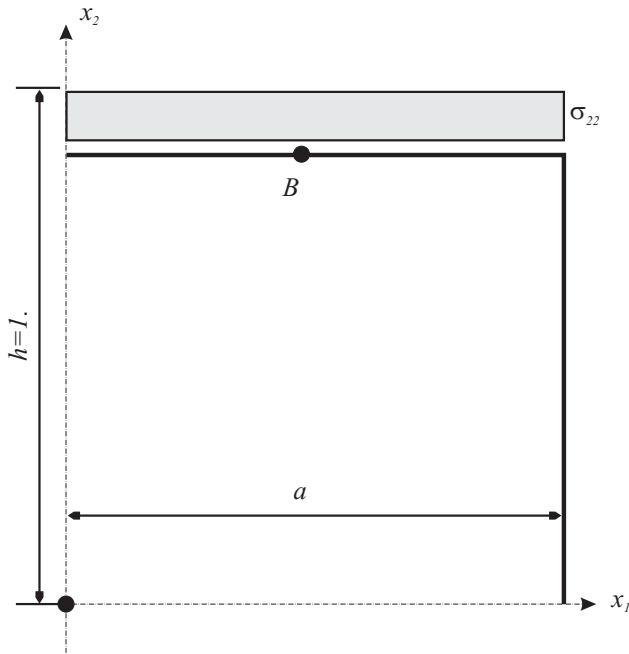


Figure 1 : A rectangular plate with varying aspect ratio a/h under a uniaxial uniform load $\sigma_{22} = 1$

5.1 Example 1. Effect of the aspect ratio in a rectangular domain

A drawback of the conventional boundary element method is that it is not convenient for analysis of domains with high aspect ratios and a special regularization approach is required [Sladek and Sladek (1998); Sladek, Sladek and Tanaka (1993)]. When the distance between the elements on close boundary faces is smaller than the size of elements, the accuracy fails due to an inaccurate evaluation of nearly singular integrals if conventional BEM formulation and/or standard numerical quadratures are employed. To investigate the dependence of the accuracy on the aspect ratio in the presented Trefftz method, we have analysed a rectangular domain with a varying aspect ratio (Fig. 1). The plate is subjected to a uniform load in x_2 direction. The following material constants are considered: Young modulus $E = 1$ and Poisson ratio $\nu = 0.3$.

Due to two axial symmetries only one quarter of the plate is modelled. The analysed domain is discretized by 8 linear elements. The height of the plate is kept constant and the plate width is varying from $a/h = 1$ to $a/h = 0.001$. Numerical results are given in Tab. 1. One can observe essentially higher stability of numerical computations by the presented method than by the conventional BEM formulation combined with using regular Gauss quadrature for numerical integration. A quadratic approximation in the BEM is used on 4 elements (8 nodes). When increasing the number of elements in the BEM, the accuracy of the results is expected to be higher.

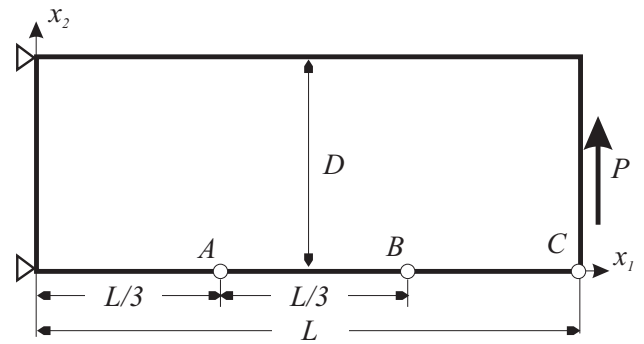


Figure 2 : Cantilever beam with a parabolic shear end load

5.2 Example 2. Cantilever beam

The behaviour of the present Trefftz BIE is also studied in the cantilever problem (Fig. 2) for which the following exact solution is given by Timoshenko and Goodier (1970). The displacements in the beam are

$$\begin{aligned} u_1 &= -\frac{P(1-\nu^2)}{6EI} \left(x_2 - \frac{D}{2}\right) \left[3x_1(2L-x_1) + \frac{2-\nu}{1-\nu}x_2(x_2-D)\right] \\ u_2 &= \frac{P(1-\nu^2)}{6EI} \left[x_1^2(3L-x_1) + \frac{3\nu}{1-\nu}(L-x_1)(x_2-D/2)^2 + \frac{4+\nu}{4-4\nu}D^2x_1\right], \end{aligned} \quad (20)$$

where $I = D^3/12$

$$\bar{E} = \begin{cases} E & \text{plane strain} \\ \frac{(1+2\nu)E}{(1+\nu)^2} & \text{plane stress} \end{cases}$$

The stresses corresponding to eq. (20) are

$$\begin{aligned} \sigma_{11} &= -\frac{P}{I}(L-x_1)(x_2-D/2), \\ \sigma_{22} &= 0, \\ \sigma_{12} &= -\frac{Px_2}{2I}(x_2-D). \end{aligned} \quad (21)$$

In the numerical analysis the following geometry and material parameters were considered: $L = 3$, $D = 1$, $E = 1$, $\nu = 0.3$, $P = 1$. The beam domain is divided into three subdomains of the same size with 12 linear elements on each subdomain boundary.

Numerical results obtained by the present Trefftz BIE method are compared with the analytical solutions at three points A, B, C (Fig. 2). The displacements at these points are given in Tab. 2.

A quite good agreement of results can be observed. A relative error for the maximum bending stresses σ_{11} at the clamped side is less than 1% (exact value: $\sigma_{11} = 18$ and the numerical value $\sigma_{11} = 18.16$).

Table 1 : Numerical results for a rectangular plate under a uniform tension

a/h	Trefftz method				BEM				Exact	
	σ_{22}^A	u_2^B	Err. σ %	Err. u %	σ_{22}^A	u_2^B	Err. σ %	Err. u %	σ_{22}^A	u_2^B
1.	1.	0.91	0.	0.	0.9993	0.9108	0.07	0.088	1.	0.91
0.5	0.9992	0.9097	0.08	0.033	0.9995	0.9104	0.05	0.044	1.	0.91
0.1	0.9989	0.9091	0.11	0.099	1.114	0.8695	11.4	4.45	1.	0.91
0.05	0.9956	0.9099	0.44	0.011	0.583	0.817	41.7	10.22	1.	0.91
0.01	1.05	0.9241	5.0	1.549	1.526	0.6372	52.6	29.97	1.	0.91
0.005	1.019	0.948	1.9	4.175	2.079	0.865	107.9	4.94	1.	0.91
0.001	1.52	0.769	52.0	15.49	2.484	9.587	148.4	953.5	1.	0.91

Table 2 : Displacements at the cantilever beam

	u_1^A	u_2^A	u_1^B	u_2^B	u_1^C	u_2^C
Analytical solution	10.50	14.25	16.80	43.95	18.90	82.05
Trefftz BIE	10.61	13.98	17.03	44.22	19.21	82.41
Error [%]	1.047	1.894	1.369	0.614	1.640	0.438

6 Conclusions

The present Trefftz BIE method is a promising computational alternative to more popular finite element or boundary element methods. A multi-domain Trefftz BIE formulation is keeping the simplicity of the FEM and the boundary character of the BEM. The number of subdomains in the presented method can be much less than in the conventional FEM due to higher modelling accuracy. As compared with the conventional BEM formulation, the treatment of singularities is avoided. Therefore, the numerical integration is much easier. Without any special effort the method is also convenient to analyse very narrow objects due to the regular integrals in the formulation.

A drawback of the Trefftz BIE formulation is the necessity to use a multi-domain approach because of the decrease of the condition number of the discretized BIE with increasing number of Trefftz functions. Consequently, this leads to an increase of the number of unknowns on the subdomain interfaces also in physically homogeneous problems. Of course, in the case of piecewise nonhomogeneous problems this effect disappears because of the necessity of domain subdivision in such problems.

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Appendix A:

In this Appendix, we collect the explicit expressions for several Trefftz functions $u_i^{*(n)}$ and $t_i^{*(n)}$ in terms of polynomials with respect to cartesian coordinates of a field point in a global coordinate system. First of all, let us introduce two parameters e and \tilde{v}

$$e = \begin{cases} E & \text{plane stress} \\ \frac{E}{1-\nu^2} & \text{plane strain} \end{cases} \quad (22)$$

$$\tilde{v} = \begin{cases} \nu & \text{plane stress} \\ \frac{\nu}{1-\nu} & \text{plane strain} \end{cases} \quad (23)$$

which allow to write the Trefftz functions in a compact form for both the plane stress and plane strain problems. Recall that the Hooke law takes the form

$$\sigma_{ij} = \frac{e}{1+\tilde{v}} \left(\varepsilon_{ij} + \frac{\tilde{v}}{1-\tilde{v}} \delta_{ij} \varepsilon_{kk} \right), \quad \varepsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}).$$

The tractions $t_i^{*(n)}$, corresponding to the Trefftz displacements $u_i^{*(n)}$, can be obtained by contraction of stresses $\sigma_{ij}^{*(n)}$ by the normal vector n_j as

$$t_i^{*(n)} = \sigma_{ij}^{*(n)} n_j \quad (24)$$

Trefftz functions:

$$u_1^{*(1)} = 1, \quad (25)$$

$$u_2^{*(1)} = 0, \quad (26)$$

$$t_1^{*(1)} = 0, \quad (27)$$

$$t_2^{*(1)} = 0 \quad (28)$$

$$u_1^{*(2)} = x_1, \quad (29)$$

$$u_2^{*(2)} = 0, \quad (30)$$

$$t_1^{*(2)} = \frac{e}{1-\tilde{\nu}^2} n_1, \quad (31)$$

$$t_2^{*(2)} = \frac{e\tilde{\nu}}{1-\tilde{\nu}^2} n_2, \quad (32)$$

$$u_1^{*(3)} = x_2/2, \quad (33)$$

$$u_2^{*(3)} = x_1/2, \quad (34)$$

$$t_1^{*(3)} = \frac{e}{2(1+\tilde{\nu})} n_2, \quad (35)$$

$$t_2^{*(3)} = \frac{e}{2(1+\tilde{\nu})} n_1, \quad (36)$$

$$u_1^{*(4)} = 0, \quad (37)$$

$$u_2^{*(4)} = 1, \quad (38)$$

$$t_1^{*(4)} = 0, \quad (39)$$

$$t_2^{*(4)} = 0, \quad (40)$$

$$u_1^{*(5)} = 0, \quad (41)$$

$$u_2^{*(5)} = x_2, \quad (42)$$

$$t_1^{*(5)} = \frac{e\tilde{\nu}}{1-\tilde{\nu}^2} n_1, \quad (43)$$

$$t_2^{*(5)} = \frac{e}{1-\tilde{\nu}^2} n_2, \quad (44)$$

$$u_1^{*(6)} = -x_2/2, \quad (45)$$

$$u_2^{*(6)} = x_1/2, \quad (46)$$

$$t_1^{*(6)} = 0, \quad (47)$$

$$t_2^{*(6)} = 0, \quad (48)$$

$$u_1^{*(7)} = x_1 x_2, \quad (49)$$

$$u_2^{*(7)} = -(1+\tilde{\nu})x_2^2/4, \quad (50)$$

$$\sigma_{11}^{*(7)} = \frac{e(2+\tilde{\nu})}{2(1+\tilde{\nu})} x_2, \quad (51)$$

$$\sigma_{12}^{*(7)} = \frac{e}{2(1+\tilde{\nu})} x_1, \quad (52)$$

$$\sigma_{22}^{*(7)} = -\frac{e}{2(1+\tilde{\nu})} x_2, \quad (53)$$

$$u_1^{*(8)} = \frac{1}{\tilde{\nu}-1} (2x_2^2 + \tilde{\nu}x_2^2 - x_2^2), \quad (54)$$

$$u_2^{*(8)} = 0, \quad (55)$$

$$\sigma_{11}^{*(8)} = \frac{2e}{1-\tilde{\nu}^2} x_1, \quad (56)$$

$$\sigma_{12}^{*(8)} = -\frac{2e}{1-\tilde{\nu}^2} x_2, \quad (57)$$

$$\sigma_{22}^{*(8)} = \frac{2e\tilde{\nu}}{1-\tilde{\nu}^2} x_1, \quad (58)$$

$$u_1^{*(9)} = \frac{1+\tilde{\nu}}{2(\tilde{\nu}-1)} x_2^2, \quad (59)$$

$$\begin{aligned}
u_2^{*(9)} &= x_1 x_2, \\
\sigma_{11}^{*(9)} &= \frac{e\tilde{v}}{1-\tilde{v}^2} x_1, \\
\sigma_{12}^{*(9)} &= \frac{e\tilde{v}}{1-\tilde{v}^2} x_2, \\
\sigma_{22}^{*(9)} &= -\frac{e\tilde{v}}{1-\tilde{v}^2} x_2, \\
u_1^{*(10)} &= 0, \\
u_2^{*(10)} &= (2x_1^2 + \tilde{v}x_2^2 - x_2^2)/2, \\
\sigma_{11}^{*(10)} &= -\frac{e\tilde{v}}{1+\tilde{v}} x_2, \\
\sigma_{12}^{*(10)} &= \frac{e}{1+\tilde{v}} x_1, \\
\sigma_{22}^{*(10)} &= -\frac{e}{1+\tilde{v}} x_2, \\
u_1^{*(11)} &= \frac{x_1}{6} (6x_2^2 + \tilde{v}x_1^2 - x_1^2), \\
u_2^{*(11)} &= -\frac{1+\tilde{v}}{6} x_2^3, \\
\sigma_{11}^{*(11)} &= \frac{e}{2(1+\tilde{v})} (\tilde{v}x_2^2 + 2x_2^2 - x_1^2), \\
\sigma_{12}^{*(11)} &= \frac{e}{1+\tilde{v}} x_1 x_2, \\
\sigma_{22}^{*(11)} &= -\frac{e}{2(1+\tilde{v})} (x_2^2 + \tilde{v}x_1^2), \\
u_1^{*(12)} &= \frac{x_2}{3(\tilde{v}-1)} (2x_2^2 + 3\tilde{v}x_1^2 - 3x_1^2), \\
u_2^{*(12)} &= \frac{1+\tilde{v}}{3(\tilde{v}-1)} x_1^3, \\
\sigma_{11}^{*(12)} &= \frac{2e}{1-\tilde{v}^2} x_1 x_2, \\
\sigma_{12}^{*(12)} &= -\frac{e}{1-\tilde{v}^2} (x_2^2 + \tilde{v}x_1^2), \\
\sigma_{22}^{*(12)} &= \frac{2e\tilde{v}}{1-\tilde{v}^2} x_1 x_2, \\
u_1^{*(13)} &= \frac{1+\tilde{v}}{3(\tilde{v}-1)} x_2^3, \\
u_2^{*(13)} &= \frac{x_1}{3(\tilde{v}-1)} (3\tilde{v}x_2^2 - 3x_2^2 + 2x_1^2), \\
\sigma_{11}^{*(13)} &= \frac{2e\tilde{v}}{1-\tilde{v}} x_1 x_2, \\
\sigma_{12}^{*(13)} &= -\frac{e}{1-\tilde{v}^2} (\tilde{v}x_2^2 + x_1^2), \\
\sigma_{22}^{*(13)} &= \frac{2e}{1-\tilde{v}^2} x_1 x_2, \\
u_1^{*(14)} &= -\frac{1+\tilde{v}}{6} x_1^3, \\
u_2^{*(14)} &= \frac{x_2}{6} (\tilde{v}x_2^2 - x_2^2 + 6x_1^2), \\
\sigma_{11}^{*(14)} &= -\frac{e}{2(1+\tilde{v})} (\tilde{v}x_2^2 + x_1^2), \\
\sigma_{12}^{*(14)} &= \frac{e}{1+\tilde{v}} x_1 x_2, \\
\sigma_{22}^{*(14)} &= -\frac{e}{2(1+\tilde{v})} (x_2^2 - \tilde{v}x_1^2 - 2x_1^2), \\
u_1^{*(15)} &= x_1 x_2^3, \\
u_2^{*(15)} &= -\frac{1}{2(1+\tilde{v})} (\tilde{v}x_2^4 - 3\tilde{v}x_1^2 x_2^2 + 3x_1^2 x_2^2 - x_1^4), \\
\sigma_{11}^{*(15)} &= \frac{ex_2}{(1+\tilde{v})^2} (2\tilde{v}x_2^2 + x_2^2 - 3\tilde{v}x_1^2), \\
\sigma_{12}^{*(15)} &= \frac{ex_1}{(1+\tilde{v})^2} (3\tilde{v}x_2^2 + x_1^2), \\
\sigma_{22}^{*(15)} &= -\frac{ex_2}{(1+\tilde{v})^2} (\tilde{v}x_2^2 + 3x_1^2), \\
u_1^{*(16)} &= x_2 x_1^3, \\
u_2^{*(16)} &= -\frac{1}{4(1+\tilde{v})} (\tilde{v}x_2^4 - x_2^4 + 12x_1^2 x_2^2 - \tilde{v}x_1^4 - 3x_1^4), \\
\sigma_{11}^{*(16)} &= \frac{ex_2}{(1+\tilde{v})^2} (\tilde{v}x_2^2 + 3x_1^2), \\
\sigma_{12}^{*(16)} &= -\frac{ex_1}{(1+\tilde{v})^2} (3x_2^2 - \tilde{v}x_1^2 - 2x_1^2), \\
\sigma_{22}^{*(16)} &= \frac{ex_2}{(1+\tilde{v})^2} (x_2^2 - 3\tilde{v}x_1^2 - 6x_1^2), \\
u_1^{*(17)} &= \frac{1}{4(1+\tilde{v})} (\tilde{v}x_2^4 + 3x_2^4 - 12x_1^2 x_2^2 - \tilde{v}x_1^4 + x_1^4), \\
u_2^{*(17)} &= x_1 x_2^3, \\
\sigma_{11}^{*(17)} &= -\frac{ex_1}{(1+\tilde{v})^2} (3\tilde{v}x_2^2 + 6x_2^2 - x_1^2), \\
\sigma_{12}^{*(17)} &= \frac{ex_2}{(1+\tilde{v})^2} (\tilde{v}x_2^2 + 2x_2^2 - 3x_1^2), \\
\sigma_{22}^{*(17)} &= \frac{ex_1}{(1+\tilde{v})^2} (3x_2^2 + \tilde{v}x_1^2), \\
u_1^{*(18)} &= \frac{1}{2(1+\tilde{v})} (x_2^4 + 3\tilde{v}x_1^2 x_2^2 - 3x_1^2 x_2^2 - \tilde{v}x_1^4), \\
u_2^{*(18)} &= x_2 x_1^3, \\
\sigma_{11}^{*(18)} &= -\frac{ex_1}{(1+\tilde{v})^2} (3x_2^2 + \tilde{v}x_1^2), \\
\sigma_{12}^{*(18)} &= \frac{ex_2}{(1+\tilde{v})^2} (x_2^2 + 3\tilde{v}x_1^2), \\
\sigma_{22}^{*(18)} &= -\frac{ex_1}{(1+\tilde{v})^2} (3\tilde{v}x_2^2 - 2\tilde{v}x_1^2 - x_1^2), \\
u_1^{*(19)} &= -\frac{x_2}{20} (\tilde{v}x_2^4 + 3x_2^4 - 20x_1^2 x_2^2 - 5\tilde{v}x_1^4 + 5x_1^4), \\
u_2^{*(19)} &= -\frac{1+\tilde{v}}{20} x_1 (5x_2^4 - x_1^4), \\
\sigma_{11}^{*(19)} &= \frac{e}{(1+\tilde{v})} x_1 x_2 (\tilde{v}x_2^2 + 2x_2^2 - x_1^2), \\
\sigma_{12}^{*(19)} &= -\frac{e}{4(1+\tilde{v})} (\tilde{v}x_2^4 + 2x_2^4 - 6x_1^2 x_2^2 - \tilde{v}x_1^4), \\
\sigma_{22}^{*(19)} &= -\frac{e}{(1+\tilde{v})} x_1 x_2 (x_2^2 + \tilde{v}x_1^2),
\end{aligned}
\tag{60}$$

$$u_1^{*(20)} = -\frac{1+\tilde{\nu}}{10(\tilde{\nu}-1)}x_1(5x_2^4-x_1^4), \quad (114) \quad \sigma_{22}^{*(24)} = -\frac{2ex_2}{5(1+\tilde{\nu})^2}(\tilde{\nu}x_2^4+2x_2^4-10x_1^2x_2^2-5\tilde{\nu}x_1^4). \quad (138)$$

$$u_2^{*(20)} = -\frac{x_2}{5(\tilde{\nu}-1)}(\tilde{\nu}x_2^4-5\tilde{\nu}x_2^2x_1^2+5x_2^2x_1^2-5x_1^4), \quad (115) \quad \text{Other higher order Trefftz functions can be obtained analogously.}$$

$$\sigma_{11}^{*(20)} = -\frac{e}{2(1-\tilde{\nu}^2)}(2\tilde{\nu}x_2^4+x_2^4-6\tilde{\nu}x_1^2x_2^2-x_1^4), \quad (116)$$

$$\sigma_{12}^{*(20)} = -\frac{2e}{1-\tilde{\nu}^2}x_1x_2(\tilde{\nu}x_2^2+x_1^2), \quad (117)$$

$$\sigma_{22}^{*(20)} = \frac{e}{2(1-\tilde{\nu}^2)}(\tilde{\nu}x_2^4+6x_1^2x_2^2-\tilde{\nu}x_1^4-2x_1^4), \quad (118)$$

$$u_1^{*(21)} = \frac{1+\tilde{\nu}}{20}x_2(x_2^4-5x_1^4), \quad (119)$$

$$u_2^{*(21)} = \frac{x_1}{20}(5\tilde{\nu}x_2^4-5x_2^4+20x_1^2x_2^2-\tilde{\nu}x_1^4-3x_1^4), \quad (120)$$

$$\sigma_{11}^{*(21)} = -\frac{e}{(1+\tilde{\nu})}x_1x_2(\tilde{\nu}x_2^2+x_1^2), \quad (121)$$

$$\sigma_{12}^{*(21)} = \frac{e}{4(1+\tilde{\nu})}(\tilde{\nu}x_2^4+6x_1^2x_2^2-\tilde{\nu}x_1^4-2x_1^4), \quad (122)$$

$$\sigma_{22}^{*(21)} = -\frac{e}{1+\tilde{\nu}}x_1x_2(x_2^2-\tilde{\nu}x_1^2-2x_1^2), \quad (123)$$

$$u_1^{*(22)} = -\frac{1}{15}(2x_2^6-15x_1^2x_2^4+x_1^6), \quad (124)$$

$$u_2^{*(22)} = -\frac{4x_1x_2}{15(1+\tilde{\nu})}(3\tilde{\nu}x_2^4-5\tilde{\nu}x_1^2x_2^2+5x_1^2x_2^2-3x_1^4), \quad (125)$$

$$\sigma_{11}^{*(22)} = \frac{2ex_1}{5(1+\tilde{\nu})^2}(10\tilde{\nu}x_2^4+5x_2^4-10\tilde{\nu}x_1^2x_2^2-x_1^4), \quad (126)$$

$$\sigma_{12}^{*(22)} = -\frac{2ex_2}{5(1+\tilde{\nu})^2}(2\tilde{\nu}x_2^4+x_2^4-10\tilde{\nu}x_1^2x_2^2-5x_1^4), \quad (127)$$

$$\sigma_{22}^{*(22)} = -\frac{2ex_1}{5(1+\tilde{\nu})^2}(5\tilde{\nu}x_2^4+10x_1^2x_2^2-\tilde{\nu}x_1^4-2x_1^4), \quad (128)$$

$$u_1^{*(23)} = -\frac{1}{15}(x_2^6-15x_1^4x_2^2+2x_1^6), \quad (129)$$

$$u_2^{*(23)} = -\frac{2x_1x_2}{15(1+\tilde{\nu})}(3\tilde{\nu}x_2^4-3x_2^4+20x_1^2x_2^2-9x_1^4), \quad (130)$$

$$\sigma_{11}^{*(23)} = \frac{2ex_1}{5(1+\tilde{\nu})^2}(5\tilde{\nu}x_2^4+10x_1^2x_2^2-\tilde{\nu}x_1^4-2x_1^4), \quad (131)$$

$$\sigma_{12}^{*(23)} = -\frac{2ex_2}{5(1+\tilde{\nu})^2}(\tilde{\nu}x_2^4+10x_1^2x_2^2-5\tilde{\nu}x_1^4-10x_1^4), \quad (132)$$

$$\sigma_{22}^{*(23)} = \frac{2ex_1}{5(1+\tilde{\nu})^2}(5x_2^4-10\tilde{\nu}x_1^2x_2^2-20x_1^2x_2^2+2\tilde{\nu}x_1^4+3x_1^4), \quad (133)$$

$$u_1^{*(24)} = \frac{2x_1x_2}{15(1+\tilde{\nu})}(\tilde{\nu}x_2^4+9x_2^4-20x_1^2x_2^2-3\tilde{\nu}x_1^4+3x_1^4), \quad (134)$$

$$u_2^{*(24)} = -\frac{1}{15}(2x_2^6-15x_1^2x_2^4+x_1^6), \quad (135)$$

$$\sigma_{11}^{*(24)} = \frac{2ex_2}{5(1+\tilde{\nu})^2}(2\tilde{\nu}x_2^4+3x_2^4-10\tilde{\nu}x_1^2x_2^2-20x_1^2x_2^2+5x_1^4), \quad (136)$$

$$\sigma_{12}^{*(24)} = \frac{2ex_1}{5(1+\tilde{\nu})^2}(5\tilde{\nu}x_2^4+10x_2^4-10x_1^2x_2^2-\tilde{\nu}x_1^4), \quad (137)$$