

# BEM / FEM Comparison Studies for the Inelastic Dynamic Analysis of Thick Plates on Elastic Foundation

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**Abstract:** Boundary and Finite Element methodologies for the determination of the inelastic response of thick plates resting on Winkler-type elastic foundations are compared and critically discussed. For comparison reasons the domain/boundary element and the finite element methodology use isoparametric elements of the same accuracy level. After a discretization of the integral equations of motion in both methodologies an efficient step-by-step time integration algorithm is used to solve the resulting matrix equations. Comparison studies are shown for impacted elastoplastic thick plates with smooth boundaries and supported on different Winkler-type foundations. The numerical results reveal that boundary element method appears to be a better choice in modelling the plate-foundation interaction.

**keyword:** Boundary Element Method, Finite Element Method, Dynamic Analysis, Elastoplasticity, Thick Plates, Winkler type elastic foundation

## 1 Introduction

The interaction analysis between structural elements such as plates and elastic media is of great importance to several areas of engineering. In the context of civil engineering this kind of problems provides useful analogues for the study of the interaction between structural foundation and the supporting soil medium. Among the various numerical methods used for the solution of this kind of problems the finite element (FE) and the boundary element (BE) methods are proved to be the most effective. In general, the finite element method (FEM) is well-suited to problems with finite domains in which the material properties are inhomogeneous and where nonlinear behaviour exists (e.g. Bathe(1982), Zienkiewicz and Taylor(1991)).

The boundary element method (BEM) is being explored as a possible alternative to FEM for this class of problems and particular attention is being paid to the accuracy and efficiency of this approach, as explained in the book on plates and shells edited by Beskos(1991). There have been a lot of papers concerning the application of the boundary element method to thin plate bending problems resting on different types of foundations. More on the subject can be found in the recent review article of Proidakis and Beskos(1999a). The effects of shear deformations and rotatory inertia are taken into account

in the Reissner - Mindlin plate theory (Reissner(1945) and Mindlin(1951)). Several papers have been published describing the boundary element analysis of thick plates resting on different foundations models as it is evident in the recent book of Aliabadi(1998). However, all of these studies have been limited to static loading and has neglected the dynamic interaction between plate and foundation deformation system. Proidakis(1999a, b) developed for the first time a time domain BEM named as Domain/Boundary Element Method (D/BEM) to determine the dynamic response of elastoplastic thick plates on a Winkler-type elastic foundation. It employs the elastostatic fundamental solution of the problem and can be thought as an appropriate combination of the work of Proidakis and Beskos(1999b) on thick plate elasto-plasto-dynamics with the work of Bezine(1988) on the effect of the elastic foundation in plate elastostatics. However, the employment of the static fundamental solution requires in addition to the boundary discretization an interior discretization due to the presence of two kinds of domain integrals, namely inelastic and inertial ones. The employment of the Winkler type elastic foundation effect is achieved by considering this effect as an additional loading of the plate. Thus, in addition to inertial and inelastic domain integrals one has also to consider the domain integrals due to elastic foundation effect.

A very important question has to do with the accuracy and efficiency of the BEM in relation to the FEM for this class of problems. The objective of this paper is to present a comparison of these two methods as applied to several selected numerical examples. The boundary element formulation uses quadratic elements on the boundary and as well as in the interior domain of the plate. The finite element formulation, on the other hand, uses Mindlin type quadratic isoparametric elements with 3 degrees of freedom per node. Both of these methods use a Prandtl-Reuss stress strain law based on Von-Mises' yield condition to describe the hardening elastoplastic material behaviour. The resultant algebraic system is solved by the step-by-step time integration algorithm of the central predictor method. The present FEM program used to generate the results in this paper has been constructed by the author in order to avoid programming bias and make the comparison as meaningful as possible.

In the following sections both the BEM and FEM formulations are briefly described in conjunction with their solution strategies. Numerical results, obtained by these two methods, are

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compared on the basis of certain representative dynamic thick plate bending problems with smooth plate boundaries.

## 2 Boundary Element Formulation

The following developments are based on the works by Proviidakis(1999a, b) and Proviidakis and Beskos(1999b). Consider an elastoplastic plate of constant thickness  $h$  occupying a two dimensional domain  $S$  bounded by a smooth boundary  $\Gamma$ , resting on a Winkler-type foundation and undergoing a lateral motion response. The plane  $x$ - $y$  is assumed to coincide with the mean surface of the plate. Following Reissner-Mindlin's plate theory, the equations of dynamic equilibrium of an elastic plate in lateral motion (Leissa(1969)) can be reformulated in incremental form to include bending plastic strain increments as

$$\begin{aligned} \frac{\partial \delta M_x}{\partial x} + \frac{\partial \delta M_{xy}}{\partial y} - \delta Q_x - \frac{\rho h^3}{12} \delta \ddot{\phi}_x &= 0 \\ \frac{\partial \delta M_{xy}}{\partial x} + \frac{\partial \delta M_y}{\partial y} - \delta Q_y - \frac{\rho h^3}{12} \delta \ddot{\phi}_y &= 0 \\ \frac{\partial \delta Q_x}{\partial x} + \frac{\partial \delta Q_y}{\partial y} - \delta \bar{q} - \rho h \delta \ddot{w} &= 0 \end{aligned} \quad (1)$$

where  $\rho$ ,  $h$  and  $\bar{q}$  are the mass density per unit area, the plate thickness and the transient dynamic loading per unit area, respectively. In addition,  $\delta \ddot{\phi}_x$ ,  $\delta \ddot{\phi}_y$  and  $\delta \ddot{w}$  indicate increments of the accelerations of the two slopes  $\phi_x$ ,  $\phi_y$  and of the lateral deflection  $w$ , respectively,  $\delta M_x$ ,  $\delta M_y$  and  $\delta M_{xy}$  represent increments of the bending and twisting moments,  $\delta Q_x$  and  $\delta Q_y$  represent increments of the shear forces and overdots denote time differentiation.

In the case of a plate resting on a Winkler-type foundation, the transient dynamic loading  $\bar{q}$  is given by

$$\bar{q} = -kw + q \quad (2)$$

where  $k$  is the foundation rigidity and  $q$  the load applied on the plate. Consequently, the equations of dynamic equilibrium (1) in their incremental form are given by

$$\begin{aligned} \frac{\partial \delta M_x}{\partial x} + \frac{\partial \delta M_{xy}}{\partial y} - \delta Q_x - \frac{\rho h^3}{12} \delta \ddot{\phi}_x &= 0 \\ \frac{\partial \delta M_{xy}}{\partial x} + \frac{\partial \delta M_y}{\partial y} - \delta Q_y - \frac{\rho h^3}{12} \delta \ddot{\phi}_y &= 0 \\ \frac{\partial \delta Q_x}{\partial x} + \frac{\partial \delta Q_y}{\partial y} - \delta q - k \delta w - \rho h \delta \ddot{w} &= 0 \end{aligned} \quad (3)$$

The increments of the total bending and shear strains can be given as

$$\begin{aligned} \delta \epsilon_x &= \delta \epsilon_x^e + \delta \epsilon_x^p & \delta \psi_x &= \delta \psi_x^e \\ \delta \epsilon_y &= \delta \epsilon_y^e + \delta \epsilon_y^p & \delta \psi_y &= \delta \psi_y^e \\ \delta \epsilon_{xy} &= \delta \epsilon_{xy}^e + \delta \epsilon_{xy}^p \end{aligned} \quad (4)$$

where  $\epsilon_x$ ,  $\epsilon_y$ , and  $\epsilon_{xy}$  represent bending strains,  $\psi_x$  and  $\psi_y$  represent shear strains and the superscripts  $e$  and  $p$  indicate the elastic and plastic part of the strains, respectively. Within the small strain theory the increments of the total bending and shear strains can also be expressed in terms of the increments of the generalized displacements as

$$\begin{aligned} \delta \epsilon_x &= \frac{\partial \delta \phi_x}{\partial x}, \quad \delta \psi_x = \delta \phi_x + \frac{\partial \delta w}{\partial x} \\ \delta \epsilon_y &= \frac{\partial \delta \phi_y}{\partial y}, \quad \delta \psi_y = \delta \phi_y + \frac{\partial \delta w}{\partial y} \\ \delta \epsilon_{xy} &= \frac{1}{2} \left( \frac{\partial \delta \phi_x}{\partial y} + \frac{\partial \delta \phi_y}{\partial x} \right) \end{aligned} \quad (5)$$

Following the initial plastic moment procedure of Karam and Telles(1992), the increments of the bending moments and shear forces can be expressed as

$$\begin{aligned} \delta M_x &= D \left[ \frac{\partial \delta \phi_x}{\partial x} + \nu \frac{\partial \delta \phi_y}{\partial y} \right] + \frac{\nu \delta q}{(1-\nu^2)\lambda^2} - \delta M_x^p \\ \delta Q_x &= \frac{D(1-\nu)\lambda^2}{2} \left( \delta \phi_x + \frac{\partial \delta w}{\partial x} \right) \\ \delta M_y &= D \left[ \frac{\partial \delta \phi_y}{\partial y} + \nu \frac{\partial \delta \phi_x}{\partial x} \right] + \frac{\nu \delta q}{(1-\nu^2)\lambda^2} - \delta M_y^p \\ \delta Q_y &= \frac{D(1-\nu)\lambda^2}{2} \left( \delta \phi_y + \frac{\partial \delta w}{\partial y} \right) \\ \delta M_{xy} &= \frac{D(1-\nu)}{2} \left[ \frac{\partial \delta \phi_x}{\partial y} + \frac{\partial \delta \phi_y}{\partial x} \right] - \delta M_{xy}^p \end{aligned} \quad (6)$$

where  $\delta M_x^p$ ,  $\delta M_y^p$  and  $\delta M_{xy}^p$  are the increments of the plastic moments which can be defined in initial stress form by the expressions

$$\begin{aligned} \delta M_x^p &= D \left[ \frac{\partial \delta \phi_x^p}{\partial x} + \nu \frac{\partial \delta \phi_y^p}{\partial y} \right] \\ \delta M_y^p &= D \left[ \frac{\partial \delta \phi_y^p}{\partial y} + \nu \frac{\partial \delta \phi_x^p}{\partial x} \right] \\ \delta M_{xy}^p &= \frac{D(1-\nu)}{2} \left[ \frac{\partial \delta \phi_x^p}{\partial y} + \frac{\partial \delta \phi_y^p}{\partial x} \right] \end{aligned} \quad (7)$$

In the above,  $D = Eh^3/12(1-\nu^2)$  is the plate flexural rigidity with  $E$  and  $\nu$  being the elastic modulus and Poisson's ratio, respectively and  $\lambda^2 = 10/h^2$  is the shear correction factor of Reissner's theory. The shear correction factor  $\kappa^2$  of Mindlin's theory is usually taken as  $5/6$  in order for the two theories to coincide provided that  $\lambda^2 = 12\kappa^2/h^2$ . Substituting equations (6) into (1) one can obtain the following incremental equations of motion in terms of the increments of the generalized

displacements and plastic moments

$$\begin{aligned} & \frac{D}{2} \left[ (1-\nu)\nabla^2\delta\phi_x + (1+\nu)\frac{\partial}{\partial x} \left( \frac{\partial\delta\phi_x}{\partial x} \frac{\partial\delta\phi_y}{\partial y} \right) \right] - \\ & \frac{D(1-\nu)\lambda^2}{2} \left( \delta\phi_x - \frac{\partial\delta w}{\partial x} \right) = \frac{\rho h^3}{12} \delta\ddot{\phi}_x + \\ & \frac{\nu}{(1-\nu)\lambda^2} \frac{\partial\delta q}{\partial x} - \frac{\partial M_x^p}{\partial x} - \frac{\partial M_{xy}^p}{\partial y} \\ \\ & \frac{D}{2} \left[ (1-\nu)\nabla^2\delta\phi_y + (1+\nu)\frac{\partial}{\partial x} \left( \frac{\partial\delta\phi_x}{\partial x} \frac{\partial\delta\phi_y}{\partial y} \right) \right] - \\ & \frac{D(1-\nu)\lambda^2}{2} \left( \delta\phi_y - \frac{\partial\delta w}{\partial y} \right) = \frac{\rho h^3}{12} \delta\ddot{\phi}_y + \\ & \frac{\nu}{(1-\nu)\lambda^2} \frac{\partial\delta q}{\partial y} - \frac{\partial M_{xy}^p}{\partial x} - \frac{\partial M_y^p}{\partial y} \\ & \frac{D(1-\nu)\lambda^2}{2} \left( \nabla^2\delta w - \frac{\partial\delta\phi_x}{\partial x} - \frac{\partial\delta\phi_y}{\partial y} \right) = \rho h\delta\ddot{w} - \delta q + k\delta w \quad (8) \end{aligned}$$

The following boundary conditions can be defined on the boundary  $\Gamma = \Gamma_1 + \Gamma_2$ .

$$u_k = \bar{u}_k \quad \text{at } \Gamma_1, \quad p_k = \bar{p}_k \quad \text{at } \Gamma_2 \quad (9)$$

where overbars denote prescribed values  $u_k = \{\phi_x, \phi_y, w\}$  and  $p_k$  is the generalized traction defined as  $p_\alpha = M_{\alpha\beta}n_\beta$  and  $p_3 = Q_\beta n_\beta$  with  $\alpha, \beta = x$  or  $y$  and  $n_\beta$  being the outward normal vector at the boundary  $\Gamma$ . Initial conditions have the form

$$\begin{aligned} u_k(x, y, t=0) &= \bar{u}_k(x, y) \\ \dot{u}_k(x, y, t=0) &= \bar{\dot{u}}_k(x, y) \end{aligned} \quad (10)$$

The static-like form of eqs (8) with the inertial terms being in their right hand sides, suggests using known integral identities as described in Karam and Telles(1988) for the elasto-plasto-static Reissner plate problem. By using the elastostatic fundamental solution of the problem, one can obtain for a point  $\underline{\xi}$  inside the domain  $S$  of the plate the integral equations

$$\begin{aligned} C_{ij}(\underline{\xi})\delta u_j(\underline{\xi}) &= \int_{\Gamma} \left\{ u_{ij}^*(\underline{\xi}, \underline{X})\delta p_j(\underline{X}) - \right. \\ & \left. p_{ij}^*(\underline{\xi}, \underline{X})\delta u_j(\underline{X}) \right\} \delta\Gamma(\underline{X}) + \\ & \int_S \left[ u_{i3}^*(\underline{\xi}, \underline{x}) - \frac{\nu}{(1-\nu)\lambda^2} u_{i\alpha\alpha}^*(\underline{\xi}, \underline{x}) \right] \delta q(\underline{x}) dS(\underline{x}) + \\ & \frac{\rho h^3}{12} \int_S u_{i\alpha}^*(\underline{\xi}, \underline{x}) \delta \ddot{u}_\alpha(\underline{x}) dS(\underline{x}) - \rho h \int_S u_{i3}^*(\underline{\xi}, \underline{x}) \delta \ddot{u}_3(\underline{x}) dS(\underline{x}) + \\ & \int_S E_{\alpha\beta i}^*(\underline{\xi}, \underline{x}) \delta M_{\alpha\beta}^p dS(\underline{x}) - k \int_S u_{i3}^*(\underline{\xi}, \underline{x}) \delta w(\underline{x}) dS(\underline{x}) \quad (11) \end{aligned}$$

where  $i, j = 1, 2, 3$ ,  $\alpha, \beta = 1, 2$  and  $\underline{X}$  or  $\underline{x}$  represent the field point at the boundary or in the interior of the plate, respectively. The matrix  $C_{ij}$  depends only upon the geometry of the boundary at point  $\underline{\xi}$  and equals  $\delta_{ij}$  for internal points  $\underline{\xi}$  and

$\delta_{ij}/2$  for boundary points  $\underline{\xi}$  at smooth boundaries. The tensors  $u_{ij}^*$ ,  $p_{ij}^*$ , and  $E_{\alpha\beta i}^*$  represent the fundamental solution at the field point  $\underline{x}$  of an infinite plate when a unit couple (for  $i = 1$  and  $2$ ) or a unit force (for  $i = 3$ ) is applied at the source point  $\underline{\xi}$ . Thus the generalized displacements, the corresponding surface tractions, the expressions for  $u_{i\alpha\alpha}^*$  and the one for  $E_{\alpha\beta i}^*$  are given explicitly in Providakis and Beskos(1999b).

### 3 Boundary element discretization

Integral equations (11) can be expressed in discrete form by dividing the boundary  $\Gamma$  and the interior of the plate domain  $\Omega$  into a number of three noded quadratic boundary elements and eight noded quadrilateral interior elements, respectively. Thus the discretized version of the boundary integral equations (11) and after using the boundary conditions and eliminating the boundary unknown variables yields in the following matrix form

$$\begin{aligned} ([I] - [\tilde{K}^*]) \delta \{U\} + [\tilde{M}^*] \delta \{\dot{U}\} &= [\tilde{A}^*] \delta \{Y\} + \\ [\tilde{S}^*] \delta \{Q\} + [\tilde{E}^*] \delta \{M^p\} \end{aligned} \quad (12)$$

where  $[I]$  is the identity matrix and

$$\begin{aligned} [\tilde{A}^*] &= [A^*] - [B^*][B^*]^{-1}[A^*] \\ [\tilde{M}^*] &= [M^*] + [B^*][B^*]^{-1}[M^*] \\ [\tilde{S}^*] &= [S^*] + [B^*][B^*]^{-1}[S^*] \\ [\tilde{E}^*] &= [E^*] + [B^*][B^*]^{-1}[E^*] \\ [\tilde{K}^*] &= [K^*] + [B^*][B^*]^{-1}[K^*] \end{aligned} \quad (13)$$

In the above  $\delta \{Y\}$  is the vector of the known increments of the nodal boundary values,  $\delta \{U\}$  and  $\delta \{\dot{U}\}$  are the vectors of the unknown increments of the nodal generalized displacements and accelerations, respectively,  $\delta \{Q\}$  is the vector of the known increments of the nodal load values and  $\delta \{M^p\}$  is the vector of the plastic moment terms. The influence matrix  $[\tilde{A}^*]$  can be considered as the sum of certain element matrices that describe the influence of the boundary element layers on the collocation point. All the boundary integrals in (12) are singular (due to the fundamental kernel singularities as  $r \rightarrow 0$ ), they must be understood in the sense of a Cauchy principal value and can be evaluated according to the procedure presented in Providakis(1999b). The matrices  $[A^*]$  and  $[B^*]$  are boundary element integral matrices, while  $[M^*]$ ,  $[E^*]$  and  $[K^*]$  are domain element integral matrices related to the inertial, plastic moment and foundation effect terms.

### 4 Finite element formulation

The following developments are based on the work of Hinton, Owen and Shantaram (1977). The dynamic equilibrium of a plate in motion and supporting on elastic foundation and in the absence of body forces, can be expressed by the following

principle of virtual work at time station  $t_n$ ,

$$\int_S [\delta \epsilon_n]^t \{ \sigma_n \} dS - \int_S [\delta u_n]^t [k_{fn} - \rho_n u_{n,t}] dS - \int_{\Gamma_1} [\delta u_n]^t \delta t_n d\Gamma = 0 \quad (14)$$

irrespective of material behaviour, where  $\delta u_n$  is the vector of virtual displacements,  $\delta \epsilon_n$  the vector of associated virtual strains,  $k_{fn}$  the vector of foundation effect considered as an additional loading with  $k_f$  being the foundation rigidity,  $\delta t_n$  the vector of boundary tractions,  $\sigma_n$  the vector of stresses,  $\rho_n$  the mass density and the subscripts  $(,t)$  denote differentiation with respect to time. The boundary  $\Gamma_i$  is the part of  $\Gamma$  on which boundary tractions  $t_n$  are specified. The discretization process in this formulation is carried out by dividing the entire body  $S$  into a discrete set of structural elements. The resulting dynamic equilibrium equation (14) at the node  $i$  can be written as

$$\{ p_i \}_n - \{ f_i^p \}_n - \{ f_i^t \}_n - \{ f_i^f \}_n - \{ f_i^t \}_n = 0 \quad (15)$$

where  $\{ p_i \}_n$  is the vector of nodal internal resisting or restoring forces,  $\{ f_i^f \}_n$  the vector of consistent nodal forces due to the foundation effect,  $\{ f_i^t \}_n$  the vector of nodal inertial forces, and  $\{ f_i^t \}_n$  the vector of consistent nodal forces associated with the boundary actions.

Thus equation (15), reads

$$[M] \{ \ddot{d}_n \} + \{ F(d_n) \} = \{ F_n \} \quad (16)$$

where  $[M]$  is the mass matrix,  $\{ F_n \}$  the vector of external nodal actions and  $\{ \ddot{d}_n \}$ ,  $\{ d_n \}$  are, respectively, the vectors of the increments of the nodal accelerations and displacements, respectively. The internal resisting forces can be expressed by the use of the principle of virtual work as

$$\{ F(d_n) \} = \sum_{j=1}^{n_e} \int_{S_e} [\delta \epsilon]^t \{ \sigma^e \} dS \quad (17)$$

in which  $n_e$  is the number of elements. The internal resisting forces can be written as

$$\{ F(d_n) \} = [K] \{ d_n \} \quad (18)$$

where  $[K]$  is the global stiffness matrix assembled from individual element contributions. It is well known that the application of the finite element procedure to solving plate problems, under the assumptions of the so called “thin plate” theory, introduces considerable difficulties due to the requirements of slope continuity between adjacent elements. In the present work, a finite element approach based on the assumptions proposed in Mindlin (1951) is adopted to avoid the continuity requirements, which have made the solution of thin plates so difficult. The displacement field  $\{ d_n \}$  can thus be uniquely

specified by an independent variation of the increments of the lateral deflection  $\delta w$  and of the two angles  $\delta \theta_x$ ,  $\delta \theta_y$  defining the direction of the line originally normal to the midsurface of the plate, as described in the work of Hinton, Owen and Shantaram(1977).

### 5 Finite element discretization

In the 8-noded isoparametric plate bending finite element used here, the geometry is defined by the expression

$$\begin{bmatrix} x \\ y \end{bmatrix} = \sum_{i=1}^8 N_i \begin{bmatrix} x_i \\ y_i \end{bmatrix} \quad (19)$$

where  $N_i$  is the shape function associated with node  $i$ . The displacement variation over any element is defined, in terms of the increments of the nodal displacement components, as

$$\begin{Bmatrix} \delta w \\ \delta \theta_x \\ \delta \theta_y \end{Bmatrix} = \sum_{i=1}^8 N_i d_i \quad (20)$$

where  $d_i = [ \delta w_i \quad \delta \theta_x \quad \delta \theta_y ]^T$  is the vector of increments of displacements at node  $i$ . For small displacement analysis of plate deformation the strain-displacement relation, in global coordinates, is given by

$$\begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{xy} \end{Bmatrix} = -z \begin{Bmatrix} \frac{\partial \delta \theta_x}{\partial x} \\ \frac{\partial \delta \theta_y}{\partial y} \\ \frac{\partial \delta \theta_x}{\partial y} + \frac{\partial \delta \theta_y}{\partial x} \end{Bmatrix} = -z \delta \{ \chi \} \quad (21)$$

where  $\delta \{ \chi \}$  is the vector of the increments of the partial derivatives of the lateral deflections which can be expressed in matrix form as

$$\delta \{ \chi \} = \sum_{i=1}^8 [B_i] \{ d_i \} \quad (22)$$

where

$$[B_i] = \begin{bmatrix} 0 & -\frac{\partial N_i}{\partial x} & 0 \\ 0 & 0 & -\frac{\partial N_i}{\partial y} \\ 0 & -\frac{\partial N_i}{\partial y} & -\frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial x} & -N_i & 0 \\ \frac{\partial N_i}{\partial y} & 0 & -N_i \end{bmatrix}$$

### 6 Constitutive relations

The elastoplastic formulation given by Zienkiewicz and Taylor(1991) is adopted here. After some manipulations established in Providakis and Beskos(1999b) and Providakis(1999b), the plastic strains are given by the matrix equation

$$\delta \{ \epsilon \}^p = [D]^* \delta \{ \epsilon \} \quad (23)$$

where  $[D]^* = [I] - [D]^{e-1} [D]^{ep}$  with  $[I]$  being the identity matrix,  $[D]^e$  elasticity matrix and

$$[D]^{ep} = [D]^e - [D]^e \left\{ \frac{\partial F}{\partial \{\sigma\}} \right\} \left\{ \frac{\partial F}{\partial \{\sigma\}} \right\} [D]^e \times \left( H_I + \left\{ \frac{\partial F}{\partial \{\sigma\}} \right\}^T [D]^e \left\{ \frac{\partial F}{\partial \{\sigma\}} \right\} \right)^{-1} \quad (24)$$

In the above  $H_I$  is the slope of the uniaxial effective stress versus plastic strain curve,  $\{\sigma\} = \{\sigma_{xx}, \sigma_{yy}, \sigma_{xy}\}^T$  is the stress vector and  $F$  is the yield surface (Von Mises in the present case) given by the equation

$$F = [\sigma_{xx}^2 + \sigma_{yy}^2 - \sigma_{xx}\sigma_{yy} + 3\sigma_{xy}^2]^{1/2} - \bar{\sigma} \quad (25)$$

where  $\bar{\sigma}$  is the uniaxial effective stress and  $\sigma_{ij}$  ( $i, j = x, y$ ) the components of the stress tensor.

## 7 Time integration algorithms

### 7.1 Boundary element method

The values of the nodal generalized displacements at every time station are obtained by integrating forward in time through the use of an explicit central difference predictor scheme. The initial distribution of generalized displacements, velocities and accelerations are prescribed and set to zero. The generalized displacements can thus be determined at the end of the first time step. These are now used in a discretized version of an equation analogous to (21) and the incremental strain can be obtained through the plate. The increments of the stresses are then obtained from the strain increments and the incremental plastic moments calculations follow from equations (7) after an appropriate checking at yielding. Thus the total and incremental generalized displacements are then found at time  $\Delta t$  and so on, and the time histories of all the variables are obtained. For more details one can consult Providakis (1999a, b).

### 7.2 Finite element method

The initial values of displacements, velocities and accelerations are used for the calculations of the displacement at the end of the first time step. Next, the incremental strain and stress are obtained at every node from the use of equations (21), (22) and the usual stress/strain relations, respectively. These are now used in the discretized version of equation (23) to compute the plastic strain increments. The plastic contribution to the internal resisting forces vector can now be computed and thus the incremental acceleration and the total acceleration are calculated from the solution of equation (16). These increments are used to find the values of the variables at a new time  $\Delta t$ , and so on, and in this way the time histories of the relevant variables are obtained. For more details one can consult Hinton, Owen and Shantaram (1977).

## 8 Numerical examples - comparisons

To carry out the comparisons between D/BEM and FEM, a series of numerical examples involving elastic and elastoplastic dynamic analyses of plates subjected to impulsive loads are presented and discussed.

### 8.1 Example 1

Consider a uniformly loaded clamped circular elastoplastic plate resting on a Winkler-type elastic foundation. The following geometric and material parameters are assumed: radius  $R = 1.0 \text{ m}$ , modulus of elasticity  $E = 2.06 \times 10^{11} \text{ N/m}^2$ , Poisson's ratio  $\nu = 0.3$ , uniaxial effective stress  $\bar{\sigma} = 0.025D/h^3$ . The plate is subjected to a rectangular impulse  $q(r, t) = q_0(H(t) - H(t - t_1))$  of intensity  $q_0 = 1.3D/R^3$  and time duration  $t_1 = \frac{t_0}{2} = \frac{R^2}{2} \sqrt{\frac{\rho h}{D}}$ . The foundation rigidity is given

again in dimensionless form by  $\alpha = R/\sqrt{D/k^4}$ . The history of the central elastic and elastoplastic dynamic deflection of this thin plate resting on a Winkler-type foundation with different rigidities is shown in Figure 1 obtained as a special case of the proposed thick plate D/BEM in conjunction with 16 boundary and 20 interior elements per quadrant. In the same figure the results of the FEM formulation are also depicted. The finite element mesh consists of two different interior discretization with 20 and 40 interior elements, respectively. The same problem has been analyzed by Fotiu, Irschik and Ziegler (1994) using a special thin plate D/BEM in conjunction with modal analysis, characterized by high accuracy but limited generality. The results of the present D/BEM are in very good agreement with those of Fotiu, Irschik and Ziegler (1994), where use is made of a mesh consisting of 100 elements along the radius and 10 elements over half the thickness. The FEM needs more elements to give results coincident with Fotiu's.

### 8.2 Example 2

Consider a simply supported circular thick plate of  $R = 1.0 \text{ m}$  and  $h = 0.15 \text{ m}$  subjected to a suddenly applied uniformly distributed load of intensity  $100 \text{ N/m}^2$  resting on a Winkler-type foundation with foundation parameter equals to  $\alpha = 0$  and  $\alpha = 2$ . The plate material is assumed to be elastoplastic with strain hardening and material constants  $E = 2.06 \times 10^{11} \text{ N/m}^2$ ,  $\nu = 0.3$ ,  $\bar{\sigma} = 488000 \text{ N/m}^2$ ,  $\rho = 76900 \text{ Nsec}^2/\text{m}^4$  and  $H = 0.6D$ , where  $D = Eh^3/12(1 - \nu^2)$ . The central elastoplastic deflection history of the plate is portrayed in Figure 2 as obtained by the present D/BEM in conjunction with 16 boundary and 32 interior elements and  $\Delta t = 1 \times 10^{-6}$  secs. In figure 2 is also shown the central deflection history of the same plate as obtained for dimensionless foundation parameter  $\alpha = 2$  by using the Finite Element Method with 40 Mindlin finite elements and  $\Delta t = 1 \times 10^{-6}$  secs. The D/BEM results are again close to the FEM ones. However, the D/BEM uses a smaller number of equations for the final algebraic matrix system.

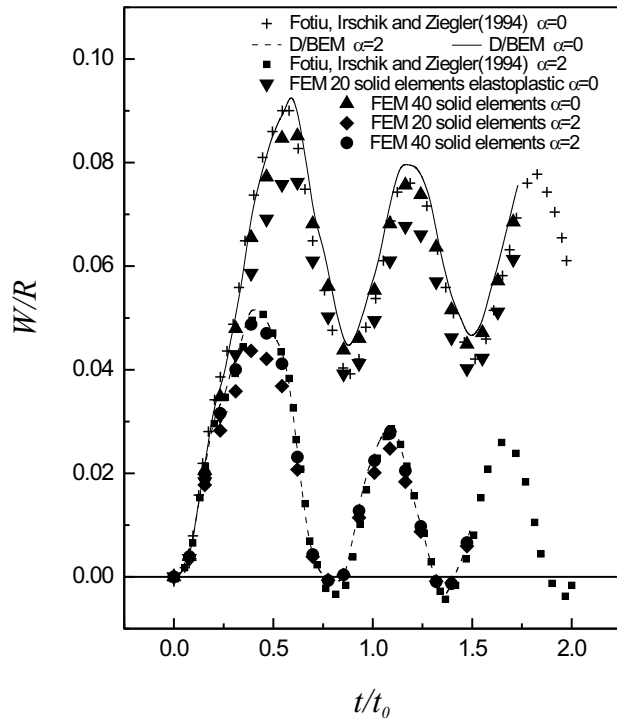


Figure 1 : Central deflection history of a clamped thin circular plate resting on different Winkler-type foundation

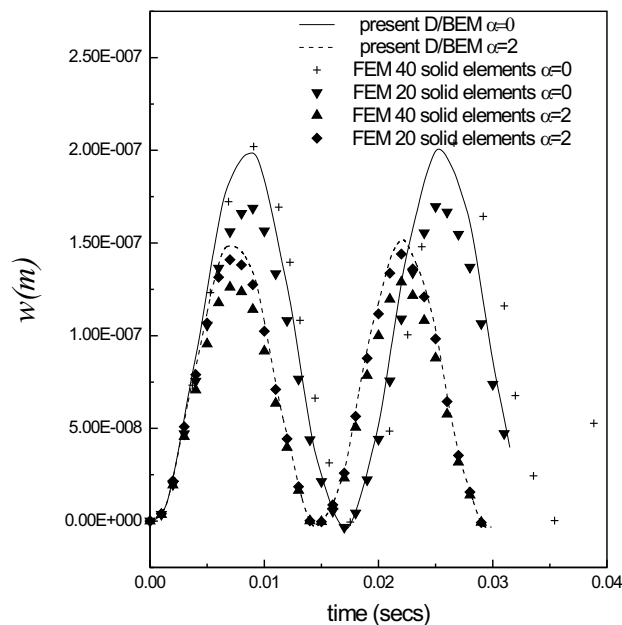


Figure 2 : Central deflection history for a simply supported thick plate resting on different Winkler-type foundations

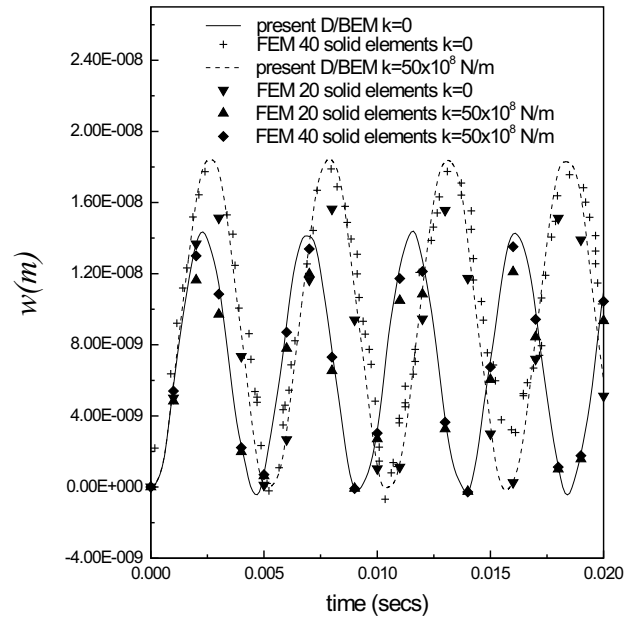


Figure 3 : Central deflection history for a simply supported thick elliptical plate resting on different Winkler-type foundation

### 8.3 Example 3

Consider a simply supported thick plate with elliptical boundary geometry, subjected to a suddenly applied load uniformly distributed over the whole plate with intensity  $100 \text{ N/m}^2$  and resting on a Winkler-type foundation with foundation rigidities  $k = 0$  and  $k = 50. \times 10^8 \text{ N/m}$ . The two semi-axes  $a$  and  $b$  of the elliptical shape are equal to  $0.5 \text{ m}$  and  $0.6 \text{ m}$ , respectively and the plate thickness  $h = 0.15 \text{ m}$ . The material parameters are  $E = 2.06 \times 10^{11} \text{ N/m}^2$ ,  $\nu = 0.3$ ,  $\bar{\sigma} = 488000 \text{ N/m}^2$ ,  $\rho = 76900 \text{ Nsec}^2/\text{m}^4$  and  $H = 0.6D$ . Figure 3 depicts the central elasto-plastic deflection history of the elliptical plate as obtained by the present D/BEM in conjunction with 16 boundary and 32 interior elements per quadrant and  $\Delta t = 1 \times 10^{-6} \text{ secs}$ . The central deflection of the same plate is also obtained by using the FEM with foundation rigidity  $k = 50. \times 10^8 \text{ N/m}$  and a mesh consisting of 40 Mindlin finite elements. The agreement between the two results appears to be close but D/BEM has a final matrix system with fewer equations compared to the FEM.

### 8.4 Computing time comparisons

The CPU time on a PC required for the solution of the above numerical results are given in Table 1. One can see there, that for this kind of problem, the D/BEM program requires less computing time as compared to FEM. This can be expected since the number of d.o.f. in the resulting algebraic system is less as compared to FEM. Besides, the topology and the num-

**Table 1** : D/BEM vs FEM computing time comparisons

	D/BEM (CPU in secs)				FEM (CPU in secs)			
	BN	IN	DF	CPU	N	E	DF	CPU
A	33	43	129	140	79	20	237	152
B	33	43	129	140	147	40	541	184
C	33	43	129	144	79	20	237	153
D	33	43	129	144	147	40	541	185
E	33	121	363	165	79	20	237	155
F	33	121	363	165	147	40	541	195
G	33	121	363	165	79	20	237	158
H	33	121	363	165	147	40	541	194
I	33	121	363	165	79	20	237	156
J	33	121	363	165	147	40	541	198
K	33	121	363	165	79	20	237	157
L	33	121	363	165	147	40	541	195

A and B: Clamped circular thin plate with  $\alpha = 0$

C and D: Clamped circular thin plate with  $\alpha = 2$

E and F: Simply supported circular thick plate,  $\alpha = 0$

G and H: Simply supported circular thick plate,  $\alpha = 2$

I and J: Simply supported elliptical thick plate,  $k=0$

K and L: Simply supported elliptical thick plate,  $k=50$

bering scheme for D/BEM can be arbitrary and consequently, it is expected that the total time required for a FEM program run become even greater as the geometrical configuration of the problem increases.

## 9 Conclusions

A D/BEM and a FEM are presented for the dynamic elastoplastic analysis of thick plates with smooth boundaries and supporting on Winkler-type elastic foundation. The D/BEM is not actually a pure boundary element method since it also requires domain discretization. However, the method still retains most of the advantages over the FEM. On the basis of the preceding developments the following conclusions can be drawn :

1. The number of unknowns in the resulting algebraic system is proportional to the number of either the boundary or the interior nodes in D/BEM as opposed to the total number of boundary and interior nodes in FEM.
2. The topology of the internal elements is much simpler than for the FEM and the numbering of internal nodes can be arbitrary without loss of computational efficiency.
3. With the D/BEM program very good results were obtained using a coarse element discretization. This fact allowed considerable savings in solution time with respect to the FEM program.

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