A Boundary-only Solution to Dynamic Analysis of Non-homogeneous Elastic Membranes

J.T. Katsikadelis,[1] **M.S. Nerantzaki**[1]

Abstract: A boundary-only method is presented for the solution of the vibration problem of non-homogeneous membranes. Both free and forced vibrations are considered. The presented method is based on the Analog Equation Method (AEM). According to this method the second order partial differential equation with variable coefficients of hyperbolic type, which governs the dynamic response of the membrane, is substituted by a Poisson's equation describing a quasi-static problem for the homogeneous membrane subjected to a fictitious time dependent load. The fictitious load is established using BEM. Several numerical examples are presented which illustrate the efficiency and the accuracy of the method.

keyword: elasticity, membrane, non-homogeneous, BEM, analog equation

1 Introduction

The vibrations of membranes have been considered by many investigators for over a century. Starting with Rayleigh (1945) and Morse (1948) there exists an extensive literature on the subject. With few exceptions, e.g. De (1971); Sato (1980) the existing analytical solutions are restricted to homogeneous membranes having simple geometry such as circular, rectangular, elliptical or annular, for which the method of separation of variables can be applied. For arbitrary shaped membranes, however, solution to the problem is feasible only by numerical methods e.g. FDM and FEM. The BEM has also been employed to non-homogeneous membranes under uniform stress. In the latter case the problem has been solved using D/BEM [Katsikadelis and Sapountzakis (1988)].

The standard BEM formulation can not be employed since it is not possible to establish the fundamental solution either in the time domain or in the transformed one. The dual reciprocity method (DRM) could be employed as a boundary-only BEM [Patridge, Brebbia and Wrobel (1992)]. However, this method is subject to the limitation that for a non-standard governing differential operator a standard dominant operator can be extracted, whose fundamental solution is known or can be established. Except for special cases, this is not feasible for the second order partial differential equation with variable coefficients. The solution procedure using DRM becomes even more complicated, if the coefficients are not given by analytical expressions but pointwise, e.g. when the prestress of the membrane is established from the numerical solution of the plane stress problem.

In this paper a boundary-only BEM is developed for solving initial-boundary value problems for systems governed by the second order partial differential equation with variable coefficients of hyperbolic type as it is the case of non-homogeneous membranes. The shape of the membrane is arbitrary. The inhomogenuity may be due to non-homogeneous material, to variable thickness or to non-uniform stretching.

The standard BEM formulation can not be employed since it is not possible to establish the fundamental solution either in the time domain or in the transformed one.

The presented method is based on the concept of the analog equation [Katsikadelis (1994)]. According to this method the governing equation is replaced by a Poisson's equation with a fictitious time dependent source. The fictitious source is established as function of time using BEM. Subsequently, the solution at any instant is obtained from the integral representation of the Poisson's equation. Several numerical examples are presented for both free and force vibrations, which illustrate the efficiency and accuracy of the proposed method.

2 Problem statement

Consider a thin flexible membrane consisting of non-homogeneous material having surface mass density $\rho(x,y)$ occupying the two-dimensional multiply connected domain Ω in the x,y plane bounded by the $K+1$ curves $\Gamma_0,\ \Gamma_1, \dots \Gamma_K$ (see Fig. 1). The membrane is prestressed and fixed or elastically supported on the boundary $\Gamma = \cup_{i=0}^{i=K} \Gamma_i$. Without restricting the generality, the prestress is imposed by the force $T(s)$ acting along the boundary and in the direction normal to it. Hence, the membrane is under plane stress, the components of which $T_x(x,y)$, $T_y(x,y)$ and $T_{xy}(x,y)$ are established from the solution of the respective plane elasticity problem, namely

$$T_{x,x} + T_{xy,y} = 0 \tag{1}$$

$$T_{xy,x} + T_{y,y} = 0 \qquad \text{in } \Omega \tag{2}$$

$$T_n = T(s), \ \ T_t = 0 \qquad \text{on } \Gamma \tag{3}$$

The plane elasticity problem (1)-(3) can be solved either by means of Airy's stress function or in terms of the inplane displacements by integrating the Navier equations [Katsikadelis (1999)]. It is assumed that the stretching forces T_1 and T_2 in

[1] NTUA, Athens, Greece

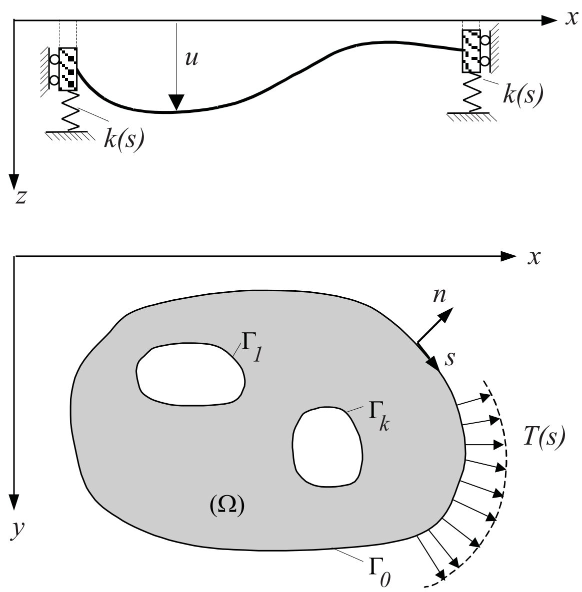


Figure 1 : Domain Ω occupied by the membrane

the principal directions at any point inside Ω are tensile, so that wrinkling of the membrane is avoided [Katsikadelis (2000)], i.e. it is

$$T_{1,2} = \frac{T_x + T_y}{2} \pm \sqrt{\left(\frac{T_x - T_y}{2}\right)^2 + T_{xy}^2} > 0 \tag{4}$$

The initially flat membrane is deflected to a surface $u(x,y,t)$ when subjected to a transverse load with intensity $g(x,y,t)$. Due to the lateral defection of the membrane, additional strains are produced in the middle surface. For linear deflection theory of membranes the following assumptions are made:

a. The prestressing of the membrane is large enough so that the forces T_x, T_y, T_{xy} remain unchanged during the deflection.

b. The additional strains of the middle surface of the membrane due to its elastic deformation are negligible as compared with those due to the slopes of the middle surface.

The second assumption implies that the strain components are given as

$$\varepsilon_x = \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 \tag{5}$$

$$\varepsilon_y = \frac{1}{2}\left(\frac{\partial u}{\partial y}\right)^2 \tag{6}$$

$$\gamma_{xy} = \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} \tag{7}$$

On the basis of the above assumptions the strain energy of the deflected membrane is written as

$$U = \frac{1}{2}\int_\Omega \left[T_x\left(\frac{\partial u}{\partial x}\right)^2 + 2T_{xy}\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + T_y\left(\frac{\partial u}{\partial y}\right)^2\right] d\Omega \tag{8}$$

The equation of motion of the membrane can be derived using the Hamilton's principle, which requires that the functional

$$I(u) = \int_{t_1}^{t_2}\int_\Omega \left[U - \frac{1}{2}\rho\left(\frac{\partial u}{\partial t}\right)^2 - gu\right] d\Omega\, dt + \int_{t_1}^{t_2}\int_\Gamma \left[\frac{1}{2}k(s)u^2 - \gamma(s,t)u\right] ds\, dt \tag{9}$$

with $\delta u = 0$ at instants t_1 and t_2, takes a stationary value, i.e. $\delta I = 0$. In Eq. 9 $k(s)$ is the stiffness modulus of the elastic support and $\gamma(s,t)$ is the density of the external line force along the boundary.

Using operations of the calculus of variations and integration by parts we readily obtain from Eq. 9 the following initial boundary value problem.

$$T_x u_{,xx} + 2T_{xy}u_{,xy} + T_y u_{,yy} - \rho\ddot{u} = -g \quad \text{in } \Omega \tag{10}$$

$$\beta_1 u + \beta_2 q = \beta_3 \quad \text{on } \Gamma \tag{11}$$

$$u(x,y,0) = f_1(x,y) \tag{12}$$

$$\dot{u}(x,y,0) = f_2(x,y) \quad \text{in } \Omega \tag{13}$$

where $\beta_1 = k(s)$, $\beta_2 = T(s)$, $\beta_3 = \gamma(s)$; $q_n = \partial u/\partial n$ is the derivative of u normal to the boundary, and f_1, f_2 are functions specified in Ω

3 The Analog equation method for non-homogeneous membranes

The initial boundary value problem (10)-(13) is solved using the concept of the analog equation [Katsikadelis (1994)], which is applied to the problem at hand as follows.

Let $u = u(x,y,t)$ be the sought solution to the problem (10)-(13). This function is two times continuously differentiable in Ω. Thus, if the Laplacian operator is applied to it, we have

$$\nabla^2 u = b(x,y,t) \tag{14}$$

Eq. 14 is a quasi-static equation and indicates that the solution of Eq. 10 at instant t could be established by solving this equation under the boundary conditions (11), if the fictitious time dependent source $b(x,y,t)$ were known.

The fictitious source can be established by working as follows.

Following the idea of Nardini and Brebbia (1982) to approximate time dependent field quantities by radial base functions series and a procedure similar to that presented by Katsikadelis and Nerantzaki (1998 and 1999) for the static problem, we assume

$$b = \sum_{j=1}^{M} \alpha_j f_j \tag{15}$$

where $f_j = f_j(x,y)$ is a set of approximation functions and $\alpha_j = \alpha_j(t)$ time dependent coefficients to be determined.

The solution of Eq. 14 at instant t can be written as a sum of the homogeneous solution $\overline{u} = \overline{u}(x,y,t)$ and a particular solution $u_p = u_p(x,y,t)$ of the non-homogeneous equation. Thus, we have

$$u = \overline{u} + u_p \tag{16}$$

The particular solution is obtained from

$$\nabla^2 u_p = \sum_{j=1}^{M} \alpha_j f_j \tag{17}$$

which yields

$$u_p = \sum_{j=1}^{M} \alpha_j \hat{u}_j \tag{18}$$

where $\hat{u}_j$ $(j = 1,2,...M)$ is a particular solution of the equation

$$\nabla^2 \hat{u}_j = f_j \qquad j = 1,2,...M \tag{19}$$

The particular solution of Eq. 19 can always be established, if f_j is specified.

The homogeneous solution $\overline{u}$ is obtained from the boundary value problem

$$\nabla^2 \overline{u} = 0 \qquad \text{in } \Omega \tag{20}$$

$$\beta_1 \overline{u} + \beta_2 \overline{q} = \beta_3 - \left(\beta_1 \sum_{j=1}^{M} \alpha_j \hat{u}_j + \beta_2 \sum_{j=1}^{M} \alpha_j \hat{q} \right) \qquad \text{on } \Gamma \tag{21}$$

where $\hat{q}_j = \partial \hat{u}_j / \partial n$.

The boundary value problem (20), (21) is solved using the BEM. Thus, the integral representation of the solution $\overline{u}$ is given as

$$c\overline{u}(P,t) = -\int_\Gamma (u^* \overline{q} - \overline{u} q^*) ds \qquad P\{x,y\} \in \Omega \cup \Gamma \tag{22}$$

in which

$$u^* = \frac{1}{2\pi} \ell n r \tag{23}$$

is the fundamental solution to Eq. 20 and

$$q^* = u^*_{,n} \tag{24}$$

its derivative normal to the boundary with

$$r = |Q - P| = \left[(\xi - x)^2 + (y - \eta)^2 \right]^{1/2} \tag{25}$$

being the distance between any two points $P(x,y)$ in $\Omega \cup \Gamma$, $Q(\xi,\eta)$ on Γ; c is a constant which takes the values $c = 1$ if $P \in \Omega$ and $c = \alpha/2\pi$ if $P \in \Gamma$; α is the interior angle between the tangents of boundary at point P. Note that $c = 1/2$ for points where the boundary is smooth.

On the basis of Eq. 16, 18 and 22 the solution of Eq. 14 is written as

$$cu = -\int_\Gamma (u^* \overline{q} - \overline{u} q^*) \; ds + \sum_{j=1}^{M} \alpha_j \hat{u}_j \tag{26}$$

Differentiating the above equation for $P \in \Gamma$ $(c = 1)$ yields

$$u_{,xx} = -\int_\Gamma (u^*_{,xx} \overline{q} - \overline{u} q^*_{,xx}) \; ds + \sum_{j=1}^{M} (\hat{u}_j)_{,xx} \alpha_j \tag{27}$$

$$u_{,yy} = -\int_\Gamma (u^*_{,yy} \overline{q} - \overline{u} q^*_{,yy}) \; ds + \sum_{j=1}^{M} (\hat{u}_j)_{,yy} \alpha_j \tag{28}$$

$$u_{,xy} = -\int_\Gamma (u^*_{,xy} \overline{q} - \overline{u} q^*_{,xy}) \; ds + \sum_{j=1}^{M} (\hat{u}_j)_{,xy} \alpha_j \tag{29}$$

The final step of AEM is to apply Eq. 10 to M discrete points inside Ω. We, thus, obtain a set of M equations

$$T_x^i u^i_{,xx} + 2T_{xy}^i u^i_{,xy} + T_y^i u^i_{,yy} - \rho^i \ddot{u}^i = -g^i \qquad i = 1,2,...M \tag{30}$$

Using Eq. 26-29 to evaluate u and its derivatives at points $i = 1,2,...M$ and substituting them into Eq. 26 yields a set of linear equations of motion

$$F_i(\alpha_j, \ddot{\alpha}_j) = -g^i \qquad i = 1,2,...M \tag{31}$$

from which the coefficients α_j can be established. The final step of AEM can be implemented only numerically as it is shown in the following section.

4 Numerical Implementation

The BEM with constant elements is used to approximate the boundary integrals in Eq. 26-29. If N is the number of the boundary nodal points (Fig. 2), then Eq. 26 is written as

$$c^i \overline{u}^i = \sum_{k=1}^{N} \tilde{H}_{ik} \overline{u}^k - \sum_{k=1}^{N} G_{ik} \overline{q}^k \tag{32}$$

where

$$\tilde{H}_{ik} = \int_k q^*(r_{ik}) \; ds \tag{33}$$

$$G_{ik} = \int_k u^*(r_{ik}) \; ds \tag{34}$$

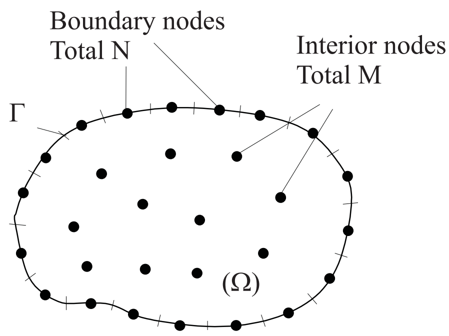


Figure 2 : Boundary discretization and domain nodal points

Applying Eq. 32 to all boundary nodal points and using matrix notation yields

$$[H]\{\overline{u}\} - [G]\{\overline{q}\} = 0 \tag{35}$$

where

$$[H] = [\tilde{H}] - [I]\{c\} \tag{36}$$

with $\{c\}$ being a vector including the values of the coefficient c^i and $[I]$ the unit matrix. The boundary condition (21), when applied to the boundary nodal points, yields

$$(\beta_1)^i \overline{u}^i + (\beta_2)^i \overline{q}^i = (\beta_3)_i - \left[(\beta_1)^i \sum_{j=1}^{M} \alpha_j \hat{u}_j^i + (\beta_2)^i \sum_{j=1}^{M} \alpha_j \hat{q}_j^i \right] \tag{37}$$

or using matrix notation

$$[\beta_1]\{\overline{u}\} + [\beta_2]\{\overline{q}\} = \{\beta_3\} - \left([\beta_1]\{\hat{U}\} + [\beta_2]\{\hat{Q}\}\right)\{\alpha\} \tag{38}$$

in which $[\hat{U}] = [\hat{u}_j^i]$, $[Q] = [\hat{q}_j^i]$ are $N \times M$ matrices; $[\beta_1], [\beta_2]$ are $N \times N$ diagonal matrices and $\{\alpha\}$ the vector of the coefficients to be determined.

Eq. 35 and 38 may be combined to express $\{\overline{u}\}$ and $\{\overline{q}\}$ in terms of $\{\alpha\}$. Thus, we may write

$$\begin{bmatrix} [H] & -[G] \\ [\beta_1] & [\beta_2] \end{bmatrix} \begin{Bmatrix} \{\overline{u}\} \\ \{\overline{q}\} \end{Bmatrix} = \begin{bmatrix} [0] \\ [T] \end{bmatrix} \{\alpha\} + \begin{Bmatrix} \{0\} \\ \{\beta_3\} \end{Bmatrix} \tag{39}$$

where

$$[T] = -([\beta_1][\hat{U}] + [\beta_2][\hat{Q}]) \tag{40}$$

Solving Eq. 39 we obtain

$$\{\overline{u}\} = [S_u]\{\alpha\} + \{d_u\} \tag{41}$$
$$\{\overline{q}\} = [S_q]\{\alpha\} + \{d_q\} \tag{42}$$

in which $[S_u], [S_q]$ are known $N \times M$ rectangular matrices and $[d_u], [d_q]$ known vectors.

Eq. 26-29 when discretized and applied to the M nodal points inside Ω give

$$\{u\} = [H]\{\overline{u}\} - [G]\{\overline{q}\} + [\hat{U}]\{\alpha\} \tag{43}$$
$$\{u_{xx}\} = [H_{xx}]\{\overline{u}\} - [G_{xx}]\{\overline{q}\} + [\hat{U}_{xx}]\{\alpha\} \tag{44}$$
$$\{u_{yy}\} = [H_{yy}]\{\overline{u}\} - [G_{yy}]\{\overline{q}\} + [\hat{U}_{yy}]\{\alpha\} \tag{45}$$
$$\{u_{xy}\} = [H_{xy}]\{\overline{u}\} - [G_{xy}]\{\overline{q}\} + [\hat{U}_{xy}]\{\alpha\} \tag{46}$$

in which $[G]$, $[H]$, $[G_{xx}]$, $[H_{yy}]$, $\ldots$, $[H_{xy}]$ are known square matrices having dimensions $M \times M$ and originating from the integration of the kernel functions u^* and q^* and their respective derivatives; $\left[\hat{U}\right]$ $\left[\hat{U}_{xx}\right], \ldots, \left[\hat{U}_{xy}\right]$ are known matrices having dimensions $M \times M$, the elements of which result from the functions $\hat{u}_j$ and their derivatives.

Substituting Eq. 41-42 into Eq. 43-46 yields

$$\{u\} = [W]\{\alpha\} + \{w\} \tag{47}$$
$$\{u_{xx}\} = [W_{xx}]\{\alpha\} + \{w_{xx}\} \tag{48}$$
$$\{u_{yy}\} = [W_{yy}]\{\alpha\} + \{w_{yy}\} \tag{49}$$
$$\{u_{xy}\} = [W_{xy}]\{\alpha\} + \{w_{xy}\} \tag{50}$$

where

$$[W] = [H][S_u] - [G][S_q] + [\hat{U}] \tag{51}$$
$$[W_{xx}] = [H_{xx}][S_u] - [G_{xx}][S_q] + [\hat{U}_{xx}] \tag{52}$$
$$[W_{yy}] = [H_{yy}][S_u] - [G_{yy}][S_q] + [\hat{U}_{yy}] \tag{53}$$

$$[W_{xy}] = [H_{xy}][S_u] - [G_{xy}][S_q] + [\hat{U}_{xy}] \tag{54}$$

and

$$\{w\} = [H][d_u] - [G][d_q] \tag{55}$$
$$\{w_{xx}\} = [H_{xx}][d_u] - [G_{xx}][d_q] \tag{56}$$
$$\{w_{yy}\} = [H_{yy}][d_u] - [G_{yy}][d_q] \tag{57}$$
$$\{w_{xy}\} = [H_{xy}][d_u] - [G_{xy}][d_q] \tag{58}$$

Differentiation of (47)-(50) with respect to time yields

$$\{\ddot{u}\} = [W]\{\ddot{\alpha}\} \tag{59}$$

Finally, writing Eq. 30 in matrix form and substituting Eq. 47-50 and 55-58 in it, we obtain the typical semidiscretized equation of motion

$$[M]\{\ddot{\alpha}\} + [K]\{\alpha\} = \{F\} \tag{60}$$

where

$$[M] = -[\rho][W] \tag{61}$$
$$[K] = [T_x][W_{xx}] + 2[T_{xy}][W_{xy}] + [T_y][W_{yy}] \tag{62}$$
$$[F] = -\{g\} - ([T_x][w_{xx}] + 2[T_{xy}][w_{xy}] + [T_y][w_{yy}]) \tag{63}$$

Table 1 : Eigenfrequencies $\Omega_{m,n} = \alpha\omega\sqrt{\rho_c/T}$ of a circular homogeneous membrane. For $m \geq 1$ the computed values are double (N=100, M=64).

	$\overline{\beta}=0$		$\overline{\beta}=1$	
$\Omega_{m,n}$	Exact	Comp.	Exact	Comp.
$\Omega_{0,1}$	2.405	2.406	1.256	1.256
$\Omega_{1,1}$	3.832	3.827	2.405	2.405
$\Omega_{2,1}$	5.136	5.151	3.518	3.519
$\Omega_{0,2}$	5.520	5.525	4.080	4.080
$\Omega_{3,1}$	6.380	6.532	4.613	4.628
$\Omega_{1,2}$	7.016	7.015	5.520	5.511

Table 3 : Values of the mode shape $u_{1,1}$ along the line $y=0$ of the rectangular membrane of Example 2.

	Case (i)		Case (iii)
x	Exact	Comp.	Comp.
-5.1429	0.223	0.223	0.225
-3.4286	0.623	0.623	0.601
-1.7143	0.901	0.901	0.882
0.000	1.000	1.000	1.000
1.7143	0.901	0.901	0.882
3.4286	0.623	0.623	0.601
5.1429	0.223	0.223	0.225

are the generalized mass matrix, stiffness matrix and force vector, respectively. Note that $[\rho]$, $[T_x]$, $[T_y]$ and $[T_{xy}]$ in Eq. 61-63 are diagonal matrices. The initial conditions for $\{\alpha\}$ are obtained from Eq. 47 taking into account that the vector $\{w\}$ is constant. Thus, we have

$$\{a(0)\} = [W]^{-1}(\{f_1\} - \{w\}) \tag{64}$$

$$\{\dot{\alpha}(0)\} = [W]^{-1}\{f_2\} \tag{65}$$

4.1 Forced vibrations

For forced vibrations, Eq. 60 is solved using any time step integration method taking into account the initial conditions (64) and (65). Once α_j are computed, the deflection of the membrane at any point is given by the discretized counterpart of Eq. 26.

4.2 Free vibrations

For free vibrations it is $\{F\} = \{0\}$, and by setting $\{\alpha(t)\} = \{A\}\,e^{i\omega t}$ Eq. 60 becomes

$$\left(-\omega^2\,[M] + [K]\right)\{A\} = 0 \tag{66}$$

which is a generalized eigenvalue problem of linear algebra from which the eigenfrequencies ω and the respective mode shapes $\{A\}$ are established.

Table 2 : Eigenfrequencies $\Omega_{m,n} = \omega\alpha\sqrt{\rho_c/T}$ of the rectangular membrane of Example (N=82, M=100).

	Case (i)		Case (ii)		Case (iii)	
$\Omega_{m,n}$	Exact	Comp.	Comp.	Rayl.	Comp.	Rayl.
$\Omega_{1,1}$	3.850	3.850	4.454	4.459	4.259	4.272
$\Omega_{2,1}$	5.448	5.448	6.687	6.710	6.328	6.362
$\Omega_{1,2}$	6.667	6.667	8.011	8.063	7.870	7.905
$\Omega_{3,1}$	7.378	7.381	9.029	8.984	8.315	8.336
$\Omega_{2,2}$	7.700	7.701	9.832	9.905	9.979	9.951
$\Omega_{3,2}$	9.168	9.171	11.728	11.659	11.463	11.515
$\Omega_{4,1}$	9.439	9.446	11.553	11.587	11.851	11.624

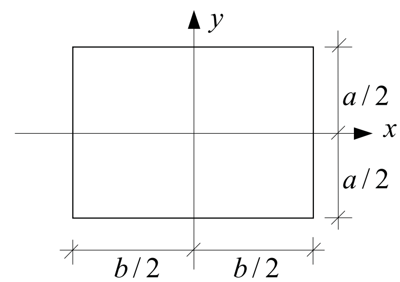


Figure 3 : Rectangular Membrane

4.3 The static problem

In this case its $\{\ddot{\alpha}\} = \{0\}$ and Eq. 60 becomes

$$[K]\{\alpha\} = \{F\} \tag{67}$$

from which the coefficients $\{\alpha\}$ are established by solving a system of linear algebraic equations.

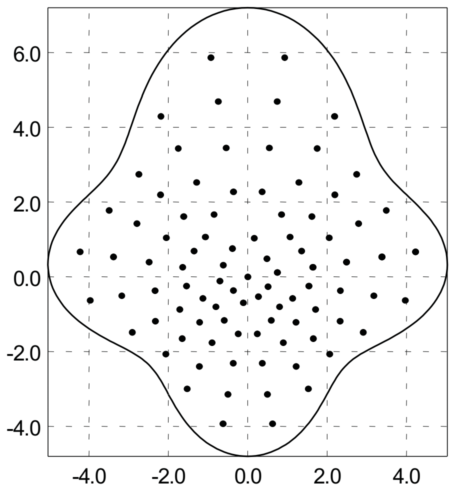
5 Numerical Examples

On the basis of the procedure presented in previous sections a FORTRAN code has been written and several membranes have been analyzed. The obtained numerical results are compared with those from existing analytical solutions or approximate ones validating, thus, the accuracy and efficiency of the proposed method. The employed approximation functions f_j are the multiquadrics [Goldberg, Chen, and Kapur (1996)] which are defined as

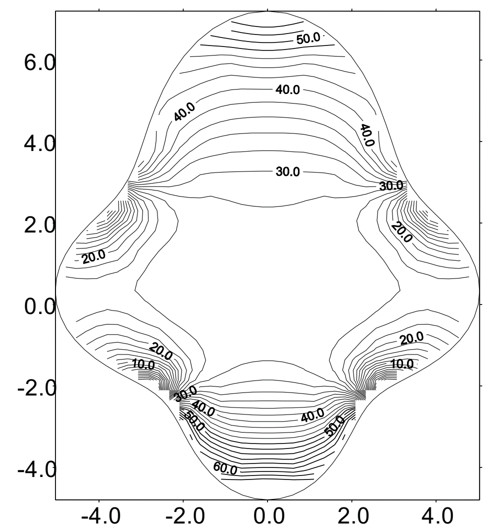
$$f_j = \sqrt{r^2 + c^2} \tag{68}$$

where c is an arbitrary constant and

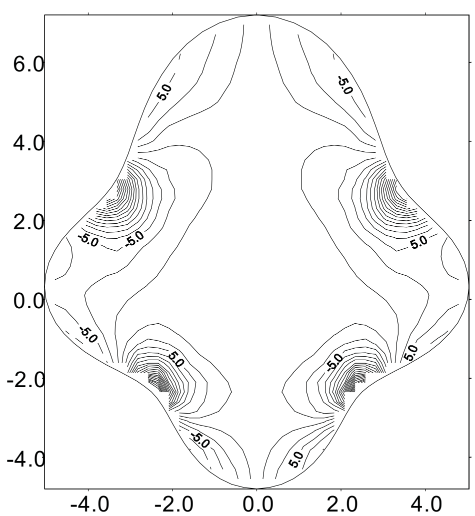
$$r = \sqrt{(x - x_j)^2 + (y - y_j)^2} \qquad (j = 1, 2, \ldots M) \tag{69}$$



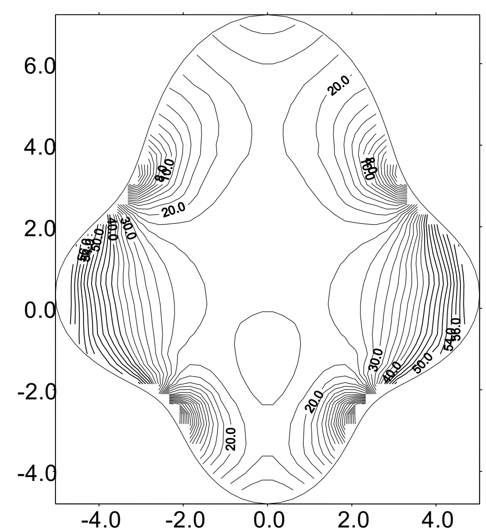
(a) Membrane of arbitrary shape with distribution of the $M = 101$ domain nodal points



(b) Contours of T_x



(c) Contours of T_y



(d) Contours of T_{xy}

Figure 4 : The membrane of arbitrary shape and distribution of the stretching forces of Example 3.

with x_j, y_j being the collocation point. The particular solution of Eq. 19 are obtained is

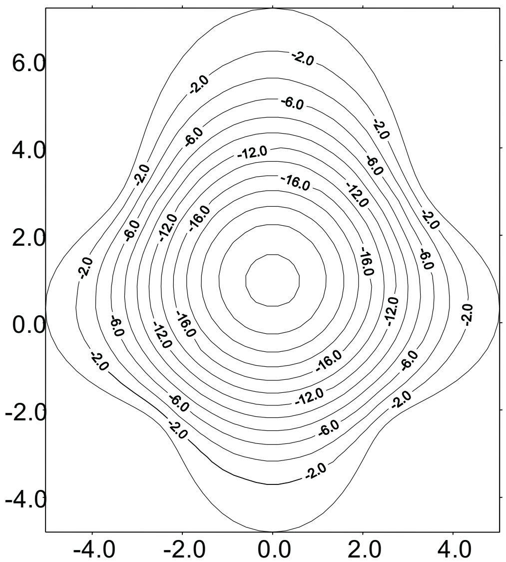
$$\hat{u}_j = -\frac{c^3}{3}\left[\ell n(2c^2) - \frac{4}{3}\right] \qquad \text{for } r = 0 \tag{70}$$

$$\hat{u}_j = -\frac{c^3}{3}\ell\left(c\sqrt{r^2+c^2}+c^2\right) + \frac{1}{9}\left(r^2+4c^2\right)\sqrt{r^2+c^2} \quad \text{for } r \neq 0 \tag{71}$$

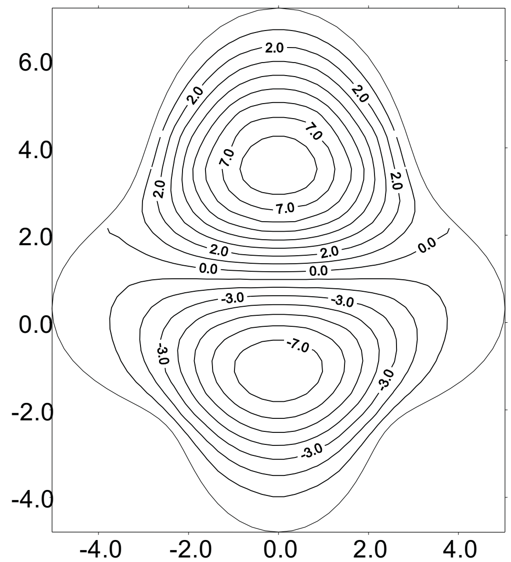
The radial base functions of polynomial type did not work.

5.1 *Example 1. Free vibrations of a circular homogeneous membrane*

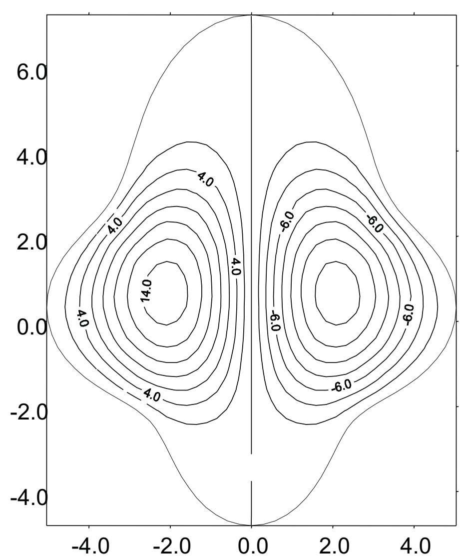
The free vibrations of a uniformly stretched homogeneous circular membrane with radius a and surface mass density $\rho = \rho_c$ have been studied. The numerical results have been obtained for (i) $\overline{\beta} = \beta_2/\beta_1\alpha = 0$ (fixed edge) and for (ii) $\overline{\beta} = \beta_2/\beta_1\alpha = 1$ (elastically supported edge) and they are given in Tab. 1. The exact results have been obtained from the analytic solution



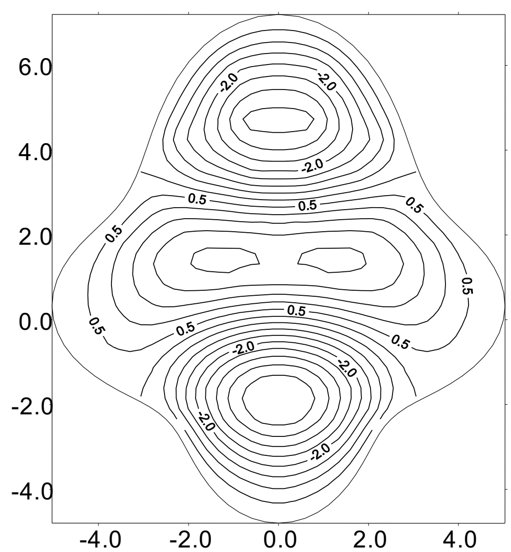
(a) Model 1, $\omega_1 = 2.803$



(b) Model 2, $\omega_2 = 4.119$



(c) Model 3, $\omega_3 = 4.591$



(d) Model 4, $\omega_4 = 5.227$

Figure 5 : Mode shapes and eigenfrequencies of the membrane of Example 3.

[Morse and Feshbach (1953)].

$$A_{m,n}(r,\theta) = J_m\left(\Omega_{m,n} r/\alpha\right)\left(C_m \cos m\theta + C_m \sin m\theta\right)$$
$$m = 0,1,2,3\ldots \tag{72}$$

where J_m is the m-th order Bessel function of the first kind.

The eigenfrequencies $\Omega_{m,n}$ are calculated by numerical evaluation of the frequency equation, which using Eq. 72 is obtained as

$$(1+\overline{\beta}m)J_m(\Omega) - \overline{\beta}\Omega J_m(\Omega) = 0 \qquad m = 0,1,2,3\ldots \tag{73}$$

It is apparent that for a fixed edge $(\overline{\beta} = 0)$ $\Omega_{m,n}$ are the zeros of the Bessel function $J_m(\Omega)$.

It is shown from Tab. 1 that the computed results are in excellent agreement with the exact ones. The domain nodal points have been taken uniformly distributed on concentric circles.

5.2 *Example 2. Free vibrations of a rectangular membrane*

The free vibrations of a rectangular membrane with side ratio $a/b = 8.5/12$ (Fig. 3) have been studied for the following three cases : (i) homogeneous membrane with $\rho = \rho_c$,

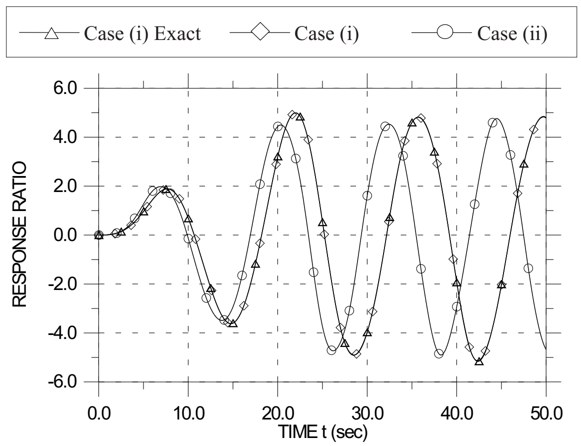


Figure 6 : Time history of the displacement response ratio at the center of the rectangular membrane in Example 4 subjected $g = g_0 \sin(0.4t)H(20-t)$. Cases (i) and (ii)

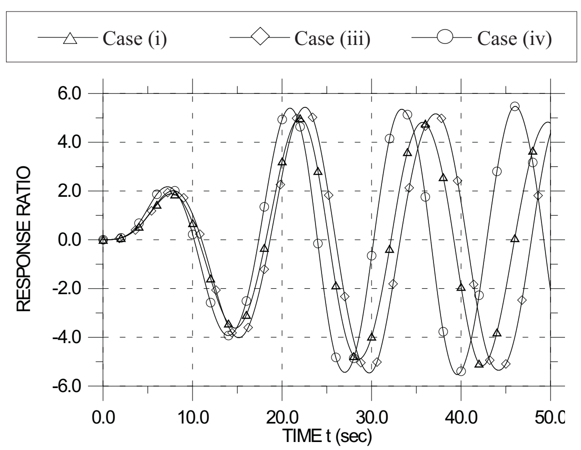


Figure 7 : Time history of the displacement response ratio at the center of the rectangular membrane in Example 4 subjected $g = g_0 \sin(0.4t)H(20-t)$. Cases (i), (iii) and (iv)

$T_x = T_y = T =$ constant, $T_{xy} = 0$; (ii) non-homogeneous membrane with $\rho/\rho_c = \exp(-0.1\{|x|+|y|\})$ and $T_x = T_y = T =$ constant, $T_{xy} = 0$; (iii) non-homogeneous membrane with $\rho/\rho_c = \exp(-0.1\{|x|+|y|\})$ and $T_x = (y^2 - x^2 + T)/T$, $T_y = (x^2 - y^2 + T)/T$, $T_{xy} = 2xy/T$.

The obtained results are presented in Tab. 2 as compared with those available from other solutions. The exact values are computed from the relation $\Omega_{m,n} = \pi\left[n^2 + m^2(a/b)^2\right]^{1/2}$. The approximate results have been obtained by the Rayleigh method with shape functions the mode shapes of the homogeneous membrane, i.e $u_{m,n} = \sin m\pi x/a \, \sin n\pi y/b$.

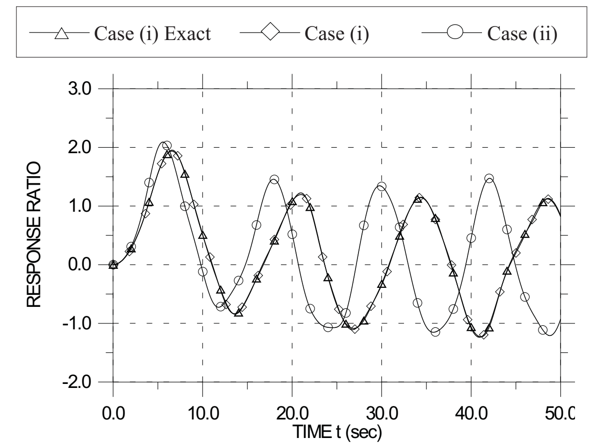


Figure 8 : Time history of the displacement response ratio at the center of the rectangular membrane in Example 4 subjected $g = g_0(1 - t/20)H(20-t)$. Cases (i) and (ii)

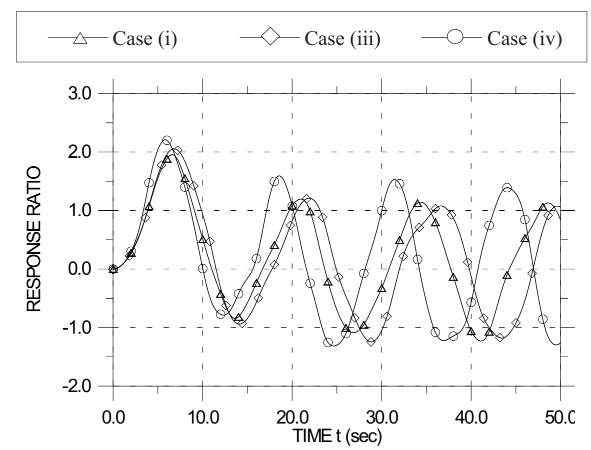


Figure 9 : Time history of the displacement response ratio at the center of the rectangular membrane in Example 4 subjected $g = g_0(1 - t/20)H(20-t)$. Cases (i), (iii) and (iv)

5.3 *Example 3. Membrane of arbitrary shape*

The membrane of arbitrary shape, shown in Fig. 4a, has been studied. Its boundary is defined by the curve $r = (5 + \sin\theta)(1.2\sin^4\theta + \cos^2\theta)$, $0 \le \theta \le 2\pi$. The prestressing of the membrane is due to imposed displacements $u_n = 0.005\ m$ along the boundary and in the direction normal to it, while in the tangential direction it is $u_t = 0$. The material of the membrane is homogeneous isotropic and linearly elastic with $E = 20000\ kN/m^2$ and $\nu = 0.20$.

The stretching forces T_x, T_y and T_{xy} have been evaluated from the solution of the plane stress problem by integrating the Navier equations using the BEM [Katsikadelis (1999)] with $N = 300$ constant boundary elements. Their distributions are shown in Fig. 4. The resulting forces in the principal directions are tensile $(T_{1,2} > 0)$. Therefore, no wrinkling occurs.

The first four mode shapes and the respective eigenfrequencies are shown in Fig. 5. They have been computed using $N = 100$ constant boundary elements and $M = 101$ domain nodal points placed as shown in Fig. 4a.

5.4 Example 4. Forced vibrations of a rectangular membrane

The forced vibrations of the rectangular membrane of Example 2 have been examined when subjected to the load cases: (a) $g = g_0 \sin \overline{\omega} t H(t_1 - t)$; (b) $g = g_0(1 - t/t_1)H(t_1 - t)$; $g_0 = 10$, $\overline{\omega} = 0.4$, $t_1 = 20$. $H(\ )$ denotes the Heaviside step function. The following four cases have been studied for each load case: (i) homogeneous membrane with $\rho = \rho_c$, $T_x = T_y = T = 1$, $T_{xy} = 0$; (ii) non-homogeneous membrane with $\rho/\rho_c = \exp(-0.1\{|x| + |y|\})$ and $T_x = T_y = T = 1$, $T_{xy} = 0$; (iii) homogeneous membrane with $\rho = \rho_c$ and $T_x = \left(y^2 - x^2\right)/50 + 1$, $T_y = \left(x^2 - y^2\right)/50 + 1$, $T_{xy} = 2xy/50$; (iv) non-homogeneous membrane with $\rho/\rho_c = \exp(-0.1\{|x| + |y|\})$ and $T_x = \left(y^2 - x^2\right)/50 + 1$, $T_y = \left(x^2 - y^2\right)/50 + 1$, $T_{xy} = 2xy/50$. The exact solution for the homogeneous membrane has been obtained using the modal superposition method. The computed time histories of the displacement response ratio $R(t) = u(0,0,t)/u_{st}$ at the center of the membrane are shown in Fig. 6 through Fig. 9. is the static deflection at the center of the membrane under the uniform load $g_0 = 10$. The direct integration of the equation of motion has been performed using the mean acceleration method.

6 Conclusions

In this paper the BEM has been developed for the dynamic analysis of non-homogeneous membranes. The presented method, which is based on the concept of the analog equation, is boundary-only in the sense that only boundary discretization is required. The collocation domain nodal points do not spoil the pure boundary character of the method. From the worked out examples one concludes that the method is accurate. A major advantage of the developed method is the use of the simple Laplace fundamental solution. This overcomes the problem of establishing the fundamental solution of Eq. 10, which is impossible to derive except for some special cases. Thus, the proposed method renders BEM a versatile computational method for solving difficult engineering problems.

References

De, S. (1971): Solution to the equation of a vibrating annular membrane of non-homogeneous material. *Pure and Applied Geophysics*, vol. 90, pp. 89–95.

Goldberg, M. A.; Chen, C. S.; Kapur, S. P. (1996): Improved multiquadric approximation for partial differential equations. *Engineering Analysis with Boundary Elements*, vol. 18, pp. 9–17.

Katsikadelis, J. T. (1994): The analog equation method: A powerful bem–based solution technique for solving linear and nonlinear engineering problems. In Brebbia, C. A.(Ed): *Boundary Element Method XVI*, pp. 167–182. Computational Mechanics Publications, Southampton.

Katsikadelis, J. T. (1999): *Boundary Elements in Engineering: Theory and Applications*. Symeon Publications Co., Athens (in Greek).

Katsikadelis, J. T. (2000): Determination of the initial prestress of the membranes to avoid wrinkling. *(to be published)*.

Katsikadelis, J. T.; Nerantzaki, M. S. (1998): A boundary only bem for linear and non-linear problems. In Kassab, A.; A., B. C.; Chopra, M.(Eds): *Boundary Elements XX*, pp. 309–320. Computational Mechanics publications, Southampton.

Katsikadelis, J. T.; Nerantzaki, M. S. (1999): The boundary element method for nonlinear problems. *Engineering Analysis with Boundary Elements*, vol. 23, pp. 365–373.

Katsikadelis, J. T.; Sapountzakis, E. J. (1988): An approach to the vibration problem of homogeneous, non-homogeneous and composite membranes based on the boundary element method. *International Journal for Numerical Methods in Engineering*, vol. 26, pp. 2439–2455.

Morse, P. M. (1948): *Vibration and Sound*. McGraw-Hill, New York.

Morse, P. M.; Feshbach, H. (1953): *Methods of Theoretical Physics*. McGraw-Hill, New York.

Nardini, D.; Brebbia, C. A. (1982): New approach to vibration analysis using boundary elements. In *Boundary Element Methods in Engineering*. Computational Mechanics Publications, Southampton and Springer-Verlag, Berlin.

Rayleigh, L. (1945): *Theory of Sound, Vol.1*. Dover Publications, New York.

Sato, K. (1980): Vibrations of non-homogeneous membranes with time dependent boundary conditions. *Journal of Sound and Vibrations*, vol. 72, pp. 273–278.