Static and Dynamic Analysis of Shell Panels Using the Analog Equation Method

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Abstract: The Analog Equation Method is applied to the static and dynamic analysis of thin cylindrical shell panels. The Flügge theory is adopted. The three displacement components are established by solving two membrane and one plate bending problems under the same boundary conditions subjected to "appropriate" (equivalent) fictitious loads. Numerical results are presented which illustrate the efficiency and the accuracy of the proposed method.

keyword: analog equation method, boundary element method, Flügge theory, fictitious loads

1 Introduction

Static and dynamic analysis of linear elastic thin shells characterized by complex geometry, loading and boundary conditions require the use of numerical methods, such as the Finite Difference Method (FDM) and especially the Finite Element Method (FEM). Both of these methods, in spite of some shortcomings, have been successfully employed for the solution of a variety of static and dynamic shell problems.

During the last few years the Boundary Element Method (BEM) has been proven an efficient alternative to the domain type methods and has been employed to study mainly thin elastic shallow shells. In the class of general shells Antes (1981) presented a direct boundary integral formulation BEM for the static analysis of general shells based on reciprocity. Explicit expressions for the fundamental solution of the special case of circular cylindrical shells were given, but no numerical examples were reported. The literature is much richer in problems concerned with thin elastic shallow shells. There are basically two BEM approaches for the static analysis of shallow shells. The first approach is the conventional BEM in its direct or indirect form, which employs the fundamental solution of the problem and is based on the displacement or the flexural displacement-membrane stress function formulation of shallow shells. Although this method requires only a boundary discretization, it is not computationally efficient due to the high complexity of the fundamental solution [Simmonds and Bradley (1976); Matsui and Matsuoka (1978)]. This approach, may be advantageous only for special cases dealing with spherical and circular cylindrical shallow shells, for which simpler fundamental solutions exist [Newton and Tottenham (1968); Tosaka and Miyake (1983); Gospodinov (1984)].

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The second approach is the direct domain/boundary element method D/BEM, which employs the plate fundamental solution in flexure and stretching in its formulation. This creates boundary as well domain integrals due to the flexure-stretching coupling terms and therefore requires a boundary as well as an interior discretization. However, the simplicity of the fundamental solution results in a more general and efficient scheme [Forbes and Robinson (1969); Zhang and Atluri (1986); Wang, Ye and Wang (1986); Ye (1988)].

Although the literature on the static analysis of shallow shells by the BEM and the D/BEM is rather rich, this is not the case with the dynamic analysis of these structures. Free and forced vibrations of shallow shells have been recently studied by the D/BEM which employs the static fundamental solution of a plate in flexure and stretching. This creates domain integrals due to the flexure-stretching coupling terms as well as due to the inertia terms. This is apparently the only viable solution to the problem, since there is no any shallow shell fundamental solution for the dynamic case in the literature, while the shallow shell static fundamental solution is very complicated and domain integrals are necessarily present in the formulation due to the existence of inertia terms. Zhang and Atluri (1986), were the first to employ the D/BEM to the dynamic analysis of shallow shells both for small and large deformations. Providakis and Beskos (1987), developed a D/BEM solution for the transient forced vibrations of linear elastic shallow shells in the frequency domain.

In this paper a novel solution approach to the static and dynamic analysis of thin shells is presented. It is illustrated without restricting the generality by applying it for cylindrical shell panels. The proposed method is based on the concept of the Analog Equation Method AEM. This method has been employed to a variety of engineering problems including potential problems [Katsikadelis (1994)], elasticity problems [Katsikadelis and Kandilas (1997)], plate problems [Katsikadelis and Nerantzaki (1994) (1996)], linear and nonlinear [Katsikadelis and Nerantzaki (1999)] static as well as dynamic ones [Katsikadelis, Nerantzaki and Kandilas (1993)]. In the present investigation the Flügge type differential equations in terms of the three displacement components are used. According to the AEM the problem governed by the three coupled Flügge type differential equations is substituted by two membrane and a plate problems subjected to "appropriate" equivalent fictitious time dependent load distributions under the same boundary conditions. Subsequently, using BEM for these three linear problems (two Poisson and one biharmonic) the deflections as well as their derivatives involved in the original Flügge equations are expressed in terms of the unknown domain fictitious loads. Substitution of these quantities (displacements and derivatives) into the Flügge equations yields a set of linear algebraic equations, which permit the determination of the fictitious loads. Finally, the displacements are obtained from the integral representations of the analog equations. The method is utilized to analyse certain static and dynamic example problems. The obtained results are in good agreement when compared with those obtained from other analytical or computational techniques. In its present form is D/BEM and requires domain discretization. Nevertheless the method can be developed as boundary-only, either by using the Dual Reciprosity Method [Partridge, Brebbia and Wrobel (1991)] or the Particular Solution Method presented by Katsikadelis and Nerantzaki (1999).

2 Governing equations

The Flügge type differential equations are used, which for a typical thin-walled cylindrical shell of uniform thickness h, made of an isotropic, linearly elastic material, are written as [Koumousis (1981)].

$$\begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \begin{Bmatrix} u(x,s,t) \\ v(x,s,t) \\ w(x,s,t) \end{Bmatrix} = -\frac{(1-\nu^2)r_o^2}{Eh} \begin{Bmatrix} X(x,s,t) \\ S(x,s,t) \\ Z(x,s,t) \end{Bmatrix} \tag{1}$$

where u, v, w are the axial, circumferential and radial displacements, and $L_{ij}\,(i,j=1,3)$ are linear differential operators given as

$$L_{11} = \frac{\partial^2}{\partial x^2} + \frac{1-\nu}{2}\frac{\partial^2}{\partial s^2} + \frac{h^2}{12r_0^2}\frac{1-\nu}{2}\frac{1}{\overline{r}^2}\frac{\partial^2}{\partial s^2} - \frac{h^2}{12r_0^2}\frac{1-\nu}{2}\frac{2\overline{r}_{,s}}{\overline{r}^3}\frac{\partial}{\partial s} - \frac{1-\nu^2}{E}\rho r_0^2\frac{\partial^2}{\partial t^2} \tag{2}$$

$$L_{12} = L_{21} = \frac{1+\nu}{2}\frac{\partial^2}{\partial x \partial s} \tag{3}$$

$$L_{13} = L_{31} = -\frac{\nu}{\overline{r}}\frac{\partial}{\partial x} + \frac{h^2}{12r_0^2}\frac{1}{\overline{r}}\frac{\partial^3}{\partial x^3} - \frac{h^2}{12r_0^2}\frac{1-\nu}{2}\frac{1}{\overline{r}}\frac{\partial^3}{\partial x \partial s^2} + \frac{h^2}{12r_0^2}\frac{1-\nu}{2}\frac{\overline{r}_{,s}}{\overline{r}}\frac{\partial^2}{\partial x \partial s} \tag{4}$$

$$L_{22} = \frac{\partial^2}{\partial s^2} + \frac{1-\nu}{2}\frac{\partial^2}{\partial x^2} + \frac{h^2}{12r_0^2}\frac{3(1-\nu)}{2}\frac{1}{\overline{r}^2}\frac{\partial^2}{\partial x^2} - \frac{h^2}{12r_0^2}\left(\frac{\overline{r}_{,s}}{\overline{r}^2}\right)^2 - \frac{(1-\nu)^2}{E}\rho r_0^2\frac{\partial}{\partial t^2} \tag{5}$$

$$L_{23} = -\frac{1}{\overline{r}}\frac{\partial}{\partial s} + \frac{\overline{r}_{,s}}{\overline{r}^2} + \frac{h^2}{12r_0^2}\frac{3-\nu}{2}\frac{1}{\overline{r}}\frac{\partial^3}{\partial x^2 \partial s} + \frac{h^2}{12r_0^2}\frac{\overline{r}_{,s}}{\overline{r}^2}\frac{\partial^2}{\partial s^2} \tag{6}$$

$$+ \frac{h^2}{12r_0^2}\frac{\overline{r}_{,s}}{\overline{r}^4} \tag{7}$$

$$L_{32} = -\frac{1}{\overline{r}}\frac{\partial}{\partial s} + \frac{h^2}{12r_0^2}\frac{3-\nu}{2}\frac{1}{\overline{r}}\frac{\partial^3}{\partial x^2 \partial s} - \frac{h^2}{12r_0^2}\frac{3-\nu}{2}\frac{\overline{r}_{,s}}{\overline{r}^2}\frac{\partial^2}{\partial x^2} - \frac{h}{12r_0^2}\frac{\overline{r}_{,s}}{\overline{r}^2}\frac{\partial^2}{\partial x^2} - \frac{h^2}{6r_0^2}\left(\frac{\overline{r}_{,s}}{\overline{r}^2}\right)_{,s}\frac{\partial}{\partial s} - \frac{h^2}{12r_0^2}\left(\frac{\overline{r}_{,s}}{\overline{r}^2}\right)_{,ss} - \frac{h^2}{12r_0^2}\frac{\overline{r}_{,s}}{\overline{r}^4} \tag{8}$$

$$L_{33} = \frac{h^2}{12r_0^2}\nabla^4 + \frac{h^4}{12r_0^2}\frac{1}{\overline{r}^2}\frac{\partial^2}{\partial s^2} + \frac{1}{\overline{r}^2} + \frac{h^2}{12r_0^2}\frac{1}{\overline{r}^2}\frac{\partial^2}{\partial s^2} - \frac{h^2}{3r_0^2}\frac{\overline{r}_{,s}}{\overline{r}^3}\frac{\partial}{\partial s} - \frac{h^2}{6r_0^2}\left(\frac{\overline{r}_{,s}}{\overline{r}^3}\right)_{,s} + \frac{h^2}{12r_0^2}\frac{1}{\overline{r}^4} + \frac{1-\nu^2}{E}\rho r_0^2\frac{2}{t^2} \tag{9}$$

with $\overline{r} = r^*/r_0$, r^* being the radius of curvature of the cross-section of the shell; r_0 is the radius of a circle whose arc length is equal to that of the cross-section of the non-circular shell; ρ is the mass density and are the components of the body force in the axial, circumferential and X, S, Z radial directions, respectively. Note that in the above equations, the inertia forces have been included in the differential operators.

The shell panel may be subjected to the following boundary conditions

- Curved edge simply supported $N_x = v = w = M_x = 0$
- Curved edge clamped $N_x = v = w = w_{,x} = 0$
- Straight edge simply supported $u = v = w = M_s = 0$
- Straight edge clamped $u = v = w = w_{,s} = 0$

An edge may be movable or immovable in the direction of the middle surface and simply supported or clamped in the transverse direction. In the present analysis we consider cylindrical shells with rectangular plan form under the following boundary conditions.

a) *Curved edge*

$$\begin{aligned} &N_x = 0, \quad v = 0, \quad w = 0, \quad M_x = 0, \\ &N_x = 0, \quad v = 0, \quad w = 0, \quad w_{,x} = 0 \end{aligned} \tag{10}$$

b) *Straight edge*

$$\begin{aligned} &\alpha_1 u + \alpha_2 N_{sx} = \alpha_3, \quad \beta_1 v + \beta_2\left(N_s - \frac{M_s}{\overline{r}}\right) = \beta_3, \\ &\gamma_1 w + \gamma_2 Q_{s_{eff}} = \gamma_3, \quad \delta_1 w_{,s} + \delta_2 M_s = \delta_3, \\ &Q_{s_{eff}} = Q_s + M_{sx,x} \end{aligned} \tag{11}$$

where $\alpha_i(p), \beta_i(p), \gamma_i(p), \delta_i(p)$ $p \in \Gamma\,(i=1,2,3)$ are functions specified on the boundary. Note that all types of conventional boundary conditions are obtained by specifying appropriety these functions, e.g for a movable simply supported edge it is $\alpha_2 = 1$, $\alpha_1 = \alpha_3 = 0$, $\beta_2 = 1$, $\beta_1 = \beta_3 = 0$, $\gamma_1 = 1$, $\gamma_2 = \gamma_3 = 0$, $\delta_1 = \delta_3 = 0$, $\delta_2 = 1$. Moreover N_x, N_s, N_{xs}, N_{sx}, M_x, M_s, M_{xs}, M_{sx}, Q_x, Q_s are the stress resultants given as [Koumousis (1981)].

$$N_x = \frac{Eh}{(1-\nu^2)r_0}\left[u_{,x} + \nu\left(v_{,s} - \frac{w}{\bar{r}}\right) + \frac{1}{12r_0^2}\frac{1}{\bar{r}}w_{,xx}\right] \tag{12}$$

$$N_s = \frac{Eh}{(1-\nu^2)r_0}\left[v_{,s} + \nu u_{,x} - \frac{w}{\bar{r}} - \frac{h^2}{12r_0^2}\frac{c}{\bar{r}}\left(w_{,ss} + \frac{w}{\bar{r}^2} - \frac{\bar{r}_{,s}}{\bar{r}^2}v\right)\right] \tag{13}$$

$$N_{xs} = \frac{Eh}{2(1+\nu)r_0}\left[u_{,s} + v_{,s} + \frac{h^2}{12r_0^2}\frac{1}{\bar{r}}\left(\frac{1}{\bar{r}}v_{,x} + w_{,xs}\right)\right] \tag{14}$$

$$N_{sx} = \frac{Eh}{2(1+\nu)r_0}\left[u_{,s} + v_{,s} - \frac{h^2}{12r_0^2}\frac{c}{\bar{r}}\left(w_{,xs} - \frac{1}{\bar{r}}u_{,x}\right)\right] \tag{15}$$

$$M_x = -\frac{D}{r_0^2}\left\{w_{,xx} + \nu\left[w_{,ss} + \left(\frac{v}{\bar{r}}\right)_{,s}\right] + \frac{1}{\bar{r}}u_{,x}\right\} \tag{16}$$

$$M_x = -\frac{D}{r_0^2}\left[c\left(w_{,ss} + \frac{w}{\bar{r}^2} - \frac{\bar{r}_{,s}}{\bar{r}^2}v\right) + \nu w_{,xx}\right] \tag{17}$$

$$M_{xs} = \frac{D}{r_0^2}(1-\nu)\left(w_{,xs} + \frac{1}{\bar{r}}v_{,x}\right) \tag{18}$$

$$M_{sx} = -\frac{D}{r_0^2}\frac{1-\nu}{2}\left[(1+c)w_{,xs} + \frac{1}{\bar{r}}v_{,x} - \frac{c}{\bar{r}}u_{,s}\right] \tag{19}$$

$$Q_x = -\frac{D}{r_0^3}\left\{w_{,xxx} + \frac{1}{2}\left[1 + c + \nu(1-c)\right]w_{,xss} + \frac{1-\nu}{2}c_{,s}w_{,xs} + \frac{1}{\bar{r}}u_{,xx} - \frac{1-\nu}{2}\left(\frac{cu_{,s}}{\bar{r}}\right)_{,s} + \frac{1+\nu}{2}\left(\frac{v}{\bar{r}}\right)_{,xs}\right\} \tag{20}$$

$$Q_s = -\frac{D}{r_0^3}\left[cw_{,sss} + w_{,xxs} + c_{,s}\left(w_{,ss} + \frac{w}{\bar{r}^2} - \frac{\bar{r}_{,s}}{\bar{r}^2}v\right) + c\left(\frac{w}{\bar{r}^2}\right)_{,s} + (1-\nu)\frac{1}{\bar{r}}v_{,xx} - c\left(\frac{\bar{r}_{,s}}{\bar{r}^2}v\right)_{,s}\right] \tag{21}$$

$$D = \frac{Eh^3}{12(1-\nu^2)}$$

$$c = 12\left(\bar{r}r_0/h\right)^2\left[\left(\bar{r}r_0/h\right)\log\frac{1+h/2\bar{r}r_0}{1-h/2\bar{r}r_0} - 1\right] \tag{22}$$

For thin shells c differs negligibly from unity. Thus, in the sequel c will be taken equal to one.

For the dynamic problem the displacements and the velocities must satisfy the following initial conditions

$$\begin{aligned} u(x,s;0) &= u_1(x,s) \quad \dot{u}(x,s;0) = \dot{u}_1(x,s) \\ v(x,s;0) &= v_1(x,s) \quad \dot{v}(x,s;0) = \dot{v}_1(x,s) \\ w(x,s;0) &= w_1(x,s) \quad \dot{w}(x,s;0) = \dot{w}_1(x,s) \\ \phi &= \text{angle}\,(r,\mathbf{n}) \end{aligned} \tag{23}$$

3 The Analog Equation Method

The initial-boundary value problem described by Eq. 1, 10-11, 23 can be solved using the Analog Equation Method (AEM) developed by Katsikadelis (1994). In this particular problem the method is applied as follows.

Let u, v, w be the sought solution of the problem. If the Laplacian operator is applied to the functions u, v and the biharmonic operator to w we obtain

$$\begin{aligned} \nabla^2 u &= q_1(s,x,t) \\ \nabla^2 v &= q_2(s,x,t) \\ \nabla^4 w &= q_3(s,x,t) \end{aligned} \tag{24}$$

Eq. 24 indicate that the solution of the original shell equations can be obtained as the deflection surfaces of two membranes and that of a plate with unit stiffness, subjected to the fictitious time dependent loads q_1, q_2, q_3 under the given boundary and initial conditions.

According to the AEM the unknown fictitious loads q_1, q_2, q_3 can be established using the BEM as following.

For the functions u, v, w satisfying the Eq. 24 the following integral representations are obtained [Katsikadelis (1989), (1994)].

$$\varepsilon u(P,t) = \int \ln r q_1 d\Omega - \int \left[X_1 \ln r - \Omega_1 (\ln r)_{,n}\right] ds \tag{25}$$

$$\varepsilon v(P,t) = \int \ln r q_2 d\Omega - \int \left[X_2 \ln r - \Omega_2 (\ln r)_{,n}\right] ds \tag{26}$$

$$\varepsilon w(P,t) = \int q_3 \Lambda_4(r) d\Omega - \int \left[\Omega_3 \Lambda_1(r) + X_3 \Lambda_2(r) + \Phi_3 \Lambda_3(r) + \Psi_3 \Lambda_4(r)\right] ds \tag{27}$$

$$\varepsilon \nabla^2 w(P,t) = \int q_3 \Lambda_2(r) d\Omega - \int \left[\Phi_3 \Lambda_1(r) + \Psi_3 \Lambda_2(r)\right] ds \tag{28}$$

$\varepsilon = 2\pi$, π, 0 depending on whether the point P is inside the domain Ω, on the boundary Γ or outside Ω, respectively. The boundary is assumed to be smooth at the point P; r is the distance between this point P and the point that varies during integration.

The kernels $\Lambda_i = \Lambda_i(r)$, $i = 1,4$, $r = |P - q|$ $P \in \Gamma$, $q \in \Omega$ are given as

$$\Lambda_1(r) = -\frac{\cos\phi}{r} \tag{29}$$

$$\Lambda_2(r) = \ln r + 1 \tag{30}$$

$$\Lambda_3(r) = -\frac{1}{4}(2r\ln r + r)\cos\phi \tag{31}$$

$$\Lambda_4(r) = \frac{r^2 \ln r}{4} \tag{32}$$

$$\phi = \text{angle}\,(r,\mathbf{n}) \tag{33}$$

In Eq. 25-28 the following notation has been employed

$$\Omega_1 = u, \quad X_1 = \frac{\partial u}{\partial n}, \tag{34}$$

$$\Omega_2 = v, \quad X_2 = \frac{\partial v}{\partial n}, \tag{35}$$

$$\Omega_3 = w, \quad X_3 = \frac{\partial w}{\partial n}, \tag{36}$$

$$\Phi_3 = \nabla^2 w, \quad \Psi_3 = \frac{\partial \nabla^2 w}{\partial n} \tag{37}$$

Using the expressions in Eq. 12-21 for the stress resultants and the notation in Eq. 34-37, the boundary conditions in Eq. 10 for the curved edge become

$$X_1 = 0, \quad \Omega_2 = 0, \quad \Omega_3 = 0, \quad \Phi_3 = 0 \tag{38}$$

and

$$X_1 = 0, \quad \Omega_2 = 0, \quad \Omega_3 = 0, \quad X_3 = 0 \tag{39}$$

Moreover the boundary conditions in Eq. 11 for the straight edge become

$$\Omega_1 = 0, \quad \Omega_2 = 0, \quad \Omega_3 = 0, \quad \delta_1 X_3 + \delta_2 \Phi_3 = 0 \tag{40}$$

The values $\delta_1 = 1$ and $\delta_2 = 0$ correspond to the clamped edge, while $\delta_1 = 0$ and $\delta_2 = 1$ to the simply supported edge.

The boundary conditions in Eq. 38-40 together with the integral equations in Eq. 25-28, when $P \in \Gamma$, constitute a set of boundary equations which permit the establishment of the unknown boundary quantities in terms of the fictitious sources q_i $(i = 1, 2, 3)$. This can be achieved numerically as follows.

The boundary Γ is discretized into N elements and the domain into M cells on which the unknown quantities are assumed constant. Collocation of the boundary equations at the boundary nodal points yields

$$\begin{bmatrix} [A_{11}] & [A_{12}] \\ [A_{21}] & [A_{22}] \end{bmatrix} \begin{Bmatrix} \{\Omega_1\} \\ \{X_1\} \end{Bmatrix} = \begin{Bmatrix} \{0\} \\ \{0\} \end{Bmatrix} + \begin{Bmatrix} [0] \\ [C_1] \end{Bmatrix} \{q_1\} \tag{41}$$

$$\begin{bmatrix} [B_{11}] & [B_{12}] \\ [B_{21}] & [B_{22}] \end{bmatrix} \begin{Bmatrix} \{\Omega_2\} \\ \{X_2\} \end{Bmatrix} = \begin{Bmatrix} \{0\} \\ \{0\} \end{Bmatrix} + \begin{Bmatrix} [0] \\ [C_2] \end{Bmatrix} \{q_2\} \tag{42}$$

$$\begin{bmatrix} [\Gamma_{11}] & [\Gamma_{12}] & [0] & [0] \\ [0] & [0] & [\Gamma_{23}] & [\Gamma_{24}] \\ [\Gamma_{31}] & [\Gamma_{32}] & [\Gamma_{33}] & [\Gamma_{34}] \\ [0] & [0] & [\Gamma_{43}] & [\Gamma_{44}] \end{bmatrix} \begin{Bmatrix} \{\Omega_3\} \\ \{X_3\} \\ \{\Phi_3\} \\ \{\Psi_3\} \end{Bmatrix} = \begin{Bmatrix} \{0\} \\ \{0\} \\ \{0\} \\ \{0\} \end{Bmatrix} + \begin{Bmatrix} [0] \\ [0] \\ [C_3] \\ [C_4] \end{Bmatrix} \{q_3\} \tag{43}$$

where $[A_{ij}]$ $(i, j = 1, 2)$, $[B_{ij}]$ $(i, j = 1, 2)$, $[\Gamma_{ij}]$ $(i, j = 1, 2, 3, 4)$ are $N \times N$ known coefficient matrices, $\{C_i\}$ $(i = 1, 3, 4)$, $N \times M$ known coefficient matrices.

Subsequent collocation of the integral equations at the domain nodal points and elimination of the boundary quantities using Eq. 41-43 yields

$$\{u\} = [G_1]\{q_1\} \tag{44}$$
$$\{v\} = [G_2]\{q_2\} \tag{45}$$
$$\{w\} = [G_3]\{q_3\} \tag{46}$$

where $\{u\}$, $\{v\}$, $\{w\}$ and q_1, q_2, q_3 are vectors, including the values of the respective functions at the M domain nodal points and $[G_1]$, $[G_2]$, $[G_3]$are $M \times M$ known coefficient matrices. For nonhomogeneous boundary conditions a constant known vector will appear in the right hand side of equations Eq. 44-46.

Similar expressions are obtained for the derivatives of u, v, w involved in Eq. 1 by direct differentiation of their integral representations Eq. 25-27. Thus we have (See Appendix A and B)

$$\{u_x\} = [G_{1_x}]\{q_1\} \tag{47}$$
$$\{u_s\} = [G_{1_s}]\{q_1\} \tag{48}$$
$$\{u_{xs}\} = [G_{1_{xs}}]\{q_1\} \tag{49}$$
$$\{u_{xx}\} = [G_{1_{xx}}]\{q_1\} \tag{50}$$
$$\{u_{ss}\} = [G_{1_{ss}}]\{q_1\} \tag{51}$$
$$\{u_{xxx}\} = [G_{1_{xxx}}]\{q_1\} \tag{52}$$
$$\{u_{xss}\} = [G_{1_{xss}}]\{q_1\} \tag{53}$$
$$\{v_s\} = [G_{2_s}]\{q_2\} \tag{54}$$
$$\{v_{xx}\} = [G_{2_{xx}}]\{q_2\} \tag{55}$$
$$\{v_{ss}\} = [G_{2_{ss}}]\{q_2\} \tag{56}$$
$$\{v_{xs}\} = [G_{2_{xs}}]\{q_2\} \tag{57}$$
$$\{v_{xxs}\} = [G_{2_{xxs}}]\{q_2\} \tag{58}$$
$$\{w_x\} = [G_{3_x}]\{q_3\} \tag{59}$$
$$\{w_s\} = [G_{3_s}]\{q_3\} \tag{60}$$
$$\{w_{xs}\} = [G_{3_{xs}}]\{q_3\} \tag{61}$$
$$\{w_{xx}\} = [G_{3_{xx}}]\{q_3\} \tag{62}$$
$$\{w_{ss}\} = [G_{3_{ss}}]\{q_3\} \tag{63}$$
$$\{w_{xxx}\} = [G_{3_{xxx}}]\{q_3\} \tag{64}$$
$$\{w_{xxs}\} = [G_{3_{xxs}}]\{q_3\} \tag{65}$$
$$\{w_{xss}\} = [G_{3_{xss}}]\{q_3\} \tag{66}$$

Finally, the application of Eq. 1 at the domain nodal points and the use of Eq. 44-66 yields

$$[M]\{\ddot{q}\} + [C]\{\dot{q}\} + [K]\{q\} = \{g\} \tag{67}$$

where $[M]$, $[C]$, $[K]$ are generalized mass, damping, and stiffness matrices with dimensions $3M \times 3M$ and $\{g\}^T = \{\{X\}, \{S\}, \{Z\}\}$ is a vector including the $3M$ values of the external force.

Note that in the absence of inertia and damping forces $[M] =$

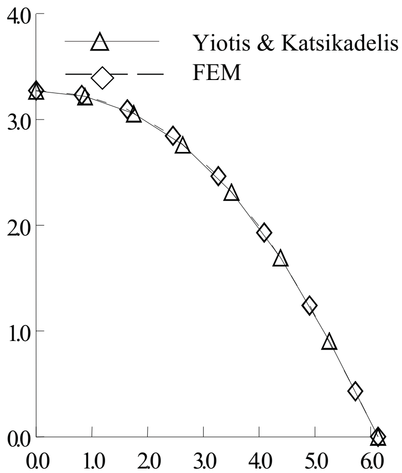

Figure 1 : Radial deflection ($w \times 10^4$ m) of a shell with simply supported the curved edges and the straight edges under uniform normal pressure.

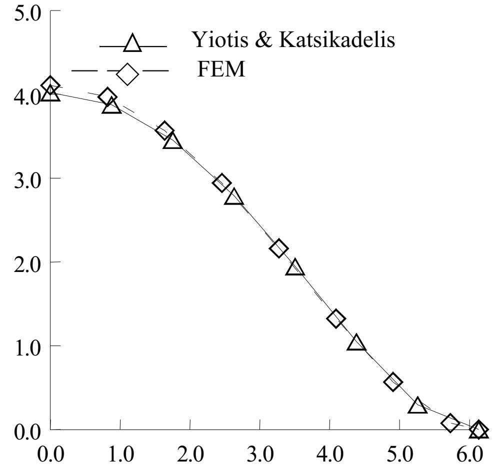

Figure 2 : Radial deflection of a shell with simply supported curved edges and clamped the straight edges under uniform normal pressure ($w \times 10^4$ m)

$[C] = 0$ and eqn Eq. 67 becomes

$$[K]\{q\} = \{g\} \tag{68}$$

which represents the static problem.

For free, undamped vibrations it is $\{g\} = 0$ and setting $q(x,s,t) = Q(x,s)e^{i\omega t}$ we obtain

$$\left([K] - \omega^2[M]\right)\{Q\} = \{0\} \tag{69}$$

from which the eigenfrequencies and mode shapes are established by solving a typical eigenvalue problem of linear algebra.

4 Numerical Examples

4.1 Example 1

A circular cylindrical shell (barrel vault problem) with all edges simply supported and subjected to uniform normal pressure has been analyzed. The numerical results have been obtained with $N = 84$, $M = 14 \times 28 = 392$. The radial deflections at the mid-shell have been calculated and compared with a FEM solution. The results from both solutions are shown in graphical form in Fig.1 and are found in good agreement with the FEM results (using a $15 \times 15 = 225$ mesh in one quarter of the shell). The employed data are $E = 2.1 \times 10^7$ KN/m^2, $h = 0.20$ m, $r_0 = 10.00$ m, $l = 24.54$ m, $\nu = 0.25$, $\phi = 70.30°$.

4.2 Example 2

The cylindrical shell of Example 1 but with different boundary conditions, namely the curved edges simply supported and the straight ones clamped, under uniform normal pressure has been analyzed. Again the transverse deflections at the curved axis of symmetry of the shell have been calculated and compared with those obtained from a FEM solution using the same discretization again. The results from both solutions are shown in graphical form in Fig.2 and are found in good agreement.

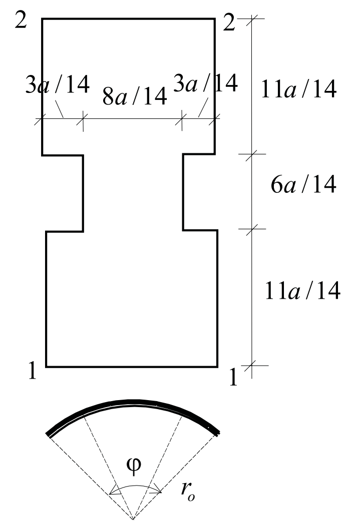

Figure 3 : Shell with two orthogonal cutouts symmetrically placed to its longitudinal axis.

4.3 Example 3

The third example is the same circular cylindrical shell but now cut by two parallel planes symmetrically to its longitudinal axis, so that in planform it has two orthogonal cutouts (Fig. 3). Its curved edges 1-1 and 2-2 are simply supported, whereas all other edges are clamped. It is subjected to grav-

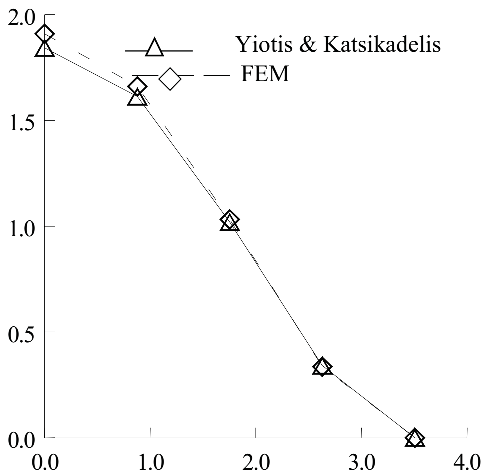

Figure 4 : Radial deflection of a shell with cutouts, simply supported the curved edges 1-1, 2-2 and clamped the other edges under gravity loading ($w \times 10^4$ m)

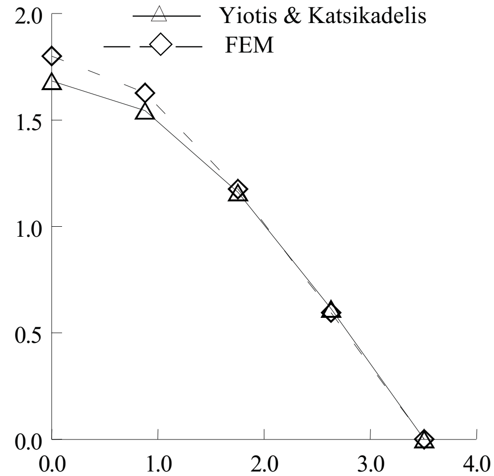

Figure 5 : Radial deflection ($w \times 10^4$ m) of a shell with cutouts, with simply supported all edges under gravity loading.

ity loading. The numerical results have been obtained with $N = 96$, $M = 356$. The radial deflections at the mid-shell have been calculated and compared with a FEM solution using 99 elements in one quarter of the shell The results from both solutions are shown in graphical form in Fig.4 and are found in good agreement. The employed data are those of the first example.

4.4 Example 4

The circular cylindrical panel with the two cutouts of example 3 has been analyzed under gravity loading but with all edges simply supported. The same discretization has been used again. The transverse deflections at the mid-shell have been calculated and compared with a FEM solution. The results from both solutions are shown in graphical form in Fig.5 and are found in good agreement.

4.5 Example 5

The free undamped vibrations of a simply supported circular cylindrical shell with $l/r_0 = 1$, $v = 0.30$, $r_0/h = 20$ and subtended angle $2\pi/3$ have been studied. The computed first 6 eigenfrequencies are presented in Tab. 1 as compared with existing results from other solutions [Koumousis (1981]. The numerical results have been obtained with $M = 400$ and $N = 100$. The results are in good agreement

4.6 Example 6

The free undamped vibrations of the circular cylindrical shell of Example 2 with the straight edges clamped have been studied. The computed first 6 eigenfrequencies are presented in Tab. 2 as compared with results obtained with a FEM solution

Table 1 : Eigenfrequencies $\Omega = r_0\omega\sqrt{(1-\nu^2)\rho/E}$ of the shell of the example 5

n	Present	Koumousis
1	0.539	0.536
2	0.589	0.582
3	0.774	0.731
4	0.872	0.858
5	0.932	1.010
6	1.020	1.234

using a 20×20 mesh over the whole shell. In the present analysis it has been used $M = 800$ and $N = 120$. The results from the two solutions are very close.

4.7 Example 7

The free undamped vibrations of the shell of Example 3 with its curved edges 1-1 and 2-2 simply supported, but with all other edges clamped, have been studied. It has been used $N = 96$, $M = 356$ again. The computed first 6 eigenfrequencies are presented in Tab. 3 as compared with results obtained with a FEM solution using 356 elements in the whole shell. The results from the two solutions are in good agreement

4.8 Example 8

The free undamped vibrations of the shell of Example 4 with all edges simply supported, have been studied. The discretization remains the same with that of example 7 for both methods. The computed first 6 eigenfrequencies are presented in Tab. 4 as compared with results obtained with a FEM solution. The

Table 2 : Eigenfrequencies $\Omega = r_0\omega\sqrt{(1-\nu^2)\rho/E}$ of the shell of example 2

n	Present	FEM
1	0.304	0.290
2	0.441	0.417
3	0.447	0.427
4	0.493	0.465
5	0.579	0.547
6	0.580	0.554

Table 3 : Eigenfrequencies $\Omega = r_0\omega\sqrt{(1-\nu^2)\rho/E}$ of the shell of example 3

n	Present	FEM
1	0.510	0.472
2	0.511	0.478
3	0.560	0.508
4	0.558	0.510
5	0.738	0.699
6	0.775	0.733

Table 4 : Eigenfrequencies $\Omega = r_0\omega\sqrt{(1-\nu^2)\rho/E}$ of the shell of example 8

n	Present	FEM
1	0.428	0.429
2	0.479	0.441
3	0.493	0.445
4	0.494	0.465
5	0.636	0.599
6	0.731	0.702

results from the two solutions are again very close.

5 Conclusions

The AEM, a BEM-based method is developed for the static and dynamic analysis of thin shells. The method is illustrated without restricting the generality for cylindrical shell panels, which may have cutouts. The basic conclusions that can be drawn from this investigation are the following:

1. The shell analysis problem is converted to the solution of two Laplacian and one biharmonic problems, which can be solved with the BEM using the well known fundamental solutions. Thus, the BEM, alleviated from the difficult task of establishing the fundamental solution of the coupled partial differential equations with variable coefficients, becomes a useful and versatile computational tool for shell analysis.

2. The proposed formulation can be converted to a boundary-only BEM by expanding the fictitious load into radial base function series using either the Dual Reciprocity Method or the Particular Solution Method.

3. The static problem results from the dynamic one as a special case (zero inertia and damping forces).

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Appendix A: Derivatives of the kernels $\Lambda_i(r)$

$$\Lambda_{1s} = -\frac{\sin(\omega-\phi)}{r^2} \tag{70}$$

$$\Lambda_{1xs} = -\frac{2\sin(2\omega-\phi)}{r^3} \tag{71}$$

$$\Lambda_{1xx} = -\frac{2\cos(2\omega-\phi)}{r^3} \tag{72}$$

$$\Lambda_{1ss} = \frac{2\cos(2\omega-\phi)}{r^3} \tag{73}$$

$$\Lambda_{1xxx} = -\frac{-6\cos\omega\cos\phi(3-4\cos^2\omega)}{r^4} - \frac{6\sin\omega\sin\phi(4\cos^2\omega-1)}{r^4} \tag{74}$$

$$\Lambda_{1xxs} = -\frac{6\sin\omega\cos\phi(4\cos^2\omega-1)}{r^4} - \frac{6\cos\omega\sin\phi(3-4\cos^2\omega)}{r^4} \tag{75}$$

$$\Lambda_{1xss} = -\frac{-6\sin\omega\sin\phi(4\cos^2\omega-1)}{r^4} - \frac{6\cos\omega\cos\phi(3-4\cos^2\omega)}{r^4} \tag{76}$$

$$\Lambda_{2s} = -\frac{\sin\omega}{r} \tag{77}$$

$$\Lambda_{2xx} = -\frac{\cos 2\omega}{r^2} \tag{78}$$

$$\Lambda_{2ss} = \frac{\cos 2\omega}{r^2} \tag{79}$$

$$\Lambda_{2xs} = -\frac{\sin 2\omega}{r^2} \tag{80}$$

$$\Lambda_{2xxx} = -\frac{2\cos\omega}{r^3}\left(1-4\sin^2\omega\right) \tag{81}$$

$$\Lambda_{2xxs} = -\frac{2\sin\omega}{r^3}\left(3\cos^2\omega-\sin^2\omega\right) \tag{82}$$

$$\Lambda_{2xss} = -\frac{2\cos\omega}{r^3}\left(3\sin^2\omega-\cos^2\omega\right) \tag{83}$$

$$\Lambda_{3s} = \frac{1}{4}\left[(2\ln r+3)\sin\omega\cos\phi+(2\ln r+1)\cos\omega\sin\phi\right] \tag{84}$$

$$\Lambda_{3xx} = \frac{\sin 2\omega\sin\phi}{2r} - \frac{\cos\phi}{2r} \tag{85}$$

$$\Lambda_{3ss} = -\frac{\sin 2\omega\sin\phi}{2r} - \frac{\cos\phi}{2r} \tag{86}$$

$$\Lambda_{3xs} = \frac{\cos 2\omega\sin\phi}{2r} \tag{87}$$

$$\Lambda_{3xxx} = -\frac{\cos(\omega-\phi)(1+2\sin^2\omega)}{2r} + \frac{2\cos\omega\sin 2\omega\sin\phi}{2r^2} \tag{88}$$

$$\Lambda_{3xxs} = -\frac{-\sin\omega\cos(2\omega-\pi)+\sin\phi\cos 3\omega}{2r} \tag{89}$$

$$\Lambda_{4s} = -\frac{1}{4}r\,(2\ln r+1)\sin\omega \tag{90}$$

$$\Lambda_{4xx} = \frac{1}{2}(\ln r+1)+\frac{1}{4}\cos 2\omega \tag{91}$$

$$\Lambda_{4ss} = \frac{1}{2}(\ln r + 1) - \frac{1}{4}\cos 2\omega \tag{92}$$

$$\Lambda_{4xs} = \frac{\sin 2\omega}{4} \tag{93}$$

$$\Lambda_{4xxx} = \frac{\cos\omega}{2r}(\cos 2\omega - 2) \tag{94}$$

$$\Lambda_{4xxs} = \frac{\cos 2\omega \sin\omega}{2r} \tag{95}$$

$$\Lambda_{4xss} = -\frac{\cos 2\omega \cos\omega}{2r} \tag{96}$$

$\phi = \text{angle}(r, \mathbf{n}) \quad \omega = \text{angle}(r, x)$

Appendix B: Evaluation of singular domain integrals

The kernels in Eq. 47-53 for small values of the argument r behave as in $\ln r$, $\frac{1}{r}$, $\frac{1}{r^2}$, $\frac{1}{r^3}$.

Therefore, we have to evaluate singular and hypersingular domain integrals on the internal cells. This can be effectively done by converting the domain singular integrals over a domain cell into regular integrals along its boundary using Green's reciprocal identity.

$$\int\int_\Omega \left(u\nabla^2 U - U\nabla^2 u\right) d\sigma = \int_\Gamma \left(u\frac{\partial U}{\partial n} - U\frac{\partial u}{\partial n}\right) ds \tag{97}$$

Eq. 97 for $u = 1$ and

$$\nabla^2 U = v^* \tag{98}$$

with $v^* = \ln r + 1$ or $v^* = \frac{r^2 \ln r}{4}$ yields

$$\int\int_\Omega v^* d\sigma = \int_\Gamma \frac{\partial U}{\partial n} ds \tag{99}$$

The function U is obtained by integrating equation in Eq. 98. Thus, we have

For $v^* = \ln r + 1$, $U = \frac{r^2 \ln r}{4}$

For $v^* = \frac{r^2 \ln r}{4}$, $U = \frac{(2r^4 \ln r - r^4)}{128}$

For the domain integrals involving derivatives, U must be replaced by its corresponding derivative in Eq. 99, namely

$$\int\int_\Omega v_m^* d\sigma = \int_\Gamma \frac{\partial^2 U}{\partial^2 mn} ds \quad m = x, s, xx, sx, ss, \ldots, xss. \tag{100}$$